

INEQUALITIES FOR POISSON INTEGRALS WITH SLOWLY GROWING DIMENSIONAL CONSTANTS

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Abstract

Let P_t be the Poisson kernel. We study the following L^p inequality for the Poisson integral $Pf(x, t) = (P_t * f)(x)$ with respect to a Carleson measure μ :

$$\|Pf\|_{L^p(\mathbb{R}_+^{n+1}, d\mu)} \leq c_{p,n} \kappa(\mu)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n, dx)},$$

where $1 < p < \infty$ and $\kappa(\mu)$ is the Carleson norm of μ . It was shown by Verbitsky [V] that for $p > 2$ the constant $c_{p,n}$ can be taken to be independent of the dimension n . We show that $c_{2,n} = O((\log n)^{\frac{1}{2}})$ and that $c_{p,n} = O(n^{\frac{1}{p} - \frac{1}{2}})$ for $1 < p < 2$ as $n \rightarrow \infty$. We observe that standard proofs of this inequality rely on doubling properties of cubes and lead to a value of $c_{p,n}$ that grows exponentially with n .

1. Introduction

The object of study in this article is the following Carleson measure inequality [C1], [C2], valid for $1 < p < \infty$

$$(1.1) \quad \|Pf\|_{L^p(\mathbb{R}_+^{n+1}, d\mu)} \leq c_{p,n} \kappa(\mu)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$

where μ is a Carleson measure on \mathbb{R}_+^{n+1} with norm $\kappa(\mu)$ and P is the Poisson integral of a function f on \mathbb{R}^n .

We begin by recalling these notions and establishing notation. We denote by $\bar{B}(x_0, r)$ the closed ball in \mathbb{R}^n with radius r centered at x_0 . The Carleson tent $T[\bar{B}(x_0, r)]$ over the ball $\bar{B}(x_0, r)$ is defined as the

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set of all points $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$ such that $|x - x_0|^2 + t^2 < r^2$. Denote by $|K|$ the Lebesgue measure of a set K in \mathbb{R}^n . A Borel measure μ on \mathbb{R}_+^{n+1} is called a Carleson measure if its Carleson norm $\kappa(\mu) = \sup \left\{ \frac{\mu(T[\overline{B}(x_0, r)])}{|\overline{B}(x_0, r)|} : x_0 \in \mathbb{R}^n, r > 0 \right\}$ is a finite number.

We define the Poisson integral Pf , i.e., the harmonic extension of f to \mathbb{R}_+^{n+1} , as the convolution

$$(1.2) \quad Pf(x, a) = P_a * f(x) = \int_{\mathbb{R}^n} P_a(x - y)f(y) dy,$$

where the Poisson kernel is defined, for all $a > 0$, $x \in \mathbb{R}^n$ by

$$(1.3) \quad P_a(x) = \frac{\gamma_n a}{(a^2 + |x|^2)^{\frac{n+1}{2}}}, \quad \text{with } \gamma_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The Lebesgue measure of the closed unit ball $\overline{B}(0, 1)$ and the surface measure of its boundary are denoted by

$$(1.4) \quad \Omega_n = |\overline{B}(0, 1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad \omega_{n-1} = |\partial\overline{B}(0, 1)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} = n\Omega_n.$$

Verbitsky [V] gave an elegant proof of (1.1) that yields a constant $c_{p,n}$ independent of n whenever $p > 2$. The starting point of Verbitsky's argument is the use of interpolation and duality to derive (1.1) from the equivalent inequalities

$$(1.5) \quad \|Pf\|_{L^{2,\infty}(\mathbb{R}_+^{n+1}, d\mu)} \leq c' \kappa(\mu)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^n)},$$

$$(1.6) \quad \|P^*(g, \mu)\|_{L^2(\mathbb{R}^n)} \leq c' \kappa(\mu)^{\frac{1}{2}} \|g\|_{L^{2,1}(\mathbb{R}_+^{n+1}, d\mu)},$$

for some absolute constant c' independent of n , where P^* is the Balayage operator defined for functions g on \mathbb{R}_+^{n+1} by

$$P^*(g, \mu)(x) = \int_{\mathbb{R}_+^{n+1}} P_b(x - y)g(y, b) d\mu(y, b).$$

Using the semigroup property for the Poisson kernel,

$$(1.7) \quad \int_{\mathbb{R}^n} P_a(\tau - x)P_b(\tau - y) d\tau = P_{a+b}(x - y),$$

Verbitsky obtained (1.6) with c' independent of n for characteristic functions of subsets of \mathbb{R}_+^{n+1} and this is enough to establish (1.6) for general μ -measurable functions g (see [SW]).

The following theorem is the main result of this article, which grew out of our attempts to extend Verbitsky's theorem to the case $1 < p \leq 2$:

Theorem 1.1. *For $1 < p < 2$ the following Carleson measure inequality holds:*

$$(1.8) \quad \|Pf\|_{L^p(\mathbb{R}_+^{n+1}, d\mu)} \leq C_p n^{\frac{1}{p}-\frac{1}{2}} \kappa(\mu)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$

with C_p independent of n . For $p = 2$ the following holds:

$$(1.9) \quad \|Pf\|_{L^2(\mathbb{R}_+^{n+1}, d\mu)} \leq C(\log n)^{\frac{1}{2}} \kappa(\mu)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^n)},$$

with C independent of n .

We observe that the estimate above is sharper than those obtained from the other known proofs of the Carleson measure inequality (1.1) (see, e.g., [A], [C2], [G], [H], [N], [S]); these proofs yield constants that grow exponentially in n .

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2. An integral formula involving Poisson kernels

In order to extend the arguments in [V] we are lead to consider the equivalent inequalities for $k = 3, 4, 5, \dots$

$$\|Pf\|_{L^{\frac{k}{k-1}, \infty}(\mathbb{R}_+^{n+1}, d\mu)} \leq \bar{c}(k, n) \kappa(\mu)^{\frac{k-1}{k}} \|f\|_{L^{\frac{k}{k-1}}(\mathbb{R}^n)},$$

$$\|P^*(g, \mu)\|_{L^k(\mathbb{R}^n)} \leq \bar{c}(k, n) \kappa(\mu)^{\frac{k-1}{k}} \|g\|_{L^{k,1}(\mathbb{R}_+^{n+1}, d\mu)},$$

$$\|P^*(\chi_E, \mu)\|_{L^k(\mathbb{R}^n)} \leq c(k, n) \kappa(\mu)^{\frac{k-1}{k}} \mu(E)^{\frac{1}{k}}, \quad \forall E \subset \mathbb{R}_+^{n+1}.$$

For $k = 2$ these inequalities hold with $c(2, n) \leq c_{2,n} \leq c_2$ independent of n , due to [V], (see (1.5) above). Our goal is to obtain good estimates for the constants $c(k, n)$ for all $k = 3, 4, \dots$ and then use the Marcinkiewicz interpolation theorem to deduce (1.1).

Since

$$\|P^*(\chi_E, \mu)\|_{L^k(\mathbb{R}^n)}^k = \int_{E^k} \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(x_j - \tau) d\tau d\mu(x_1, a_1) \dots d\mu(x_k, a_k)$$

our first task is to derive a workable formula for

$$\int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(x_j - \tau) d\tau,$$

which, in the case $k = 2$, is computed explicitly via the semigroup property. In this section we prove the following:

Proposition 2.1. *Let $P_a(x)$ be the Poisson kernel, as in definition (1.3). Then, for any $a_1, \dots, a_k \in (0, \infty)$ and for any $x_1, \dots, x_k \in \mathbb{R}^n$, we have*

$$(2.1) \quad \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(\tau - x_j) d\tau \\ = \frac{a_1 a_2 \dots a_k}{\pi^{(k-1)\frac{n}{2} + \frac{k}{2}}} \int_0^\infty \dots \int_0^\infty \frac{(u_1 u_2 \dots u_k)^{\frac{n-1}{2}}}{(u_1 + u_2 + \dots + u_k)^{\frac{n}{2}}} \\ \times \text{Exp} \left[- \sum_{j=1}^k a_j^2 u_j - \frac{\sum_{1 \leq i < j \leq k} u_i u_j |x_i - x_j|^2}{u_1 + u_2 + \dots + u_k} \right] du_1 \dots du_k.$$

In particular, when $k = 3$, the formula above can be written as

$$(2.2) \quad \int_{\mathbb{R}^n} P_a(\tau - x) P_b(\tau - y) P_c(\tau - z) d\tau \\ = \frac{abc}{\pi^{n + \frac{3}{2}}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{(uvw)^{\frac{n-1}{2}}}{(u + v + w)^{\frac{n}{2}}} \\ \times \text{Exp} \left[- \left\{ a^2 u + b^2 v + c^2 w \right. \right. \\ \left. \left. + \frac{uv|x-y|^2 + uw|x-z|^2 + vw|y-z|^2}{u + v + w} \right\} \right] du dv dw,$$

and when $k = 2$, we have

$$(2.3) \quad \int_{\mathbb{R}^n} P_a(\tau - x) P_b(\tau - y) d\tau \\ = \frac{ab}{\pi^{\frac{n}{2} + 1}} \int_0^{+\infty} \int_0^{+\infty} \frac{(uv)^{\frac{n-1}{2}}}{(u + v)^{\frac{n}{2}}} e^{-\{a^2 u + b^2 v + \frac{uv}{u+v}|x-y|^2\}} du dv.$$

Proof: We start from the ‘‘subordination’’ formula

$$P_a(x) = \int_0^{+\infty} \beta_a(u) e^{-u|x|^2} du$$

where

$$\beta_a(u) = \frac{a}{\pi^{\frac{n+1}{2}}} e^{-a^2 u} u^{\frac{n-1}{2}}$$

which can be easily deduced using the definition of the Gamma function.

The integral on the left hand side in (2.1) can now be rewritten as

$$\begin{aligned}
(2.4) \quad & \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(\tau - x_j) d\tau \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^k \left(\int_0^{+\infty} \beta_{a_j}(u_j) e^{-u_j |\tau - x_j|^2} du_j \right) d\tau \\
&= \frac{a_1 \cdots a_k}{\pi^{\frac{(n+1)k}{2}}} \int_{\mathbb{R}^n} \int_{[0, \infty)^k} (u_1 \cdots u_k)^{\frac{n-1}{2}} e^{-\sum_{j=1}^k (a_j^2 u_j + u_j |\tau - x_j|^2)} du_1 \cdots du_k d\tau \\
&= \frac{a_1 \cdots a_k}{\pi^{\frac{(n+1)k}{2}}} \int_{[0, \infty)^k} (u_1 \cdots u_k)^{\frac{n-1}{2}} e^{-\sum_{j=1}^k a_j^2 u_j} \\
&\quad \times \left[\int_{\mathbb{R}^n} e^{-\sum_{j=1}^k u_j |\tau - x_j|^2} d\tau \right] du_1 \cdots du_k.
\end{aligned}$$

Let us now rewrite the inner integral inside square brackets using Cartesian coordinates, namely $\tau = (\tau_1, \dots, \tau_n)$ and $x_j = (x_{j1}, \dots, x_{jn})$. We obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} e^{-\sum_{j=1}^k u_j |\tau - x_j|^2} d\tau \int_{\mathbb{R}^n} e^{-\{u_1(\tau_1 - x_{11})^2 + u_2(\tau_1 - x_{21})^2 + \cdots + u_k(\tau_1 - x_{k1})^2\}} \dots \\
& \dots e^{-\{u_1(\tau_n - x_{1n})^2 + u_2(\tau_n - x_{2n})^2 + \cdots + u_k(\tau_n - x_{kn})^2\}} d\tau_1 d\tau_2 \dots d\tau_n = \prod_{r=1}^n I_r
\end{aligned}$$

where, for each index $r = 1, 2, \dots, n$, we have defined

$$(2.5) \quad I_r = \int_{-\infty}^{+\infty} e^{-(A\tau_r^2 - 2B\tau_r + C)} d\tau_r$$

with

$$\begin{aligned}
A &= u_1 + u_2 + \cdots + u_k \\
B &= B_r = u_1 x_{1r} + u_2 x_{2r} + \cdots + u_k x_{kr} \\
C &= C_r = u_1 x_{1r}^2 + u_2 x_{2r}^2 + \cdots + u_k x_{kr}^2.
\end{aligned}$$

We claim that, for each choice of the coordinate index $r = 1, \dots, n$ we have

$$B^2 - AC = B_r^2 - AC_r = - \sum_{i < j} u_i u_j (x_{ir} - x_{jr})^2$$

where the sum is taken over all possible pairs of distinct indices i and j both running from 1 to k (there are $k(k-1)/2$ of such pairs). In fact this claim is easily checked by observing that in $B^2 - AC$ all the square terms of the kind $u_i^2 x_{ir}^2$ cancel out, while the remaining mixed terms can be collected in groups of three, each group giving $-u_i u_j (x_{ir}^2 - 2x_{ir} x_{jr} + x_{jr}^2)$.

Now, completing the square in (2.5) we obtain

$$I_r = e^{\frac{B^2 - AC}{A}} \int_{-\infty}^{+\infty} e^{-A(\tau_r - \frac{B}{A})^2} d\tau_r = \sqrt{\frac{\pi}{u_1 + \dots + u_k}} e^{-\frac{\sum_{i < j} u_i u_j (x_{ir} - x_{jr})^2}{u_1 + \dots + u_k}}$$

which implies that

$$\int_{\mathbb{R}^n} e^{-\sum_{j=1}^k u_j |\tau - x_j|^2} d\tau = \left(\frac{\pi}{u_1 + \dots + u_k} \right)^{\frac{n}{2}} e^{-\frac{\sum_{i < j} u_i u_j |x_i - x_j|^2}{u_1 + \dots + u_k}}.$$

Using this identity in the square brackets in (2.4) and simplifying, we obtain (2.1). \square

Remark. The reader may wonder if it is possible to obtain a formula for the left hand side in (2.1) that does not involve any integrals, something analogous to (1.7) for $k \geq 3$. When $n = 1$ using residues (plus some involved algebraic manipulations) we were able to show that

$$\begin{aligned} & \int_{\mathbb{R}} P_a(\tau - x) P_b(\tau - y) P_c(\tau - z) d\tau \\ &= \frac{ab}{(a+c)(b+c)} P_{a+c}(x-z) P_{b+c}(y-z) \\ &+ \frac{ac}{(a+b)(c+b)} P_{b+c}(y-z) P_{a+b}(x-y) \\ &+ \frac{bc}{(b+a)(c+a)} P_{a+c}(x-z) P_{a+b}(x-y) \\ &+ 4\pi \frac{abc(a+b+c)}{(a+b)(a+c)(b+c)} P_{a+b}(x-y) P_{a+c}(x-z) P_{b+c}(y-z). \end{aligned}$$

Unfortunately, for $n > 1$, an integral-free formula of this sort is difficult to obtain. It seems that the size and complexity of the formula grows quickly with n , and furthermore, there is no obvious “leading term” to be used in our estimates as $n \rightarrow \infty$. On the other hand we will show that (2.1), after some manipulations, suffices for the purposes of the proof of Theorem 1.1.

3. Proof of Theorem 1.1: the case $\frac{3}{2} < p < 2$

For clarity of exposition we first give a detailed proof of Theorem 1.1 in the case $3/2 < p < 2$. In the next section we indicate how the same technique can be adapted to the case $p \in \left(\frac{k}{k-1}, 2\right)$, any $k = 4, 5, \dots$.

Let us start by showing that the Balayage operator P^* satisfies the following estimate for all Carleson measures μ , for all μ -measurable subsets E of \mathbb{R}_+^{n+1} , and for all $n = 1, 2, \dots$,

$$(3.1) \quad \|P^*(\chi_E, \mu)\|_{L^3(\mathbb{R}^n)}^3 \leq c'' n^{\frac{1}{2}} \kappa(\mu)^2 \mu(E),$$

for some absolute constant c'' . Once (3.1) is established, using duality and [SW] we obtain for some other absolute constant c'''

$$(3.2) \quad \|Pf\|_{L^{\frac{3}{2}, \infty}(\mathbb{R}_+^{n+1}, d\mu)} \leq c''' n^{\frac{1}{6}} \kappa(\mu)^{\frac{2}{3}} \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^n)}$$

and thus, by the Marcinkiewicz interpolation and (1.5), Theorem 1.1 follows in the case $3/2 < p < 2$.

We have

$$(3.3) \quad \|P^*(\chi_E, \mu)\|_{L^3(\mathbb{R}^n)}^3 \\ = \int_{E^3} \int_{\mathbb{R}^n} P_a(\tau - x) P_b(\tau - y) P_c(\tau - z) d\tau d\mu(x, a) d\mu(y, b) d\mu(z, c).$$

We will use the following modification of formula (2.2):

Lemma 3.1. *The integral in (2.2) can also be written as*

$$\int_{\mathbb{R}^n} P_a(\tau - x) P_b(\tau - y) P_c(\tau - z) d\tau \\ = \frac{\Gamma\left(\frac{n+3}{2}\right)}{\pi^{n+\frac{3}{2}}} abc \int_0^1 \int_0^1 \frac{(1-t)^n t^{\frac{n-1}{2}} (s(1-s))^{\frac{n-1}{2}}}{\{a^2 t + (1-t)B^2 + t(1-t)|x-q|^2\}^{n+\frac{3}{2}}} dt ds$$

where

$$(3.4) \quad B^2 = B^2(s, y, z) = b^2s + c^2(1-s) + s(1-s)|y-z|^2$$

$$(3.5) \quad q = q(s, y, z) = sy + (1-s)z.$$

We will also need the following estimates:

Lemma 3.2. *The following inequalities hold for $\alpha > 0$, $\beta > 0$, $\gamma > 0$ such that $\beta > \alpha$ and $\gamma < \beta + 2$ and for any $D \in \mathbb{R}$.*

$$\begin{aligned} & \int_0^1 t^\alpha (1-t)^\beta \{a^2t + (1-t)B^2 + t(1-t)D^2\}^{-\gamma} dt \\ & \leq \frac{1}{aB} \int_0^1 t^\alpha (1-t)^\beta \{a^2t + (1-t)B^2 + t(1-t)D^2\}^{-\gamma+1} dt \\ & \leq \frac{1}{aB} \int_0^1 t^\alpha (1-t)^{\beta-\gamma+1} \{B^2 + t(a^2 + D^2)\}^{-\gamma+1} dt. \end{aligned}$$

The proofs of these lemmas are postponed until the end of this section.

Returning to the proof of (3.1), we split (3.3) in the six regions $a \leq b \leq c$, $a \leq c \leq b$, $b \leq c \leq a$, $b \leq a \leq c$, $c \leq b \leq a$, and $c \leq a \leq b$; by symmetry we need only consider the region over which $a \leq b \leq c$. Following [V] we write

$$(3.6) \quad \begin{aligned} & \|P^*(\chi_E, \mu)\|_{L^3(\mathbb{R}^n)}^3 (6\mu(E))^{-1} \\ & \leq \sup_{(z,c) \in E} \int_{\mathbb{R}^n \times (0,c]} \left[\int_{\mathbb{R}^n \times (0,b]} \int_{\mathbb{R}^n} P_a(\tau - x) \right. \\ & \quad \left. \times P_b(\tau - y) P_c(\tau - z) d\tau d\mu(x, a) \right] d\mu(y, b). \end{aligned}$$

Observe that $a \leq b \leq c$ implies $a \leq B$. Applying Lemma 3.2 for each fixed s , (y, b) , and (z, c) , with B and q as in (3.4) and (3.5), $D = |x - q|$,

$\alpha = (n-1)/2$, $\beta = n$ and $\gamma = n + 3/2$, we have

$$\begin{aligned}
(3.7) \quad & \int_{\mathbb{R}^n \times (0,b]} \frac{\Gamma(n+\frac{3}{2})}{\pi^{n+\frac{3}{2}}} abc \int_0^1 \frac{t^{\frac{n-1}{2}}(1-t)^n}{(a^2t+(1-t)B^2+t(1-t)|x-q|^2)^{n+\frac{3}{2}}} dt d\mu(x, a) \\
& \leq \int_{\mathbb{R}^n \times (0,b]} \frac{bc\Gamma(n+\frac{3}{2})}{B\pi^{n+\frac{3}{2}}} \int_0^1 \frac{t^{\frac{n-1}{2}}(1-t)^{-1/2}}{(B^2+t(a^2+|x-q|^2))^{n+\frac{1}{2}}} dt d\mu(x, a) \\
& \leq \frac{bc(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{B\pi^{n+\frac{3}{2}}} \int_0^1 \int_{\mathbb{R}_+^{n+1}} \int_{B^2+t(a^2+|x-q|^2)}^\infty \frac{t^{\frac{n-1}{2}}(1-t)^{-\frac{1}{2}}}{r^{n+\frac{3}{2}}} dt dr d\mu(x, a) \\
& \leq \frac{bc(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{B\pi^{n+\frac{3}{2}}} \int_0^1 \int_{B^2}^\infty \mu \left(T \left[\overline{B} \left(q, \left(\frac{r-B^2}{t} \right)^{\frac{1}{2}} \right) \right] \right) \\
& \quad \times \frac{t^{\frac{n-1}{2}}(1-t)^{-\frac{1}{2}}}{r^{n+\frac{3}{2}}} dt dr d\mu(x, a) \\
& \leq \Omega_n \kappa(\mu) \frac{bc(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{B\pi^{n+\frac{3}{2}}} \int_0^1 \int_{B^2}^\infty (r-B^2)^{\frac{n}{2}} \\
& \quad \times \frac{t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}}{r^{n+\frac{3}{2}}} dt dr d\mu(x, a) \\
& = \kappa(\mu) \frac{bc(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{B^{n+2}\pi^{\frac{n+3}{2}}\Gamma(\frac{n}{2}+1)} \int_0^1 \int_1^\infty \frac{(r-1)^{\frac{n}{2}}}{r^{n+\frac{3}{2}}} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt dr d\mu(x, a) \\
& = \kappa(\mu) \frac{bc(n+\frac{1}{2})\Gamma(\frac{n+1}{2})}{B^{n+2}\pi^{\frac{n+1}{2}}} \\
& = \kappa(\mu) \frac{bc(n+\frac{1}{2})\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}(b^2s+c^2(1-s)+s(1-s)|y-z|^2)^{\frac{n}{2}+1}}.
\end{aligned}$$

Next, observe that the following identity holds:

$$\begin{aligned}
(3.8) \quad & \int_0^1 \frac{bc(s(1-s))^{\frac{n-1}{2}}}{(b^2s+c^2(1-s)+s(1-s)|y-z|^2)^{\frac{n}{2}+1}} ds \\
& = \frac{\gamma_n \pi^{\frac{n}{2}+1}}{\Gamma(\frac{n}{2}+1)} \frac{(b+c)}{[(b+c)^2+|y-z|^2]^{\frac{n+1}{2}}}.
\end{aligned}$$

In fact, by the semigroup formula for the Poisson kernel and (2.3) we have

$$P_{b+c}(y-z) = \frac{bc}{\pi^{\frac{n}{2}+1}} \int_0^{+\infty} \int_0^{+\infty} \frac{(uv)^{\frac{n-1}{2}}}{(u+v)^{\frac{n}{2}}} e^{-(b^2u+c^2v+\frac{uv}{u+v}|y-z|^2)} du dv.$$

On the other hand, after the change of variables $v = \lambda u$ the above quantity becomes

$$\begin{aligned} \frac{bc}{\pi^{\frac{n}{2}+1}} \int_0^{+\infty} \frac{\lambda^{\frac{n-1}{2}}}{(1+\lambda)^{\frac{n}{2}}} d\lambda \int_0^{+\infty} u^{\frac{n}{2}} e^{-Wu} du \\ = \frac{bc\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}+1}} \int_0^{+\infty} \frac{\lambda^{\frac{n-1}{2}}}{(1+\lambda)^{\frac{n}{2}} W^{\frac{n}{2}+1}} d\lambda \end{aligned}$$

where $W = b^2 + c^2\lambda + \frac{\lambda}{1+\lambda}|y-z|^2$, and the further change of variables $s = 1/(\lambda+1)$ (and therefore $\lambda = (1-s)/s$) gives

$$P_{b+c}(y-z) = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}+1}} \int_0^1 \frac{bc(s(1-s))^{\frac{n-1}{2}}}{(b^2s+c^2(1-s)+s(1-s)|y-z|^2)^{\frac{n}{2}+1}} ds.$$

We now apply Lemma 3.1 in the inner most integral in (3.6) and then use estimate (3.7) integrated with respect to $(s(1-s))^{\frac{n-1}{2}} ds d\mu(y, b)$ on $[0, 1] \times \mathbb{R}^n \times (0, c]$ and identity (3.8) to obtain

(3.9)

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, c]} \left[\int_{\mathbb{R}^n \times (0, b]} \int_{\mathbb{R}^n} P_a(\tau-x) P_b(\tau-y) P_c(\tau-z) d\tau d\mu(x, a) \right] d\mu(y, b) \\ \leq \kappa(\mu) \frac{(n+\frac{1}{2})\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}_+^{n+1}} \frac{\gamma_n \pi^{\frac{n}{2}+1} (b+c)}{\Gamma\left(\frac{n}{2}+1\right) [(b+c)^2 + |y-z|^2]^{\frac{n+1}{2}}} d\mu(y, b) \\ \leq \kappa(\mu) \frac{2\pi^{1/2} (n+\frac{1}{2})\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \int_{\mathbb{R}_+^{n+1}} \gamma_n \frac{c}{[(b+c)^2 + |y-z|^2]^{\frac{n+1}{2}}} d\mu(y, b). \end{aligned}$$

By (13) in [V]

$$\sup_{(z, c) \in \mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} \gamma_n \frac{c}{[(b+c)^2 + |y-z|^2]^{\frac{n+1}{2}}} d\mu(y, b) \leq C\kappa(\mu)$$

for some absolute constant C independent of n . Therefore (3.9) is bounded above by some absolute constant independent of n times

$$\kappa(\mu)^2 \frac{(n+\frac{1}{2})\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \sim \kappa(\mu)^2 \sqrt{n}$$

as $n \rightarrow \infty$, by the well known expansion $\Gamma(z)/\Gamma(z+\alpha) \sim z^{-\alpha}$ for $z \rightarrow \infty$. This last bound for (3.9), together with (3.6), proves (3.1) and hence Theorem 1.1 in the case $3/2 < p < 2$.

Proof of Lemma 3.1: We start with the following n -dimensional identity

$$(3.10) \quad \frac{|y-x|^2}{\alpha} + \frac{|z-x|^2}{\beta} = \frac{|y-z|^2}{\alpha+\beta} + \frac{\alpha+\beta}{\alpha\beta} \left| x - \frac{\beta}{\alpha+\beta}y - \frac{\alpha}{\alpha+\beta}z \right|^2$$

which is valid for all x, y and z in \mathbb{R}^n and for all $\alpha, \beta > 0$ and can be verified by a straightforward calculation.

We now apply (3.10), with $\alpha = \frac{u+v+w}{uv}$ and $\beta = \frac{u+v+w}{uw}$, to the first two terms of

$$(3.11) \quad \frac{uv|x-y|^2 + uw|x-z|^2 + vw|y-z|^2}{u+v+w}$$

which appear in the exponential factor in (2.2). Adding up the third term of (3.11), simplifying, and plugging back into (2.2) we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^n} P_a(\tau-x)P_b(\tau-y)P_c(\tau-z) d\tau \\ &= \frac{abc}{\pi^{n+\frac{3}{2}}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{(uvw)^{\frac{n-1}{2}}}{(u+v+w)^{\frac{n}{2}}} \\ & \quad \times \text{Exp} \left[- \left\{ a^2u + b^2v + c^2w + \frac{vw}{v+w}|y-z|^2 \right. \right. \\ & \quad \left. \left. + \frac{u(v+w)}{u+v+w} \left| x - \frac{v}{v+w}y + \frac{w}{v+w}z \right|^2 \right\} \right] du dv dw. \end{aligned}$$

The change of variables $v = \lambda u$ and $w = \mu u$ transforms the above integral into

$$(3.12) \quad \frac{abc}{\pi^{n+\frac{3}{2}}} \int_0^{+\infty} \int_0^{+\infty} \frac{(\lambda\mu)^{\frac{n-1}{2}}}{(1+\lambda+\mu)^{\frac{n}{2}}} d\lambda d\mu \int_0^{+\infty} u^{n+\frac{1}{2}} e^{-Wu} du,$$

where

$$W = a^2 + b^2\lambda + c^2\mu + \frac{\lambda\mu}{\lambda+\mu}|y-z|^2 + \frac{\lambda+\mu}{1+\lambda+\mu} \left| x - \frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}z \right|^2.$$

Expressing the inner integral in (3.12) as a Gamma function we obtain

$$\frac{abc\Gamma\left(n+\frac{3}{2}\right)}{\pi^{n+\frac{3}{2}}} \int_0^{+\infty} \int_0^{+\infty} \frac{(\lambda\mu)^{\frac{n-1}{2}}}{(1+\lambda+\mu)^{\frac{n}{2}} W^{n+\frac{3}{2}}} d\lambda d\mu.$$

Now let us set $t = \frac{1}{1 + \lambda + \mu}$ and $s = \frac{\lambda}{\lambda + \mu}$. This is an invertible transformation of the first quadrant of the (λ, μ) plane onto the square $(0, 1) \times (0, 1)$ of the (t, s) plane. We have $\lambda = \frac{s(1-t)}{t}$ and $\mu = \frac{(1-t)(1-s)}{t}$, while the Jacobian of the transformation is $\frac{1-t}{t^3}$. Collecting and simplifying the various powers of t , $(1-t)$, s and $(1-s)$ that appear we deduce that $\int_{\mathbb{R}^n} P_a(\tau-x)P_b(\tau-y)P_c(\tau-z) d\tau$ is in fact equal to the expression in the statement of Lemma 3.1. \square

Proof of Lemma 3.2: To prove the first inequality, we scale the parameters involved so that $B = 1$ and thus $0 < a \leq 1$. We need to show that

$$\begin{aligned} 0 &\leq \int_0^1 \frac{t^\alpha(1-t)^\beta}{\{a^2t+(1-t)+t(1-t)D^2\}^{\gamma-1}} \left(\frac{1}{a} - \frac{1}{a^2t+(1-t)+t(1-t)D^2} \right) dt \\ &= \frac{1}{a} \int_0^1 \frac{t^\alpha(1-t)^\beta}{\{a^2t+(1-t)+t(1-t)D^2\}^\gamma} (a^2t+1-t-a+t(1-t)D^2) dt. \end{aligned}$$

But $a^2t+1-t-a+t(1-t)D^2 \geq a^2t+1-t-a = (1-a)(1-t(1+a))$ and therefore we need to show that

$$\int_0^1 \frac{t^\alpha(1-t)^\beta}{\{a^2t+(1-t)+t(1-t)D^2\}^\gamma} (1-(1+a)t) dt \geq 0.$$

As $a^2t+(1-t) \leq 1$ and $1-(1+a)t \geq 1-2t$, it will be sufficient to prove that

$$\int_0^1 \frac{t^\alpha(1-t)^\beta}{\{1+t(1-t)D^2\}^\gamma} (1-2t) dt \geq 0.$$

We split this integral in the parts from 0 to $\frac{1}{2}$ and from $\frac{1}{2}$ to 1. Switching variables $t \rightarrow 1-t$ in the second of these integrals we obtain

$$\int_0^{\frac{1}{2}} \frac{(t(1-t))^\alpha}{\{1+t(1-t)D^2\}^\gamma} ((1-t)^{\beta-\alpha} - t^{\beta-\alpha}) (1-2t) dt \geq 0$$

and the first inequality in the lemma is proved. Note that here we have used the hypothesis $\beta > \alpha$.

To prove the second inequality in Lemma 3.2 we start with the elementary fact

$$\begin{aligned} a^2t + (1-t)B^2 + t(1-t)D^2 &= B^2 - t(B^2 - a^2 - D^2) - t^2D^2 \\ &\geq B^2 - t(B^2 - a^2 - D^2) - t^2(a^2 + D^2) = (1-t)[B^2 + t(a^2 + D^2)] \end{aligned}$$

which yields

$$\{a^2t + (1-t)B^2 + t(1-t)D^2\}^{-\gamma+1} \leq (1-t)^{-\gamma+1} \{B^2 + t(a^2 + D^2)\}^{-\gamma+1}.$$

Multiplying both sides by $(1-t)^\beta$ and taking the integral from 0 to 1 with respect to the weight t^α we obtain the second claimed inequality in the lemma. Note that the exponent $\beta - \gamma + 1$ can be negative, but $\beta - \gamma + 1 > -1$ by our assumptions, so that integrability in $t = 1$ is guaranteed. Lemma 3.2 is proved. \square

4. Proof of Theorem 1.1: the general case

In this section we outline how to derive the estimate

$$(4.1) \quad \|P^*(\chi_E, \mu)\|_{L^k(\mathbb{R}^n)}^k \leq C^k (k!)^{\frac{3}{2}} n^{\frac{k-2}{2}} \kappa(\mu)^{k-1} \mu(E), \quad k = 4, 5, \dots$$

for some absolute constant $C > 0$, independent of n and k and for all μ -measurable subsets E of \mathbb{R}_+^{n+1} .

As a consequence of (4.1) we obtain

$$(4.2) \quad \|Pf\|_{L^{\frac{k}{k-1}, \infty}(\mathbb{R}_+^{n+1}, d\mu)} \leq C k^{\frac{3}{2}} n^{\frac{1}{2} - \frac{1}{k}} \kappa(\mu)^{\frac{k-1}{k}} \|f\|_{L^{\frac{k}{k-1}}(\mathbb{R}^n)}$$

for all $k = 3, 4, \dots$. These estimates extend (3.1) and (3.2), respectively. Once (4.2) is known, we deduce (1.8) as follows: for a given $p \in (1, 2)$ we find a positive integer $k_0 \geq 4$ such that $k_0/(k_0 - 1) < (p + 1)/2$. We use the Marcinkiewicz interpolation theorem to interpolate between estimate (1.5) (i.e. $p = 2$) and estimate (4.2) with $p = k_0/(k_0 - 1)$. Using a good value for the interpolation constant (see for instance the value of the constant in [G, p. 33]) we obtain (1.8) with $C_p = (p-1)^{-\frac{5}{2}}$. We note that the standard proofs of (1.1) yield the constant $c_{p,n} = C(p-1)^{-1}n^c$ (for some absolute $C > 0$ and $c > 0$), which grows much faster in n , but blows up at a slightly slower rate when n is fixed and $p \rightarrow 1$.

In order to work with (4.1) as in the case $k = 3$, we need a suitable generalization of Lemma 3.1 for arbitrary k . First, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(\tau - x_j) d\tau \\ &= \frac{a_1 a_2 \dots a_k}{\pi^{(k-1)\frac{n}{2} + \frac{k}{2}}} \int_{[0, \infty)^k} \frac{(u_1 u_2 \dots u_k)^{\frac{n-1}{2}}}{(u_1 + u_2 + \dots + u_k)^{\frac{n}{2}}} \\ & \quad \times \text{Exp} \left\{ - \sum_{j=1}^k a_j^2 u_j - \sum_{\ell=1}^{k-1} \frac{(u_1 + \dots + u_\ell) u_{\ell+1}}{u_1 + \dots + u_{\ell+1}} \right. \\ & \quad \left. \times \left| x_{\ell+1} - \frac{x_1 u_1 + \dots + x_\ell u_\ell}{u_1 + \dots + u_\ell} \right| \right\} du_1 \dots du_k \end{aligned}$$

which can be obtained from the n -dimensional identity (3.10) working in a recursive fashion as in the proof of Lemma 3.1, until the $k(k-1)/2$ distances appearing in the original formula (2.1) are “squeezed” together into $k-1$ distances. Note that in the case $k = 3$ we applied (3.10) just once.

Next, if Δ_k denotes the $(k-1)$ -dimensional simplex

$$\Delta_k = \left\{ v = (v_1, \dots, v_k) : v_j \geq 0, \sum_{j=1}^k v_j = 1 \right\}$$

then the change of variables $u = \rho v$ with $v \in \Delta_k$ and $\rho > 0$ yields

$$\begin{aligned} & \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(\tau - x_j) d\tau \\ &= \frac{a_1 a_2 \dots a_k \Gamma\left(\frac{k-1}{2}n + \frac{k}{2}\right)}{\pi^{\frac{k-1}{2}n + \frac{k}{2}}} \int_{\Delta_k} (v_1 \dots v_k)^{\frac{n-1}{2}} \\ & \quad \times \left(\sum_{j=1}^k a_j^2 v_j + \sum_{\ell=1}^{k-1} \frac{(v_1 + \dots + v_\ell) v_{\ell+1}}{v_1 + \dots + v_{\ell+1}} \right. \\ & \quad \left. \times |x_{\ell+1} - x_1 v_1 + \dots + x_\ell v_\ell v_1 + \dots + v_{\ell+1}| \right)^{-\frac{k-1}{2}n - \frac{k}{2}} dv_1 \dots dv_k. \end{aligned}$$

The further change of variables $v_j = (1-t)s_j$, $j = 1, 2, \dots, k-1$, $v_k = t$, with $s = (s_1, \dots, s_{k-1}) \in \Delta_{k-1}$ and $t \in [0, 1]$ gives

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(\tau - x_j) d\tau &= \frac{a_1 a_2 \dots a_k \Gamma\left(\frac{k-1}{2}n + \frac{k}{2}\right)}{\pi^{\frac{k-1}{2}n + \frac{k}{2}}} \\ &\times \int_{\Delta_{k-1}} \int_0^1 \frac{(1-t)^{\frac{k-1}{2}n + k - 2} t^{\frac{n-1}{2}} (s_1 \dots s_{k-1})^{\frac{n-1}{2}} dt ds}{\left(a_k^2 t + (1-t)B_k^2 + t(1-t)|x_k - q_k|^2\right)^{\frac{k-1}{2}n + \frac{k}{2}}} \end{aligned}$$

with

$$q_k = x_1 s_1 + \dots + x_{k-1} s_{k-1}$$

$$B_k^2 = \sum_{j=1}^{k-1} a_j^2 s_j + \sum_{\ell=1}^{k-2} \frac{(s_1 + \dots + s_\ell) s_{\ell+1}}{s_1 + \dots + s_{\ell+1}} \left| x_{\ell+1} - \frac{x_1 s_1 + \dots + x_\ell s_\ell}{s_1 + \dots + s_{\ell+1}} \right|,$$

and this is a suitable generalization of Lemma 3.1, with the property that B_k is independent of t , x_k , a_k .

We now apply Lemma 3.2 with $\alpha = \frac{n-1}{2}$, $\beta = \frac{k-1}{2}n + k - 2$, $\gamma = \frac{k-1}{2}n + \frac{k}{2}$ and proceed as we did in the previous case $k = 3$, assuming $a_k \leq a_{k-1} \leq \dots \leq a_1$. We obtain

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, a_{k-1})} a_k \int_0^1 \frac{t^{\frac{n-1}{2}} (1-t)^{\frac{k-1}{2}n + k - 2}}{\left(a_k^2 t + (1-t)B_k^2 + t(1-t)|x_k - q_k|^2\right)^{\frac{k-1}{2}n + \frac{k}{2}}} dt d\mu(x_k, a_k) \\ \leq \frac{\kappa(\mu) \pi^{n/2+1} \left(\frac{k-1}{2}n + \frac{k}{2} - 1\right) \Gamma\left(\frac{(k-2)(n+1)}{2}\right)}{\left(B_k^2\right)^{\frac{k-2}{2}n + \frac{k+1}{2}} \Gamma\left(\frac{(k-1)n}{2} + \frac{k}{2}\right)} \\ = \frac{\kappa(\mu) \pi^{n/2+1} \Gamma\left(\frac{(k-2)(n+1)}{2}\right)}{\left(B_k^2\right)^{\frac{k-2}{2}n + \frac{k+1}{2}} \Gamma\left(\frac{(k-1)(n+1)}{2} - \frac{1}{2}\right)}. \end{aligned}$$

Iterating this estimate $k - 3$ more times, and doing one last estimate as in [V, equation (13)] we get, for any $k \geq 4$, that

$$\begin{aligned} & \frac{1}{k!} \int_{E^{k-1}} \int_{\mathbb{R}^n} \prod_{j=1}^k P_{a_j}(\tau - x_j) d\tau dx_2 \dots dx_k \\ & \leq \frac{C\kappa(\mu)^{k-1}}{\left(\frac{k-1}{2}n + \frac{k}{2} - 1\right)^{-1}} \frac{\Gamma\left(\frac{(k-2)(n+1)}{2}\right)}{\Gamma\left(\frac{(k-2)(n+1)}{2} - \frac{1}{2}\right)} \frac{\Gamma\left(\frac{(k-3)(n+1)}{2}\right)}{\Gamma\left(\frac{(k-3)(n+1)}{2} - \frac{1}{2}\right)} \dots \\ & \dots \frac{\Gamma(n+1)}{\Gamma\left(n + \frac{1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} \end{aligned}$$

which is of the order of

$$\kappa(\mu)^{k-1} \sqrt{(k-1)!} (n+1)^{\frac{k-3}{2} + \frac{1}{2}} \sim C^k \kappa(\mu)^{k-1} \Gamma(k)^{\frac{1}{2}} n^{\frac{k-2}{2}}.$$

Since this estimate is uniform with respect to (x_1, a_1) we obtain (4.1).

To prove (1.9) we first note that applying the Marcinkiewicz interpolation theorem between (1.5) (i.e. $p = 2$) and the trivial $p = \infty$ estimate yields

$$(4.3) \quad \|Pf\|_{L^p(\mathbb{R}_+^{n+1}, d\mu)} \leq c' p^{\frac{1}{p}} (p-2)^{-\frac{1}{p}} \kappa(\mu)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$

for all $p > 2$. Finally, using the Riesz-Thorin interpolation theorem, we interpolate between (1.8) with $p = \frac{3}{2}$ and (4.3) with $p = \left(\frac{1}{2} - \frac{1}{\log n}\right)^{-1}$ ($n > 10$) to obtain (1.9).

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