

## SOME LOCAL PROPERTIES DEFINING $\mathcal{T}_0$ -GROUPS AND RELATED CLASSES OF GROUPS

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**Abstract:** We call  $G$  a  $\text{Hall}_{\mathcal{X}}$ -group if there exists a normal nilpotent subgroup  $N$  of  $G$  for which  $G/N'$  is an  $\mathcal{X}$ -group. We call  $G$  a  $\mathcal{T}_0$ -group provided  $G/\Phi(G)$  is a  $\mathcal{T}$ -group, that is, one in which normality is a transitive relation. We present several new local classes of groups which locally define  $\text{Hall}_{\mathcal{X}}$ -groups and  $\mathcal{T}_0$ -groups where  $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$ ; the classes  $\mathcal{PT}$  and  $\mathcal{PST}$  denote, respectively, the classes of groups in which permutability and S-permutability are transitive relations.

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### 1. Introduction

All groups considered will be finite. All unexplained notation or terminology can be found in [3] or [8].

There is a well-known theorem of Philip Hall which states that if a group  $G$  possesses a normal nilpotent subgroup  $N$  such that  $G/N'$  is nilpotent, then  $G$  is nilpotent. It has been the subject of several papers to consider what can be said about the class of groups  $G$  possessing a normal nilpotent subgroup  $N$  such that  $G/N'$  is an  $\mathcal{X}$ -group for various classes  $\mathcal{X}$ . In particular, the fourth author in [6], for the solvable case, answered this question when  $\mathcal{X}$  is any one of the classes  $\mathcal{T}$ ,  $\mathcal{PT}$ , or  $\mathcal{PST}$ , that is, respectively, the classes where normality, permutability, and S-permutability are transitive relations. Let us call  $G$  a  $\text{Hall}_{\mathcal{X}}$ -group if there exists a normal nilpotent subgroup  $N$  of  $G$  for which  $G/N'$  is an  $\mathcal{X}$ -group. Since the Fitting subgroup of a group  $G$  contains all normal nilpotent subgroups one can simply replace  $N$  in the definition just given with  $F$ , the Fitting subgroup of  $G$ .

It is a classical problem to determine the structure of a group  $G$  such that  $G/\Phi(G)$  is an  $\mathcal{X}$ -group for various classes  $\mathcal{X}$ . For a class  $\mathcal{X}$ , let us call  $G$  an  $\mathcal{X}_0$ -group if  $G/\Phi(G)$  is an  $\mathcal{X}$ -group. The fourth author in [6] characterized the solvable  $\mathcal{X}_0$ -groups when  $\mathcal{X}$  is one of the classes  $\mathcal{T}$ ,  $\mathcal{PT}$ ,

or  $\mathcal{PST}$ . In particular, it was shown that within the class of solvable groups, the classes  $\mathcal{T}_0$ ,  $\mathcal{PT}_0$ , and  $\mathcal{PST}_0$  coincide.

In this paper, we turn to the concept of a “local” characterization theorem for solvable  $\mathcal{T}_0$ -groups and Hall $_{\mathcal{X}}$ -groups for  $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$ . That is, we are interested in characterizations in terms of the different prime divisors of the orders of such groups.

## 2. Preliminaries

The classes  $\mathcal{C}_p$ ,  $\mathcal{X}_p$ , and  $\mathcal{Y}_p$ , and the theorems that follow, will be critical in giving the desired local characterization theorems.

### Definition.

- (1)  $\mathcal{C}_p$  is the class of groups  $G$  for which each subgroup of a Sylow  $p$ -subgroup  $P$  of  $G$  is normal in the normalizer,  $N_G(P)$ , of  $P$ .
- (2)  $\mathcal{X}_p$  is the class of groups  $G$  for which each subgroup of a Sylow  $p$ -subgroup  $P$  of  $G$  is permutable in the normalizer,  $N_G(P)$ , of  $P$ .
- (3)  $\mathcal{Y}_p$  is the class of groups  $G$  where  $H \leq S \leq P$  with  $P$  a Sylow  $p$ -subgroup of  $G$ , always implies  $H$  is S-permutable in the normalizer,  $N_G(S)$ , of  $S$ .

**Theorem 1.** *Let  $G$  be a group.*

- (1) (Robinson [7], see also [3, 2.2.2])  *$G$  is a solvable  $\mathcal{T}$ -group if and only if  $G$  is a  $\mathcal{C}_p$ -group for all primes  $p$ .*
- (2) (Beidleman, Brewster, and Robinson [5], see also [3, 2.2.3])  *$G$  is a solvable  $\mathcal{PT}$ -group if and only if  $G$  is an  $\mathcal{X}_p$ -group for all primes  $p$ .*
- (3) (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.9])  *$G$  is a solvable  $\mathcal{PST}$ -group if and only if  $G$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ .*

We will need some results on the classes  $\mathcal{C}_p$ ,  $\mathcal{X}_p$ , and  $\mathcal{Y}_p$  to aid in our characterization theorems.

**Lemma 1** (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.13]). *A group  $G$  is a  $\mathcal{Y}_p$ -group if and only if  $G$  is  $p$ -nilpotent or  $G$  is a  $\mathcal{C}_p$ -group with abelian Sylow  $p$ -subgroups.*

**Lemma 2** (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.4]). *A group  $G$  is an  $\mathcal{X}_p$ -group (respectively,  $\mathcal{C}_p$ -group) if and only if  $G$  is a  $\mathcal{Y}_p$ -group and the Sylow  $p$ -subgroups of  $G$  are Iwasawa (respectively, Dedekind) groups.*

In the previous lemma, Iwasawa groups (respectively, Dedekind groups) are the groups for which every subgroup is permutable (respectively, normal).

**Lemma 3.** *Let  $G$  be a group with  $N$  a normal  $p'$ -subgroup of  $G$  and  $M$  a normal  $p$ -subgroup of  $G$ .*

- (1) *If  $G$  is a  $\mathcal{Y}_p$ -group, then  $G/M$  is a  $\mathcal{Y}_p$ -group.*
- (2) *If  $G$  is an  $\mathcal{X}_p$ -group, then  $G/M$  is an  $\mathcal{X}_p$ -group.*
- (3) *If  $G$  is a  $\mathcal{C}_p$ -group, then  $G/M$  is a  $\mathcal{C}_p$ -group.*
- (4)  *$G$  is a  $\mathcal{Y}_p$ -group if and only if  $G/N$  is a  $\mathcal{Y}_p$ -group.*
- (5)  *$G$  is an  $\mathcal{X}_p$ -group if and only if  $G/N$  is an  $\mathcal{X}_p$ -group.*
- (6)  *$G$  is a  $\mathcal{C}_p$ -group if and only if  $G/N$  is a  $\mathcal{C}_p$ -group.*

*Proof:* (1), (2), and (3) follow directly from the definitions of  $\mathcal{Y}_p$ ,  $\mathcal{X}_p$ , and  $\mathcal{C}_p$ .

(4) This can be found in [4] as Lemma 6.

(5) This can be found in [1] as Lemma 4.1.

(6) Let  $G$  be a  $\mathcal{C}_p$ -group. By Lemma 2, this is equivalent to affirming that  $G$  is a  $\mathcal{Y}_p$ -group with Dedekind Sylow  $p$ -subgroups. Since the Sylow  $p$ -subgroups of  $G$  and  $G/N$  are isomorphic, by (4) we have that this is equivalent to stating that  $G/N$  is a  $\mathcal{Y}_p$ -group with Dedekind Sylow  $p$ -subgroups. By Lemma 2, this is to say that  $G/N$  is a  $\mathcal{C}_p$ -group. This completes the proof.  $\square$

*Remark.* In fact, the classes  $\mathcal{C}_p$ ,  $\mathcal{X}_p$ , and  $\mathcal{Y}_p$  are quotient closed in general, however, this is a much more difficult fact to deduce. For our purposes, quotient closure for normal  $p'$ -subgroups and  $p$ -subgroups is sufficient.

Lastly, we record some results on  $\mathcal{T}_0$ -groups which will be needed.

**Theorem 2.**

- (1) (van der Waall and Fransman [9]) *The class of solvable  $\mathcal{T}_0$ -groups is a quotient closed class of groups.*
- (2) (van der Waall and Fransman [9]) *A solvable  $\mathcal{T}_0$ -group is supersolvable.*
- (3) (Ragland [6]) *A group  $G$  is a solvable Hall $_{\mathcal{PST}}$ -group if and only if  $G$  is a solvable  $\mathcal{T}_0$ -group.*
- (4) (Ragland [6]) *A group  $G$  is a solvable  $\mathcal{PST}_0$ -group if and only if  $G$  is a solvable  $\mathcal{T}_0$ -group.*
- (5) (Ragland [6]) *A group  $G$  is a solvable  $\mathcal{T}_0$ -group with abelian nilpotent residual if and only if  $G$  is a solvable  $\mathcal{PST}$ -group.*

### 3. Main results

In this section we provide local characterization theorems for the solvable  $\mathcal{T}_0$ -groups, Hall $\mathcal{T}$ -groups, Hall $\mathcal{PT}$ -groups, and Hall $\mathcal{PST}$ -groups. We will need the following definitions:

**Definition.** Let  $p$  be a prime. For a group  $G$ , let  $\Phi(G)_p$  denote the Sylow  $p$ -subgroup of the Frattini subgroup of  $G$ . Let  $F'_p$  denote the derived subgroup of the Sylow  $p$ -subgroup of the Fitting subgroup,  $F$ , of  $G$ . Let  $L'_p$  denote the derived subgroup of a Sylow  $p$ -subgroup of the nilpotent residual,  $L$ , of  $G$ . If  $p$  does not divide the order of  $\Phi(G)$  then let us denote by  $\Phi(G)_p$  the identity subgroup. Likewise,  $F'_p = 1$  if  $p$  does not divide the order of  $F'$  and  $L'_p = 1$  if  $p$  does not divide the order of  $L'$ . Let  $\mathcal{L}$  be one of the classes  $\mathcal{C}_p$ ,  $\mathcal{X}_p$ , or  $\mathcal{Y}_p$ .

- (1) Define  $\Phi_{\mathcal{L}}$  to be the class of groups  $G$  for which  $G/\Phi(G)_p$  is a  $\mathcal{L}$ -group.
- (2) Define  $\mathcal{F}_{\mathcal{L}}$  to be the class of groups  $G$  for which  $G/F'_p$  is a  $\mathcal{L}$ -group.
- (3) Define  $\mathcal{L}_{\mathcal{L}}$  to be the class of groups  $G$  for which  $L$  is  $p'$ -nilpotent and  $G/L'_p$  is a  $\mathcal{L}$ -group.

Since the Frattini factor group of a solvable  $\mathcal{T}_0$ -group is a  $\mathcal{T}$ -group, and  $\mathcal{T}$ -groups can be locally characterized using the class  $\mathcal{C}_p$ , it stands to reason that  $\mathcal{T}_0$ -groups can be locally characterized using the class  $\mathcal{C}_p$  with certain conditions involving the Frattini subgroup. This is indeed the case, however, recall that  $\mathcal{T}_0 = \mathcal{PT}_0 = \mathcal{PST}_0$  in the class of solvable groups. So it also stands to reason that one could replace  $\mathcal{C}_p$  with  $\mathcal{X}_p$  or  $\mathcal{Y}_p$ . In view of Theorem 2, part (3), one should be able to characterize the solvable  $\mathcal{T}_0$ -groups using  $\mathcal{Y}_p$  with certain conditions placed upon the Fitting subgroup. The details can be found in the following theorem:

**Theorem 3.** *The following are equivalent for a group  $G$ .*

- (1)  $G$  is an  $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes  $p$ .
- (2)  $G$  is an  $\mathcal{F}_{\mathcal{Y}_p}$ -group for all primes  $p$ .
- (3)  $G$  is a solvable Hall $\mathcal{PST}$ -group.
- (4)  $G$  is a  $\Phi_{\mathcal{C}_p}$ -group for all primes  $p$ .
- (5)  $G$  is a  $\Phi_{\mathcal{X}_p}$ -group for all primes  $p$ .
- (6)  $G$  is a  $\Phi_{\mathcal{Y}_p}$ -group for all primes  $p$ .
- (7)  $G$  is a solvable  $\mathcal{T}_0$ -group.

*Proof:* Throughout the proof let us denote the nilpotent residual of  $G$  by  $L$  and the Fitting subgroup of  $G$  by  $F$ . Also, for any group  $H$ ,  $H_p$  will denote a Sylow  $p$ -subgroup of  $H$ .

(1)  $\Rightarrow$  (2) If  $G$  is an  $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes  $p$  then  $L$  is  $p'$ -nilpotent for all primes  $p$  and hence nilpotent. Thus  $L \leq F$  so that for any prime  $p$  we have  $L'_p \leq F'_p$ . Now since  $G/L'_p$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ , we have, by Lemma 3, part (1), that  $(G/L'_p)/(F'_p/L'_p) \simeq G/F'_p$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ .

(2)  $\Rightarrow$  (3) If  $G$  is an  $\mathcal{F}_{\mathcal{Y}_p}$ -group for all primes  $p$ , then  $G/F'_p$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ . Using Lemma 3, part (4), we have  $(G/F'_p)/(F'/F'_p) \simeq G/F'$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ . Hence by Theorem 1, part (3), we have that  $G/F'$  is a solvable  $\mathcal{PST}$ -group so that  $G$  is a solvable Hall $_{\mathcal{PST}}$ -group.

(3)  $\Rightarrow$  (4) If  $G$  is a solvable Hall $_{\mathcal{PST}}$ -group, then, by Theorem 2, part (3), we have that  $G$  is a solvable  $\mathcal{T}_0$ -group. Let  $\bar{G} = G/\Phi(G)_p$ . Then  $\bar{G}$  is a solvable  $\mathcal{T}_0$ -group by Theorem 2, part (1). By induction, if  $\Phi(G)_p \neq 1$ , we have  $\bar{G}/\Phi(\bar{G})_p$  is a  $\mathcal{C}_p$ -group. But  $\Phi(\bar{G})_p = 1$  and so  $\bar{G}$  is a  $\mathcal{C}_p$ -group. If  $\Phi(G)_p = 1$  then one only needs  $G$  itself to be a  $\mathcal{C}_p$  group. Now  $G/\Phi(G)$  is a  $\mathcal{T}$ -group and hence a  $\mathcal{C}_p$ -group by Theorem 1, part (1). Lemma 3, part (6), gives us that  $G$  is a  $\mathcal{C}_p$ -group since  $\Phi(G)$  is a  $p'$ -group.

Both (4)  $\Rightarrow$  (5) and (5)  $\Rightarrow$  (6) follow directly from the definitions of  $\mathcal{C}_p$ ,  $\mathcal{X}_p$ , and  $\mathcal{Y}_p$ .

(6)  $\Rightarrow$  (7) If  $G$  is a  $\Phi_{\mathcal{Y}_p}$ -group for all primes  $p$ , then  $G/\Phi(G)_p$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ . Since  $\Phi(G)/\Phi(G)_p$  is a  $p'$ -group for all primes  $p$ , we have, by Lemma 3, part (4), that  $(G/\Phi(G)_p)/(\Phi(G)/\Phi(G)_p) \simeq G/\Phi(G)$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ . Thus  $G/\Phi(G)$  is a  $\mathcal{PST}$ -group by Theorem 1, part (3). Hence  $G$  is a solvable  $\mathcal{PST}_0$ -group and thus a solvable  $\mathcal{T}_0$ -group by Theorem 2, part (4).

(7)  $\Rightarrow$  (1) If  $G$  is a solvable  $\mathcal{T}_0$ -group, then  $G$  is supersolvable by Theorem 2, part (2). Thus  $G'$  is nilpotent in which case so is  $L$ . Hence  $L$  is  $p'$ -nilpotent for all primes  $p$ . Note that if  $L$  is abelian then, by Theorem 2, part (5),  $G$  is a  $\mathcal{PST}$ -group and hence a  $\mathcal{Y}_p$ -group for all primes  $p$  by Theorem 1, part (3). From Lemma 3, part (1), it would follow that  $G$  is a  $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes  $p$  if  $L$  is abelian. So let us assume  $L$  is not abelian.

Let  $\bar{G} = G/L'_p$  and let  $\bar{L}$  be the nilpotent residual of  $\bar{G}$ . Now  $\bar{G}$  is a solvable  $\mathcal{T}_0$ -group by Theorem 2, part (1). By induction, if  $L'_p \neq 1$ , we have  $\bar{G}/\bar{L}'_p$  is a  $\mathcal{Y}_p$ -group. But  $\bar{L}'_p = 1$  and so  $\bar{G}$  is a  $\mathcal{Y}_p$ -group.

Assume that  $L'_p = 1$ . Since  $L$  is not abelian there exists a prime  $q \neq p$  for which  $L'_q \neq 1$ . Now let  $\bar{G}$  denote  $G/L'_q$  with  $\bar{L}$  the nilpotent residual of  $\bar{G}$  and note  $\bar{G}$  is a  $\mathcal{T}_0$ -group by Theorem 2, part (1). So, by induction,  $\bar{G}/\bar{L}'_p$  is a  $\mathcal{Y}_p$ -group. Thus  $\bar{G}$  is a  $\mathcal{Y}_p$ -group and hence  $G$  is a  $\mathcal{Y}_p$ -group by Lemma 3, part (4). It now follows that  $G$  is a  $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes  $p$  completing the proof.  $\square$

The classes of solvable Hall $\mathcal{T}$ -groups and Hall $\mathcal{PT}$ -groups admit similar local characterizations which we provide now.

**Theorem 4.** *The following are equivalent for a group  $G$ .*

- (1)  $G$  is a  $\mathcal{F}_{\mathcal{X}_p}$ -group for all primes  $p$ .
- (2)  $G$  is a solvable Hall $\mathcal{PT}$ -group.

*Proof:* (1)  $\Rightarrow$  (2) The proof is similar to the proof of (2)  $\Rightarrow$  (3) from Theorem 3. One only needs to use invoke the appropriate results corresponding to  $\mathcal{X}_p$  from Lemma 3 and Theorem 1.

(2)  $\Rightarrow$  (1) If  $G$  is a solvable Hall $\mathcal{PT}$ -group, then it is a solvable Hall $\mathcal{PST}$ -group and hence by Theorem 3 we have  $G/F'_p$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ .

Now  $G/F'$  a solvable  $\mathcal{PT}$ -group and thus an  $\mathcal{X}_p$ -group for all primes  $p$  by Theorem 1, part (2). Hence the Sylow subgroups of  $G/F'$  are Iwasawa. Let  $P$  denote a Sylow  $p$ -subgroup of  $G$ . Then  $PF'/F' \simeq P/(P \cap F') = P/F'_p$  and thus  $P/F'_p$  is an Iwasawa group.

Now one can conclude from Theorem 2 that  $G/F'_p$  is an  $\mathcal{X}_p$ -group for all primes  $p$  so that  $G$  is a  $\mathcal{F}_{\mathcal{X}_p}$ -group for all primes  $p$ .  $\square$

**Theorem 5.** *The following are equivalent for a group  $G$ .*

- (1)  $G$  is a  $\mathcal{F}_{\mathcal{E}_p}$ -group for all primes  $p$ .
- (2)  $G$  is a solvable Hall $\mathcal{T}$ -group.

*Proof:* Using the appropriate theorems and lemmas, the proof is quite similar to that of Theorem 4 and so we omit the proof.  $\square$

The next theorem shows that the difference between solvable  $\mathcal{T}_0$ -groups and Hall $\mathcal{PT}$ -groups (Hall $\mathcal{T}$ -groups) amounts to the Sylow structure of  $G/F'_p$  for all primes  $p$ .

**Theorem 6.** *Let  $G$  be a solvable group with  $F = \text{Fit}(G)$ . Then  $G$  is a  $\mathcal{T}_0$ -group where the Sylow  $p$ -subgroups of  $G/F'_p$  are Iwasawa (Dedekind) groups for all primes  $p$  if and only if  $G$  is a Hall $\mathcal{PT}$ -group (Hall $\mathcal{T}$ -group).*

*Proof:* Suppose  $G$  is a  $\mathcal{T}_0$ -group where the Sylow  $p$ -subgroups of  $G/F'_p$  are Iwasawa (Dedekind) groups. By Theorem 3, we have  $G/F'_p$  is a

$\mathcal{Y}_p$ -group for all primes  $p$ . By Lemma 2, we must have  $G/F'_p$  is an  $\mathcal{X}_p$ -group ( $\mathcal{C}_p$ -group) for all primes  $p$ . We can now deduce from Theorem 4, part (2), that  $G$  is a Hall $\mathcal{PT}$ -group (Hall $\mathcal{T}$ -group).

Suppose  $G$  is a Hall $\mathcal{PT}$ -group (Hall $\mathcal{T}$ -group). Then Theorem 4, part (5), says that  $G/F'_p$  is an  $\mathcal{X}_p$ -group ( $\mathcal{C}_p$ -group) for all primes  $p$ . Using Lemma 2, we can deduce that  $G/F'_p$  is a  $\mathcal{Y}_p$ -group for all primes  $p$  and that the Sylow  $p$ -subgroups of  $G/F'_p$  are Iwasawa (Dedekind) groups. That  $G$  is a  $\mathcal{T}_0$ -group follows from Theorem 3.  $\square$

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