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A GEOMETRIC AND STOCHASTIC PROOF OF THE TWIST POINT THEOREM

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Abstract: In this paper we will give a proof of the McMillan twist point theorem using geometry, potential theory and Ito's formula but not the Riemann mapping theorem.

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1. Introduction

Let Ω be a simply connected domain and let z_0 be some fixed point in Ω. A point $\xi \in \partial\Omega$ is a cone point of Ω if there is an open triangle contained in Ω with a vertex at ξ. If, moreover, there is a unique $\phi \in (-\pi, \pi]$ such that for any $\epsilon > 0$ there is $\delta > 0$ for which

$$\left\{\xi + re^{i\theta} : |\theta - \phi| < \frac{\pi}{2} - \epsilon,\, r < \delta\right\} \subset \Omega$$

then ξ is called an interior tangent point of Ω. A boundary point $\xi \in \partial\Omega$ is a twist point if

$$\liminf_{z\in\Omega\to\xi} \arg(z - \xi) = -\infty$$

and

$$\limsup_{z\in\Omega\to\xi} \arg(z - \xi) = +\infty$$

where $\arg(z - \xi)$ denotes a continuous branch of the argument defined in Ω with

$$-\pi < \arg(z_0 - \xi) \leq \pi.$$

The McMillan twist point theorem states that the set of boundary points of Ω which are neither inner tangent points nor twist points has harmonic measure zero. (See Section 2 for a definition of harmonic measure.)

To consider the same theorem from a function theoretic point of view, let $f : \mathbb{D} \to \Omega$ with $f(0) = z_0$ be a Riemann mapping and let Arg denote a continuously defined argument. The conformal mapping f is said to be twisting at the point $e^{i\theta} \in \partial\mathbb{D}$ if

$$\liminf_{z\in\mathbb{D}\to e^{i\theta}} \operatorname{Arg}(f(z) - f(e^{i\theta})) = -\infty$$

and

$$\limsup_{z\in\mathbb{D}\to e^{i\theta}} \operatorname{Arg}(f(z) - f(e^{i\theta})) = +\infty.$$

The function f is, by definition, conformal at every interior point of $\mathbb{D}$. It is said to be conformal at $e^{i\theta} \in \partial\mathbb{D}$ if, for z approaching $e^{i\theta}$ inside any triangle in $\mathbb{D}$ with a vertex at $e^{i\theta}$, the limit $\lim_{z\to e^{i\theta}} f'(z)$ exists and is not zero or infinite. An equivalent statement of the twist point theorem is that the set of points $e^{i\theta} \in \partial\mathbb{D}$ where f is neither conformal nor twisting has linear measure zero. In fact, this is the form in which the theorem was originally proved. McMillan's original reasoning in [**11**] relied on the properties of f' which are inherited from the conformality of f. The purely geometric statement in terms of inner tangent points and twist points of $\partial\Omega$ is shown to be equivalent at the end of the same paper.

We will give another proof of the twist point theorem using geometric and stochastic arguments. The goal is to achieve a proof in the same spirit as the arguments in Chapter 5 of Bass's book, [**3**], and thus develop some geometric and probabilistic intuition about the phenomenon described by the theorem. In particular, we would like to understand McMillan's theorem as a statement about the interplay between the geometry of the boundary of Ω and the exit distribution of Brownian motion. In [**1**], the authors show that the almost everywhere characterization of cone points and twist points in terms of the derivative of a conformal mapping can be replaced by an equivalent characterization in terms of a certain harmonic function which can be defined directly on the domain using potential theory and not the Riemann mapping theorem. The proof of the equivalence in [**1**] uses the McMillan twist point theorem, but the results obtained there open an avenue to obtaining similar results in higher dimensions or for more general domains. We hope that the method of proof of the twist point theorem given here may do the same.

McMillan also showed in [**11**] that the set of cone points which are not interior tangent points has harmonic measure zero. We refer to [**9**, pp. 208–210] for a geometric and potential theoretic proof of this fact and will be satisfied with proving the following version of the twist point theorem.

Theorem 1.

$$\partial\Omega = (\text{Cone}) \cup (\text{Twist}) \cup N$$

where Cone *denotes the set of cone points,* Twist *denotes the set of twist points and N is a set of harmonic measure zero.*

The theorem is essentially potential theoretic as it is a statement about harmonic measure. In our proof, we will make use of various results which can be proved directly by potential theoretic means in Ω without recourse to a conformal mapping from Ω to $\mathbb{D}$. In Sections 2 and the Appendix, we provide background and references for the results needed. In Section 3 we give the new proof of Theorem 1 above.

2. Preliminaries

Let $\Omega \subset \mathbb{C}$ be a domain. The harmonic measure at $z \in \Omega$ of a Borel subset $E \subset \partial\Omega$ is denoted by $\omega(z, E, \Omega)$ or by $\omega_z(E)$ when there is no confusion about what domain is being considered. There are several equivalent definitions.

(1) $\omega(z, E, \Omega)$ is the unique Perron-solution to the Dirichlet problem for the Laplacian in Ω with boundary data given by the characteristic function of E.

(2) $\omega(z, E, \Omega)$ is the probability that a Brownian motion started at z has its first exit from Ω in the set E.

(3) $\omega(z, E, \Omega)$ can be defined in terms of the trajectories of Green's function with pole at z as we will describe below.

(4) If the domain Ω is simply connected then $\omega(z, E, \Omega)$ is the normalized linear Lebesgue measure of the image of $f(E) \subset \partial\mathbb{D}$ by a Riemann mapping f of Ω to the unit disk which takes z to 0. This definition may be generalized to multiply connected domains by use of the universal covering map.

Only the last definition cannot be generalized to domains in higher dimensions. Notice that by Harnack's inequality, $\omega_{z_1} \ll \omega_{z_2}$ for any $z_1, z_2 \in \Omega$. We will therefore write $\omega(E) = 0$ when $\omega_z(E) = 0$ for some z.

The main ideas we will need in order to use definition (2) are the Strong Markov property for Brownian motion and Ito's formula. Let X denote a two dimensional Brownian motion started at z_0 and let f be a C^2 function defined in a domain $\Omega \subset \mathbb{C}$ containing z_0. Let $\tau = \inf\{t :$

$X_t \notin \Omega\}$. In the form we will use it, Ito's formula says that for $t < \tau$

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s) \cdot dX_s + \frac{1}{2} \int_0^t \triangle f(X_s)\, ds.$$

When $f = u$ is harmonic, we obtain that

$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s) \cdot dX_s$$

is a continuous martingale. We will also need the fact that any continuous martingale is a time change of a Brownian motion, perhaps stopped at a stopping time. We refer to [**3**] for detailed background.

We will require the following basic projection estimates of harmonic measure due to Beurling and Hall. The original proofs are in [**6**] and [**10**] respectively. Here we will give the statements as they appear in [**13**] where stochastic proofs and versions of the results for $\mathbb{R}^3$ are given.

Let $0 \leq R_1 < R_2 \leq \infty$ and let A denote the annulus

$$A = \{z : R_1 \leq |z| \leq R_2\}.$$

Let $K \subset A$ be compact and let $-R_2 < a < -R_1$. Put

$$K^* = \{|z| : z \in K\} \subset \mathbb{R} \subset \mathbb{C}$$

and define $U = A^\circ \setminus K$ and $V = A^\circ \setminus K^*$.

Lemma 2.1 (The Beurling projection theorem)**.**

$$\omega(a, K, U) \geq \omega(a, K^*, V).$$

With the same notations, suppose that

$$R_1 < r_1 < r_2 < R_2.$$

Lemma 2.2 (Hall's Lemma)**.** *There exists $c > 0$ such that for all compact $K \subset \{z : r_1 < |z| < r_2\}$,*

$$\omega(a, K, M) \geq c m_1(K^*)$$

where $M = A^\circ \setminus K \setminus [0, \infty)$ and m_1 denotes one dimensional Lebesgue measure on $\mathbb{R}$.

The next lemma is also standard, its proof being accomplished by a straightforward application of Hall's lemma and the strong Markov property. A complete probabilistic argument is given in [**14**] so we omit the proof here. Similar arguments have been used in many places. See for example the survey article [**4**] and the papers referred to there. Let $D(z, R)$ denote a Euclidean disk with radius R and center z. In what follows, and throughout the rest of the paper, all domains are assumed to be simply connected unless stated otherwise.

Lemma 2.3. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Given $\epsilon > 0$ there is $K > 0$ such that*

$$\omega(z, \partial\Omega \cap D(z, K\,\mathrm{dist}(z, \partial\Omega)) > 1 - \epsilon$$

where dist *denotes the Euclidean distance in the plane.*

Two more main ingredients in our proof of Theorem 1 will be the use of definition (3) above and the use of Fatou's theorem on convergence at the boundary of harmonic functions. We refer to the paper [**14**] for a probabilistic and potential theoretic proof of Fatou's theorem in an arbitrary simply connected domain. We will need the Fatou theorem for convergence in the "Green cones" described below whereas it is proved in [**14**] only for convergence along Green lines. The lemmas necessary for the required extension are stated in this section and proved in the Appendix.

The definition (3) of harmonic measure in terms of the trajectories of the gradient of Green's function is from the paper [**7**] of Brelot and Choquet. Note that the other definitions become vacuous in case Ω has no Green function and assume from now on that all domains considered possess one. Let $a \in \Omega$ be fixed and let $g_a(x)$ be Green's function for Ω with pole at a. The Green lines starting at a are the maximal orthogonal trajectories of the level lines of g_a which have a limit point at a. Each Green line has a well defined initial direction given by its unit tangent vector at a and each point on the unit sphere corresponds in this way to a Green line at a. Given a Borel subset $E \subset \partial\Omega$, the Green's measure of E is defined to be the normalized Lebesgue measure of the set of unit tangent vectors on the sphere for which the corresponding Green lines terminate at points of E. Denote the Green's measure of E by $g_a(E)$ and take the normalization so that $g_a(\partial\Omega) = 1$. With respect to Green's measure, almost every Green line terminates in a point of $\partial\Omega$ and such Green lines are called regular. Let $\overline{g_a}$ and $\underline{g_a}$ denote respectively the outer and inner Green measures and let $\overline{\omega_a}$ and $\underline{\omega_a}$ denote the outer and inner harmonic measures for general subsets of $\partial\Omega$. Brelot and Choquet proved that for any subset $A \subset \partial\Omega$

$$\overline{\omega_a}(A) \leq \underline{g_a}(A) \leq \overline{g_a}(A) \leq \overline{\omega_a}(A) \tag{1}$$

so that harmonic measurability implies Green measurability with equality of the measures. Later, Arsove, in [**2**], proved the converse of this statement, thereby showing that the two measures are the same.

We will not go into the details of the arguments in [**7**] but we remark that the main idea behind them is to carefully apply Green's theorem. One defines a tube as the union of Green lines connecting a pair of neighborhoods on disjoint level surfaces of g_a and uses Green's formula to show that the flux of the vector field of ∇g_a is the same at each end of the tube. We let one end of a tube tend to $\partial\Omega$ and the other to the singularity of g_a to get the equivalence of the normalized Lebesgue measure on a sphere centered at a with the harmonic measure of the ideal boundary points at the other end of the tube. Difficulties posed by critical points in multiply connected domains or in higher dimensions are circumvented by using the fact that the critical set of g_a corresponds to a set of measure zero in the normalized spherical measure at a. Covering subsets of the boundary by ends of tubes and attending to the details leads to (1). Let $\ell(z_0, z)$ denote the Green line connecting z_0 and z, formed by considering the Green function with pole at z_0. In what follows we will make use of properties of $\ell(z_0, z)$ which derive either from the general theory of existence and uniqueness for ordinary differential equations or from potential theoretic properties of $g(z, z_0)$, such as the symmetry $g(z, z_0) = g(z_0, z)$. For example, we shall freely use the fact that $\ell(z_0, z) = \ell(z, z_0)$. We defer an outline of the argument for this to Lemma 4.1 of the Appendix.

Definition 1. Let z_0 denote some fixed base point in a domain. The forward cone at z with aperture α is denoted by $\Lambda_\alpha(z)$ and defined to be the union of all Green lines starting at z who's tangent vectors at z make an angle less than $\frac{\alpha}{2}$ with the tangent vector of $\ell(z_0, z)$ at z.

Let $\operatorname{dist}(z, \partial\Omega)$ denote the Euclidean distance from z to $\partial\Omega$ and let $d = d(z, \alpha)$ be the Euclidean distance from z to $\partial\Lambda_\alpha(z) \cap \partial\Omega$. In the following, the base point z_0 is fixed as in Definition 1.

Lemma 2.4. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. There is $c > 0$ depending on α such that* $\operatorname{dist}(z, \partial\Omega) \geq c d(z, \alpha)$ *for all $z \in \Omega$.*

Proof: We have $\omega(z, \partial\Lambda_\alpha(z) \cap \partial\Omega, \Omega) \geq \frac{\alpha}{2\pi}$. Note that strict inequality may hold here since $\partial\Lambda_\alpha(z) \cap \partial\Omega$ may contain other ideal boundary points besides the ones represented by the Green lines in $\Lambda_\alpha(z)$. By Lemma 2.3 we may choose $K > 0$ such that

$$\omega(z, \partial\Omega \cap D(z, K \operatorname{dist}(z, \partial\Omega)), \Omega) > 1 - \frac{\alpha}{4\pi}.$$

It follows that $K \operatorname{dist}(z, \partial\Omega) > d(z, \alpha)$. □

In what follows, we will identify Green lines starting from some fixed base point $z_0 \in \Omega$, with ideal boundary points of Ω. By the Moore triod theorem of plane topology, (see [**12**] or [**16**]), the number of points in the Euclidean boundary, $\partial\Omega$, which are the endpoints of more than two regular Green lines is at most countable. In many situations, it is possible to reduce matters to the case in which there is exactly one Green line ending at each Euclidean boundary point by considering interior approximation by Jordan domains.

Let $\zeta \in \partial\Omega$ and let ℓ_ζ be a regular Green line ending at ζ, again with the fixed base point z_0. For $\lambda > 0$, let $\ell_\zeta(\lambda)$ be the unique point on ℓ_ζ such that $g(\ell_\zeta(\lambda), z_0) = \lambda$.

Definition 2. The Green cone of aperture α over $\ell_\zeta \in \partial\Omega$ is denoted $\Gamma_\alpha(\ell_\zeta)$ and defined as

$$\Gamma_\alpha(\ell_\zeta) = \{z \in \Omega : \ell_\zeta(\lambda) \in \Lambda_\alpha(z) \text{ for all sufficiently small } \lambda\}.$$

We now collect a few facts about the behavior of Green lines, forward cones and Green cones. They are more or less obvious when considered as conformally transplanted from the unit disk. We provide proofs in the Appendix which depend only on the basic definitions to keep the paper free from reliance on conformal mapping.

Given a simply connected domain $\Omega \subset \mathbb{C}$ and a fixed point $z_0 \in \Omega$, choose an angle $0 < \alpha < \pi$ and consider a regular Green line $\ell(z_0, z_1)$ starting from z_0 and passing through some other point $z_1 \in \Omega$. Denote the tangent vector to $\ell(z_0, z_1)$ at z_1 in the direction of decreasing g by v. Let $\Omega_\lambda = \{z : g(z_1, z) > \lambda\}$ and let $I_\alpha^\lambda(z_1)$ denote the set of endpoints on $\partial\Omega_\lambda$ of the regular Green lines in $\Lambda_\alpha(z_1)$. Let $u^\lambda(z) = \omega(z, I_\alpha^\lambda(z_1), \Omega_\lambda)$ and let $u(z) = \lim\limits_{\lambda\to 0} u^\lambda(z)$. The existence of this limit is an exercise with the maximum principle. One can, for example, compare $u^\lambda(z)$ to the strictly increasing sequence $v^\lambda(z)$ where $v^\lambda(z)$ is the solution to the Dirichlet problem in

$$(\Omega \setminus \Lambda_\alpha) \cup \Omega_\lambda$$

with boundary data

$$v^\lambda(w) = \begin{cases} 0 & \text{if } w \in \partial\Omega, \\ 1 & \text{if } w \in \partial(\Lambda_\alpha(z_1) \cap \Omega_\lambda^c) \cap \Omega. \end{cases}$$

We will require the following facts which are proved either in the Appendix or in the given references.

Lemma 2.5. $\frac{\nabla u}{|\nabla u|} = v$.

Let $L(z_0, z)$ denote a complete regular Green line starting from z_0 and passing through z to terminate at some ideal boundary point.

Lemma 2.6. *If $z_2 \in L(z_0, z_1)$ and $g(z_0, z_1) > g(z_0, z_2)$ then*

$$\Lambda_\alpha(z_2) \subset \Lambda_\alpha(z_1).$$

From Lemma 2.6, it follows that if $z \in \Gamma_\alpha(\ell_\zeta)$ for some Green line ℓ_ζ then the segment of $L(z_0, z)$ connecting z and z_0 is contained in $\Gamma_\alpha(\ell_\zeta)$. It also follows that if

$$C_\lambda = \{w : g(w, z_0) = \lambda\}$$

is any level curve then

Lemma 2.7. *$C_\lambda \cap \Gamma_\alpha(\ell_\zeta)$ is a connected arc.*

Let $X_t(\eta)$ denote Brownian motion started at the fixed base point $X_0(\eta) = z_0$ and let $\tau(\eta) = \inf_t \{X_t(\eta) \notin \Omega\}$ denote the first exit time from Ω. Let K be a compact subset of $\partial\Omega$ and let $C(K, \alpha) = \bigcup_{\ell_\zeta : \zeta \in K} \Gamma_\alpha(\ell_\zeta)$. Here the union is over all Green lines ending at points of K and we may assume that for each point of K there are at most two.

Lemma 2.8. *For almost every Brownian path η such that $\tau(\eta) \in K$, there is $t_0(\eta) < \tau(\eta)$ such that*

$$X_t(\eta) \in C(K, \alpha), \quad t_0 \le t < \tau.$$

Proof: See [**14**, Lemma 2.5] for the proof. □

A version of the last lemma holds for subsets K of the ideal boundary, but the simpler statement here is sufficient for our purposes. Note that in the unit disk, the lemma is equivalent to the statement that a sawtooth sub-domain over a closed set $K \subset \partial\mathbb{D}$ has an interior tangent at almost every point of K.

Given a boundary point ζ, a Green line $\ell_\zeta = \ell(z_0, \zeta)$ and $z \in \ell_\zeta$, let $\gamma(z, \zeta)$ denote the connected arc $\{w : g(w, z_0) = g(z, z_0)\} \cap \Gamma_\alpha(\ell_\zeta)$.

Lemma 2.9. *Let $0 < \alpha < \pi$ be fixed. There is a constant $c > 0$ only depending on α such that for any ℓ_ζ, $z \in \ell_\zeta$ and $w \in \gamma(z, \zeta)$,*

$$\frac{1}{c} \operatorname{dist}(w, \partial\Omega) \le \operatorname{dist}(z, \partial\Omega) \le c \operatorname{dist}(w, \partial\Omega).$$

The following lemma is a slightly more general version of a lemma from [**11**].

Lemma 2.10. *Let $E \subset \partial\Omega$ be a Borel set, E^c its complement. Suppose that for each $\zeta \in E$ there is an $\alpha < \frac{\pi}{2}$, a $c > 0$, and a sequence $\{z_n\} \to \zeta$ with $\{z_n\} \subset \Gamma_\alpha(\zeta)$, such that*

$$\omega(z_n, E^c \cap \partial\Omega, \Omega) > c > 0.$$

Then $\omega(E) = 0$.

Proof: We claim that

$$\lim_{\substack{z \to \zeta \\ z \in L(z_0, \zeta)}} \omega(z, E, \Omega) = 1$$

at ω_{z_0} a.e. $\zeta \in E$. This claim is a simple consequence of Fatou's theorem in the disk and a conformal invariance argument. To keep our argument free of the use of conformal invariance we refer to [**14**] where a stochastic and potential theoretic proof of the claim is given. Assuming the claim, note that by Lemma 2.9 and Harnack's inequality, there is $c' > 0$ and infinitely many $z'_n \in L(z_0, \zeta)$ such that $\omega(z'_n, E^c \cap \partial\Omega, \Omega) > c' > 0$. So it follows that $\omega_{z_0}(E) = 0$. □

In our proof of the twist point theorem we will need the following technical lemma on harmonic measure in order to make an approximation argument.

Lemma 2.11. *Let $w_0 \in \Omega$ be fixed and let $\zeta_0 \in \partial\Omega$ be a point such that* $\operatorname{dist}(w_0, \partial\Omega) = |w_0 - \zeta_0|$. *Let $0 < \rho < 1$ and let $L_\rho(\zeta_0)$ be a line segment of length $\rho|w_0 - \zeta_0|$ with one endpoint at ζ_0 and which is a subset of the line segment $[w_0, \zeta_0]$.*

Then in $\Omega \setminus L_\rho(\zeta_0) = \Omega_\rho$, we have that $\omega(z, L_\rho(\zeta_0), \Omega_\rho)$ increases, as a function of z, along the segment $[w_0, \zeta_0]$ in the direction from w_0 to ζ_0.

Proof: We have

$$\omega(z, L_\rho(\zeta_0), \Omega_\rho) = \frac{1}{\gamma_G(L_\rho)} \int g(z, \xi)\, d\mu(\xi)$$

where g is Green's function for Ω, μ is the equilibrium mass supported on L_ρ for the Green potential and

$$\gamma_G(L_\rho) = \iint g(z, \xi)\, d\mu(\xi)\, d\mu(z).$$

(See Chapter 3 of [**9**] for example.) Given $z \in [w_0, \zeta_0] \cap \Omega_\rho$ let H denote the half plane with normal vector $[w_0, \zeta_0]$ whose interior contains ζ_0 and whose boundary contains z. Since

$$g(z, \xi) = \log \frac{1}{|z - \xi|} - \int_{\partial\Omega} \log \frac{1}{|z - w|}\, d\omega_\xi(w)$$

we see by inspection that for $\xi \in L_\rho(\xi_0)$, $\nabla_z g$ is a vector which points into the half plane H from z. Since μ is a positive probability measure supported on L_ρ, the same is true for $\nabla_z \omega$ at z. Consequently, ω increases along $[w_0, \zeta_0]$ in the direction toward L_ρ. □

Finally, we record for reference the following result of Dahlberg from [**8**].

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then a Borel measurable set $E \subset \partial D$ is of harmonic measure zero with respect to D if and only if E is of vanishing $(n-1)$-dimensional Hausdorff measure.*

In fact, it is shown in [**8**] that there are constants $\alpha > \frac{1}{2}$ and $\beta > 0$ depending on the Lipschitz constant of the domain D, such that

$$\omega(F) \le C(\sigma(F))^{\alpha} \quad \text{and} \quad \sigma(F) \le (\omega(F))^{\beta} \tag{2}$$

for subsets F of ∂D. Dahlberg's proof is potential theoretic, holds for any dimension $n \ge 3$ and can be modified to work for $n = 2$. For a stochastic proof of Dahlberg's results see [**3**, Chapter 3].

3. The twist point theorem

As before, Ω is a simply connected domain and $z_0 \in \Omega$ is a fixed base point. For each $\zeta \in \partial\Omega$ which is the endpoint of a regular Green line starting at z_0, let $\arg(\cdot - \zeta)$ denote the continuous branch of the argument in Ω which satisfies

$$-\pi < \arg(z_0 - \zeta) \le \pi.$$

We will prove Theorem 1 in two steps. First we show

Theorem 3.

$$\omega\left(\left\{\zeta \in \partial\Omega : \lim_{\substack{z\to\zeta \\ z\in\Omega}} \arg(z-\zeta) = +\infty\right\}\right) = 0.$$

Then we let $E_M \subset \partial\Omega$ denote the set of boundary points $\zeta \in \partial\Omega$ satisfying

(1) $\arg(z - \zeta) < M$ for infinitely many $z \in \Gamma_\alpha(\zeta)$,
(2) ζ is not a cone point of $\partial\Omega$,
(3) $\arg(z - \zeta) \ge -M$ for each $z \in \Omega$

and we prove

Theorem 4. $\omega(E_M) = 0$.

Taken together, Theorems 3 and 4 show that the set of non-cone points ξ for which

$$\liminf_{\substack{z\to\zeta \\ z\in\Omega}} \arg(z-\xi) \ne -\infty$$

has harmonic measure zero, because it is contained in the union of

$$\left\{ \zeta \in \partial\Omega : \lim_{\substack{z \to \zeta \\ z \in \Omega}} \arg(z - \zeta) = +\infty \right\}$$

and the sets E_M as M ranges through the positive integers. The same arguments show that

$$\omega\left(\left\{ \zeta \in \partial\Omega : \lim_{\substack{z \to \zeta \\ z \in \Omega}} \arg(z - \zeta) = -\infty \right\}\right) = 0$$

and that with E'_M defined as the set of points with

(1) $\arg(z - \zeta) > -M$ for infinitely many $z \in \Gamma_\alpha(\zeta)$;
(2) ζ is not a cone point of $\partial\Omega$;
(3) $\arg(z - \zeta) \leq M$ for each $z \in \Omega$;

we have

$$\omega(E'_M) = 0.$$

Combining the above results shows that almost every $\zeta \in \partial\Omega$ is either a cone point or a twist point, and it suffices to prove Theorems 3 and 4.

We come to the proof of Theorem 3.

Proof: Let σ denote an open line segment contained in Ω with one end point at z_0 and the other at some point of $\partial\Omega$. Let g denote the Green's function for Ω with pole at z_0. A calculation shows that $U = \arg(g_x + i g_y)$ is harmonic in a neighborhood of any point in $\Omega \setminus \{z_0\}$. Since Ω is simply connected, U can be defined as a continuous harmonic function in $\Omega \setminus \overline{\sigma}$. Let w_0 be some fixed point in $\Omega \setminus \overline{\sigma}$. If X_t is Brownian motion started at w_0 then $U(X_t)$ is a continuous martingale by Ito's formula. With τ denoting the first exit time from $\Omega \setminus \overline{\sigma}$, the set of paths for which

$$\lim_{t \to \tau} U(X_t) = +\infty$$

has measure zero since $U(X_t)$ is a time change of another Brownian motion. We claim that

$$\liminf_{\substack{z \to \zeta \\ z \in \Gamma_\alpha(\ell_\zeta)}} U(z) < +\infty$$

for almost every Green line (with respect to the Green's measure at z_0 in Ω). If not, we could find a compact set K of positive ω_{z_0} measure and a corresponding set of Green lines, ℓ_ζ, with positive Green measure such that

$$\liminf_{\substack{z\to\zeta \\ z\in\Gamma_\alpha(\ell_\zeta)}} U(z) = +\infty.$$

Then with positive probability $X_\tau \in K$ and $\tau = \tau_0$ where τ_0 is the first exit time from Ω starting at w_0 and τ is as before. By Lemma 2.8, we would then have $\lim_{t\to\tau} U(X_t) = +\infty$ with positive probability.

With the claim established, we consider subsets of Green lines (and ideal boundary points) for which

$$\liminf_{\substack{z\to\zeta \\ z\in\Gamma_\alpha(\ell_\zeta)}} U(z) < M$$

for some fixed $M > 0$.

Let $z \in \Gamma_\alpha(\zeta)$ be a point such that $U(z) \le M$ and assume, as we may, that $d(z, \partial\Omega) < |z - z_0|/1000$. We will first describe a general construction of a boundary set F with large ω_z measure such that the twisting of $\ell(z_0, z)$ around z is nearly the same as the twisting around points of F. We will then further specialize the choice of z so that the twisting of a Green line can be related to the change in U along the Green line. Finally, we will apply Lemma 2.10 to finish the proof.

To proceed as indicated, let ζ_* denote a nearest boundary point to z and let:

$$d = |z - \zeta_*| \tag{3}$$

$$S = \left\{ w : |\zeta_* - w| = \frac{d}{8} \right\}$$

$$C \subset S = \text{ the unique crosscut of } \Omega \text{ which intersects } D(z, d)$$

$$F \subset \partial\Omega = \text{ the set of ideal boundary points separated from } z \text{ by } C.$$

The Beurling projection theorem shows that $\omega(z, F, \Omega) > c > 0$ for some absolute constant c. We construct a domain Ω_ρ as in Lemma 2.11 with z and ζ_* playing the roles of w_0 and ζ_0 respectively. Let g_ρ denote the Green's function for Ω_ρ with pole at z_0. Then g_ρ and its first derivatives converge uniformly on compact subsets $\Omega \setminus \{z_0\}$ to $g(\cdot, z_0)$ as $\rho \to 0$. With sufficiently small $\rho > 0$, the Green lines $\ell(z_0, z)$ for both Ω_ρ and Ω are uniformly close and we now work with Ω_ρ. The set F is as before but may be considered to include the segment $L_\rho(\zeta_*)$.

In Ω_ρ, the segment $L_\rho(\zeta_*)$ corresponds, under connection by Green lines, to a forward cone at z with some opening angle depending on ρ

and Ω. Each point of the segment will be the end point of two Green lines under this correspondence; one for each ideal boundary point. By Lemma 2.5, $\omega(\cdot, L_\rho, \Omega_\rho)$ increases along the Green line which is at the center of this forward cone in the direction toward L_ρ and therefore (using Lemma 2.6) decreases along the Green line in the opposite direction (at angle π) which we denote by ℓ_1. By Lemma 2.11, ℓ_1 does not intersect the segment $[z, \zeta_*]$ except at the initial point z. Notice also that, by uniqueness of Green lines, either $\ell_1 = \ell(z_0, z)$ or ℓ_1 does not intersect $\ell(z_0, z)$ except at z. Perturbing z slightly if necessary, we may assume the latter case holds.

Let $\arg_1(\cdot - z)$ denote a continuous branch of the argument defined in $\Omega \setminus \ell_1$ such that

$$-\pi < \arg_1(z_0 - z) \leq \pi$$

and for a given point $\zeta \in F$, let $\arg(\cdot - \zeta)$ denote a continuous branch of the argument defined in Ω such that

$$-\pi < \arg(z_0 - \zeta) \leq \pi.$$

We claim there is an absolute constant $k > 0$ such that

$$|(\arg_1(w - z) - \arg_1(z_0 - z)) - (\arg(w - \zeta) - \arg(z_0 - \zeta))| < k \tag{4}$$

for all w sufficiently close to z on $\ell(z_0, z)$ and for all $\zeta \in F$. To see this, divide $\ell(z_0, z)$ into three disjoint pieces $\ell(z_0, z) = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ where

$$\begin{aligned} \mathrm{I} &= \ell(z_0, z) \setminus (D_d(z) \cup \overline{D_{d/8}(\zeta_*)}) \\ \mathrm{II} &= \ell(z_0, z) \cap D_d(z) \\ \mathrm{III} &= \ell(z_0, z) \cap (D_{d/8}(\zeta_*) \setminus D_d(z))). \end{aligned}$$

Considering some regular parametrization of $\ell(z_0, z)$ from z_0 to z, let Δ_{I}, Δ_{II} and Δ_{III} denote the respective contributions from each piece to the difference (4).

It is evident that Δ_{I} is bounded. The boundedness of Δ_{II} is a consequence of the fact that Ω is simply connected. The Green line $\ell(z_0, z)$ cannot wind around ζ_* to cross the segment $[z, \zeta_*]$ and $\ell(z_0, z)$ cannot intersect ℓ_1 except at z. (Recall that ℓ_1 does not intersect $[z, \zeta_*]$ except at z.)

To see the boundedness of Δ_{III} we use the following topological argument which is made routine in [**11**].

Let γ denote a maximal subarc of $\ell(z_0, z)$ which is contained in $D_{d/8}(\zeta_*)$. As before, $\ell(z_0, z)$ is considered to be given by a regular parametrization from z_0 to z. Let s_0 and s_1 denote the endpoints of γ on S with (s_0, s_1) respecting the order given by the parametrization of $\ell(z_0, z)$. The points s_0 and s_1 may lie on different components of $S \cap \Omega$.

Consider the bounded component, denoted by V, of $(\gamma \cup S)^c$ which does not contain ζ. We can define $\arg_2(w-\zeta)$ as a continuous function of w on the closure of V so that it agrees with $\arg(w-\zeta)$ on γ and we see that $(\arg(s_1-\zeta)-\arg(s_0-\zeta))$ is equal to the change in $\arg_2(w-\zeta)$ on the circular arc $\partial V \cap S$. Summing the contributions of the various maximal arcs of $\ell(z_0,z)\cap D_{d/8}(\zeta_*)$ to the total change in $\arg(\cdot-\zeta)$ and respecting the signs of the changes shows that the total change in $\arg(\cdot-\zeta)$ on III is bounded by 2π while the total change in $\arg_1(\cdot-z)$ on III is evidently bounded by a smaller constant.

With (4) established, we wish to show that the set

$$E=\left\{\zeta\in\partial\Omega : \lim_{\substack{z\to\zeta\\ z\in\Omega}}\arg(z-\zeta)=+\infty\right\}\cap\left\{\zeta\in\partial\Omega : \liminf_{\substack{z\to\zeta\\ z\in\Gamma_\alpha(\zeta)}} U(z)<+\infty\right\}$$

has $\omega(E)=0$. Let $D_2\subset D_1$ denote nested disks with rational radii and centers. Writing E as a countable union of subsets we may assume that E satisfies the conditions below. The requirement that $D_2\subset D_1$ allows for control of the twisting of the Green lines and this explains why conditions v), vi), vii) repeat the conditions i), ii), iii).

i) E is contained in a single connected component $\mathcal{V}_1\subset\Omega\cap D_1$.

ii) Each point of E is separated from z_0 in Ω by the same connected arc $\gamma_1\subset\partial D_1$.

iii) For each $\zeta\in E$, there is a point $w_1\in\gamma_1$ such that $\ell(w_1,\zeta)\subset\ell(z_0,\zeta)\subset\mathcal{V}_1$.

iv) For each $\zeta\in E$,

$$U(z)-U(w_1)\le M$$

for infinitely many $z\in\ell(w_1,\zeta)$, where M is some positive integer.

v) The set E is contained in a single connected component $\mathcal{V}_2\subset\mathcal{V}_1\cap D_2$.

vi) Each point of E is separated from z_0 in Ω by the same connected arc γ_2 of ∂D_2.

vii) For each $\zeta\in E$ there is $w_2\in\gamma_2$ such that $\ell(w_2,\zeta)\subset\ell(z_0,\zeta)\subset\mathcal{V}_2$.

viii) For each $\zeta\in E$ and for all $z\in\mathcal{V}_2$

$$\arg(z-\zeta)-\arg(w_1-\zeta)>KM$$

for some large constant $K>0$.

Since $\ell(w_1,\zeta)$ is contained in the circle D_1 it can not twist around w_1.

We will use this fact to show that for $z \in \ell(w_1, \zeta)$ and $w \in \ell(w_1, \zeta)$ sufficiently near z,

$$|(U(w) - U(w_1)) - (\arg(w - z) - \arg(w_1 - z))| < C \tag{5}$$

for some fixed $C > 0$. In words, the twisting about z of the segment of $\ell(w_1, \zeta)$ from w_1 to z, is controlled by the total change in direction of the tangent vector U. To get (5), we follow reasoning from [**11**] and consider the arc $\ell(w_1, z)$ to be regularly parameterized by $\alpha(t)$ for $0 \leq t \leq 1$ so that $\alpha(0) = w_1$ and $\alpha(1) = z$. The function $\alpha(t) - \alpha(\tau)$ is continuous and nowhere zero on $T = \{(\tau, t) : 0 \leq t \leq 1,\ 0 \leq \tau < t\}$, so there is a branch of the argument so that

$$\phi(\tau, t) \equiv \arg(\alpha(t) - \alpha(\tau))$$

is continuous on T. Because $\alpha(0) \in \gamma_1$ and $\ell(w_1, z) \subset \mathcal{V}_1$, we can determine the branch of the argument so that $0 \leq \phi(0, t) \leq 2\pi$ for sufficiently small t and therefore so that $-\pi \leq \phi(0, t) \leq 3\pi$ for all $0 \leq t \leq 1$. Since $\alpha'(t)$ is continuous and never zero, $\phi(t_0) \equiv \lim_{(\tau,t)\to(t_0,t_0)} \phi(\tau, t)$ exists for each $0 \leq t_0 \leq 1$. The function $\phi(t)$ is continuous and $\phi(t) - U(\alpha(t))$ is constant. It follows that

$$\phi(1) - \phi(0) = U(\alpha(1)) - U(\alpha(0))$$

and

$$\phi(1) - \phi(0, 1) = U(\alpha(1)) - U(\alpha(0)) + (\phi(0) - \phi(0, 1)).$$

But $\phi(1) - \phi(0, 1)$ is the change in $\phi(\tau, 1)$ as τ increases from 0 to 1 and both

$$-\pi \leq \phi(0, 1) \leq 3\pi$$

and

$$-\pi \leq \phi(0) \leq 3\pi$$

so (5) follows. We construct, for each $z \in \ell(w_2, \zeta)$ with $U(z) - U(w_1) \leq M$, the set F as described earlier in (3), using the point w_1 as a base point. Using (4) and (5) we then have for each $\zeta' \in F$,

$$\arg(z - \zeta') - \arg(w_1 - \zeta') < KM$$

if K is large enough. Thus $F \subset E^c \cap \partial\Omega$ and since $\omega_z(F) > c > 0$, it now follows from Lemma 2.10 that $\omega(E) = 0$. □

Recall that $E_M \subset \partial\Omega$ denotes the set of boundary points $\zeta \in \partial\Omega$ satisfying

(1) $\arg(z - \zeta) < M$ for infinitely many $z \in \Gamma_\alpha(\zeta)$.
(2) ζ is not a cone point of $\partial\Omega$.
(3) $\arg(z - \zeta) \geq -M$ for each $z \in \Omega$.

As discussed at the beginning of the section, to complete the proof of the twist point theorem, it now suffices to show that $\omega(E_M) = 0$.

Proof of Theorem 2: Given $\zeta \in E_M$ and $z \in \Gamma_\alpha(\zeta)$ such that $\arg(z-\zeta) < M$, we will construct a set of boundary points $F \subset \partial\Omega$ with $\omega_z(F) > c > 0$ such that for any $\zeta' \in F$ either condition (2) or (3) in the definition of E_M fails. By Lemma 2.10, this will complete the proof.

We make a general construction which will be referred to as $\mathcal{P}$, (for "picture"). The construction of $\mathcal{P}$ will be moved around the domain Ω by translations, rotations and dilations as needed.

Let $1 \gg \epsilon > 0$ be small and fixed and let Ω be simply connected. Suppose that $(0,1) \in \partial\Omega$ and that

$$\left\{(x,y) : -2 < x < 1,\ -1 < y < 1 - \frac{\epsilon}{2}\right\} \subset \Omega.$$

Let

$$R = \left\{(x,y) : -1 \leq x \leq 0,\ 1 - \frac{\epsilon}{2} \leq y \leq 1 + \frac{\epsilon}{2}\right\}$$

and let $\mathcal{T}_0$ denote the collection of isosceles right triangles with right angle vertex in

$$\left\{(x,y) : -1 \leq x \leq 0,\ y \geq 1 - \frac{\epsilon}{2}\right\}$$

and base on the x-axis. Let $\mathcal{T}_1$ be the collection of triangles in $\mathcal{T}_0$ whose interiors are contained in Ω and let

$$\Omega' = \left(\bigcup_{T \in \mathcal{T}_1} T\right).$$

The domain Ω' is a simply connected Lipschitz subdomain of Ω with Lipschitz constant independent of Ω and it is the union of triangles in $\mathcal{T}_1$ whose interiors are not contained in the interior of any larger triangle in $\mathcal{T}_1$. We will call these maximal triangles in Ω'. Note that right angle vertices lying in Ω of maximal triangles in Ω' form a set of isolated points in Ω. Let V denote the set of right angle vertices of maximal triangles which are in $\partial\Omega$. We say that case 0 occurs if $\omega_{(-\frac{1}{2},\frac{1}{2})}(V,\Omega') \geq \epsilon$ and otherwise that we are in case I. The set $\partial\Omega' \cap \Omega$ is a union of countably many segments of edges of maximal triangles. In case I, the Dahlberg estimate (1) implies that $\omega_{(-\frac{1}{2},\frac{1}{2})}(\rho,\Omega) > c > 0$, where ρ denotes the set of right hand edge segments, looking from $(0,0)$ to $(0,1)$, of $\partial\Omega' \cap \Omega$.

$$\rho = \bigcup_j I_j.$$

Let ζ_j denote the right hand endpoint of I_j as seen from inside Ω' and let J_j be the segment of I_j with $|J_j| = \frac{1}{10}|I_j|$ and right hand endpoint ζ_j. Let p_j be the point in Ω' with $|p_j - \zeta_j| = |J_j|$ and with the segment $[p_j, \zeta_j]$ perpendicular to J_j. With the endpoint ζ_j fixed, rotate J_j and p_j clockwise about ζ_j, as far as possible, until the rotated segment hits another boundary point. Call the resulting segment J_j^* and the resulting point p_j^*. The amount of rotation can be zero. Note that

$$\omega_{(-\frac{1}{2},\frac{1}{2})}(J_j^*, \Omega \setminus J_j^*) > c_1 \omega_{(-\frac{1}{2},\frac{1}{2})}(I_j, \Omega')$$

for all j for some fixed constant $c_1 > 0$. The description of $\mathcal{P}$ is complete. We return to the setting of the theorem and let z and ζ be as in the first paragraph of the proof. Let ζ_0 be a nearest boundary point to z and choose a point p_0 on the segment joining z to ζ_0. We move $\mathcal{P}$ so that $(0,0)$ and $(0,1)$ correspond respectively to p_0 and ζ_0 and we require of p_0 that the circle $\{w : |w - p_0| = |\zeta - p_0|\}$ intersects the (moved) rectangle R only on its short edges and that conditions necessary for $\mathcal{P}$ to lie inside Ω are satisfied. We can do this with

$$|p_0 - \zeta_0| \geq c_2 |p_0 - \zeta_0| \tag{6}$$

for some constant $c_2 > 0$ depending on ϵ.

In this initial step, we write $V_0 = V$ and if case 0 occurs, we put $F = V_0$. If case I occurs, we obtain the segments J_j^* and corresponding points p_j^* and ζ_j^*. We repeat the construction of $\mathcal{P}$ for each j with $(0,0)$ corresponding to p_j^* and $(0,1)$ corresponding to ζ_j^*. For each j, if case 0 occurs we find a set of cone points $V = V_j$ and if case I occurs we find segments $J^*_{(j_1,j_2)}$ and the corresponding points $p^*_{(j_1,j_2)}$ and $\zeta^*_{(j_1,j_2)}$. Let α denote a multi index and $|\alpha|$ its length. Choose a large integer K and stop the construction when multiindices reach length KM. For such indices α we find a set F_α with

$$\omega_{p_\alpha}(F_\alpha) > c > 0 \tag{7}$$

by using the Beurling projection theorem as in the proof of Theorem 1. By Harnack's inequality and induction

$$\omega_{p_0}(F_\alpha) \geq C c^{|\alpha|}$$

and the construction shows that with $\alpha = (j_1, \ldots, j_{KM})$, a simple curve connecting $p_0, p_{(j_1)}, p_{(j_1,j_2)}, p_{(j_1,j_2,j_3)}, \ldots, p_\alpha$, winds clockwise around the points of F_α at least $10M$ times if K is sufficiently large. Another simple topological argument shows that

$$\arg(p_\alpha - \zeta') < -M$$

for each $\zeta' \in F_\alpha$, if K is sufficiently large. Put

$$F = \left(\bigcup_{\alpha \in \text{case } 0} V_\alpha \right) \cup \left(\bigcup_{\alpha \in \text{case I}} F_\alpha \right).$$

Choosing the original $\epsilon > 0$ smaller than $c > 0$ from (6), an induction shows that

$$\omega_{p_0}(F) \geq C\epsilon^{KM} > 0$$

and using Harnack's inequality and (5),

$$\omega_z(F) \geq C\epsilon^{KM} > 0.$$

The theorem is proved. □

4. Appendix

Here we prove the lemmas from Section 2 which require only reasoning with Green lines. The notation and statements are as in Section 2.

Lemma 4.1. *If $z_1 \neq z_2$ are joined by a regular Green line then $\ell(z_1, z_2) = \ell(z_2, z_1)$.*

Proof: Let $c_0 = g(z_1, z_2) = g(z_2, z_1)$ and let $g_2(z) = g(z, z_2)$ and $g_1(z) = g(z, z_1)$. Assume at first that z_1 and z_2 are sufficiently close so that all level curves $g_1 = c_1$ and $g_2 = c_2$ for $c_1, c_2 \geq c_0$ are strictly convex. For each $c > c_0$ there is a unique point $M(c)$ on the curve $g_2 = c$ such that g_1 is maximized subject to the constraint $g_2 = c$, or in other words, $g_1(M(c)) \geq g_1(w)$ for any w such that $g_2(w) = c$. The function $M(c)$ is differentiable by the implicit function theorem and $M(c)$ is the unique point of $g_2 = c$ which is tangent to a level curve $g_1 = c' = g_1(M(c))$. Write $M(c) = (x(c), y(c))$ and for a fixed $c \geq c_0$ choose coordinates so that the x-axis is tangent to $g_2 = c$ at $M(c)$ and the y-axis is normal to $g_2 = c$ at $M(c)$. For the fixed $c \geq c_0$ we must have $x'(c) = 0$ in order for ∇g_1 to be perpendicular to $g_1 = c'$ at $M(c)$.

We may, on the one hand, construct the line $\ell(z_2, z_1)$ as a limit of polygonal arcs produced by Euler's method starting at z_1, with initial increment perpendicular to the curve $g_2 = c_0$ and successive increments perpendicular to level curves of g_2 at their initial points. On the other hand, consider the following alternative construction of a polygonal arc to approximate $\ell(z_2, z_1)$. Construct the first increment as in Euler's method. One end point is z_1 and the other lies on $g_2 = c_0 + \Delta c$ for some $\Delta c > 0$. Consider successive level curves $g_2 = c_0 + j\Delta c$ for $j = 2, \dots$ and for $j \geq 2$ let the j^{th} increment connect the $(j-1)^{\text{th}}$ endpoint to $M(c + j\Delta c)$. Note that $\frac{\partial g_2}{\partial n}$ is bounded away from zero on $g_2 = c_0$

so that Δc is comparable to the length of the initial segment and the level curve of g_1 passing through the endpoint of the first increment is asymptotically circular as $\Delta c \to 0$. Let d denote the Euclidean length of $\ell(z_2, z_1)$ and let N be the greatest integer not exceeding $\frac{d}{\Delta c}$. The above discussion shows that for $j = 1, \ldots, N$ the distance between the j^{th} increments of the two methods is $O(j(\Delta c)^2)$ and so the two methods converge to the same curve $\ell(z_2, z_1)$. But the second construction shows that a tangent vector to $\ell(z_2, z_1)$ at any point is perpendicular to both a level curve of g_2 and a level curve of g_1. By symmetry, we then have $\ell(z_2, z_1) = \ell(z_1, z_2)$. The argument can clearly be continued to construct a slightly longer Green line $\ell(z_0, z_3) = \ell(z_3, z_0)$ which contains $\ell(z_2, z_1)$ as a proper subset. For a general pair of points $z_1 \neq z_2$, we can cover $\ell(z_2, z_1)$ by small disks in which our initial assumptions hold and obtain $\ell(z_2, z_1) = \ell(z_1, z_2)$ by piecing together smaller subarcs. □

Lemma 4.2. $\frac{\nabla u}{|\nabla u|} = v$.

Proof: By a limiting argument, it suffices to work in the domain $\Omega = \{g_{z_1} > \lambda\}$ where $\lambda > 0$ is small and fixed. The Green function for this domain is $g_{z_1} - \lambda$. Basic properties of the Green function imply that for large λ_0, the level curves $\{g_{z_1} = \lambda_0\}$ are asymptotically circular. It follows that, given $\epsilon > 0$, we may choose λ_0 sufficiently large so that within the domain $\{g_{z_1} > \lambda_0\}$, the harmonic measure of $\Lambda_\alpha(z_1) \cap \{g_{z_1} = \lambda_0\}$ has a gradient at z_1 pointing within angle ϵ of the vector v. Now, given $\delta > 0$, there is $\eta > 0$ such that if $|z - z_1| < \eta$, then the level curve $\{g_z = \lambda_0\}$ lies in a δ-neighborhood of $\{g_{z_1} = \lambda_0\}$. By continuous dependence of trajectories on the initial point (in the domain $\{g_{z_1} > \lambda\}$), a sufficiently small choice of $\eta > 0$ guarantees that the subset of $\{g_z = \lambda_0\}$ corresponding to $I_\alpha(z_1)$ via the Green lines at z lies in a δ-neighborhood of the subset of $\{g_{z_1} = \lambda_0\}$ which corresponds to $I_\alpha(z_1)$ via Green lines at z_1. For any z such that $|z - z_1| < \eta$, the harmonic measure $u(z)$ is equal to the harmonic measure within $\{g_z > \lambda_0\}$ of the subset of $\{g_z = \lambda_0\}$ corresponding to $I_\alpha(z_1)$ via Green lines at z. If $\eta > 0$ and therefore $\delta > 0$ is sufficiently small, it is then clear that the direction of greatest increase of u must be within angle 2ϵ of v. □

Lemma 4.3. *If $z_2 \in \ell(z_0, z_1)$ and $g(z_0, z_1) > g(z_0, z_2)$ then*

$$\Lambda_\alpha(z_2) \subset \Lambda_\alpha(z_1).$$

Proof: By a limiting argument, we may again assume that we are in a Jordan domain. Let $\ell_0 = \ell(z_0, z_1)$ and let ℓ_1 and ℓ_2 denote the Green lines starting at z_1 which make an angle of $\alpha/2$ with ℓ_0. The trajectories ℓ_1 and ℓ_2 form the sides of $\Lambda_\alpha(z_1)$. Consider the set of trajectories starting at z_1 lying in $\Lambda_\alpha(z_1)$ between ℓ_1 and ℓ_0 and denote the part of $\partial\Omega$ corresponding to them by H. We claim that the harmonic measure of H increases along ℓ_0 in the direction of the forward cone $\Lambda_\alpha(z_1)$. A similar argument will show that the harmonic measure of the other half of $\Lambda_\alpha(z_1)\cap\partial\Omega$ corresponding to the trajectories between ℓ_0 and ℓ_2 also increases along ℓ_0 in the same way. This claim implies the lemma since if $\Lambda_\alpha(z_2) \not\subset \Lambda_\alpha(z_1)$ for some z_2 on $\ell(z_0, z_1) \cap \Lambda_\alpha(z_1)$ with $g(z_0, z_2) < g(z_0, z_1)$ then the harmonic measure of one of the halves of $\Lambda_\alpha(z_1)$ would have decreased from z_1 to z_2.

To prove the claim, let $z \in \ell_0 \cap \Lambda_\alpha(z_1)$ and let ℓ_3 be a Green line starting at z and ending in the same ideal boundary point as ℓ_1. (ℓ_3 exists by a simple limiting argument.) Let $\theta < \pi$ be the angle between the forward tangent vectors to ℓ_0 and ℓ_3 at z and consider the trajectory ℓ_4 starting at z, lying between ℓ_0 and ℓ_3 in $\Lambda_\alpha(z_1)$, and at angle $\frac{\theta}{2}$ from ℓ_0. We know (by Lemma 2.5) that $\omega = \omega(\cdot, H, \Omega)$ has gradient tangent to ℓ_4 at z. Consequently the level curve of ω passing through z is orthogonal to ℓ_4 at z. It follows that ω increases on ℓ_0 in the direction of decreasing $g(\cdot, z_0)$. □

Lemma 4.4. *$C_\lambda \cap \Gamma_\alpha(\ell_\zeta)$ is a connected arc.*

Proof: We may again assume that Ω is a Jordan domain. Note that Lemma 2.6 implies that for any $w \in C_\lambda$ we have $\Lambda_\pi(w) \cap C_\lambda = \{w\}$. In fact, the angle between $\ell(z_0, w)$ and the complete Green line L which forms the edge of $\Lambda_\pi(w)$ is $\frac{\pi}{2}$. Lemma 2.6 implies that there can be no other intersection (say at a point w') at angle $\frac{\pi}{2}$ between L and another Green line starting at z_0. If there were, there would be a forward cone centered on L at w and another forward cone centered on L at w' with the same opening angle but whose edges intersected at z_0, contradicting Lemma 2.6. Considering a Green line starting at z_0 which intersects C_λ at a point very close to w, we see (in the same way by Lemma 2.6) that the forward angle of intersection with L must be less than $\frac{\pi}{2}$ and therefore, locally, L must lie in the unbounded component of the complement of C_λ except for the point $\{w\}$. Combining this with the previous observation, we see that L must lie completely outside of C_λ in $\{z : g(z, z_0) < \lambda\}$ except for the point $\{w\}$. Consequently, the only local extrema of $g(\cdot, z_0)$ restricted to Green lines $\ell(z_1, z_2)$ are local maxima.

By the local geometry of the Green lines and C_λ, the set $C_\lambda \cap \Gamma_\alpha(\ell_\zeta)$ includes an open arc of C_λ containing the point $\ell_\zeta \cap C_\lambda$.

Let $J_\lambda \subset C_\lambda$ be a maximal arc of $C_\lambda \cap \Gamma_\alpha(\ell_\zeta)$ containing $\ell_\zeta \cap C_\lambda$ and denote its endpoints by $J_\lambda = (a_\lambda, b_\lambda)$ with the standard orientation on C_λ. With the same orientation in mind, let ℓ_1 denote the Green line starting at b_λ making an angle $\frac{\alpha}{2}$ clockwise from the forward tangent vector to $\ell(z_0, b_\lambda)$ at the point b_λ. Since J_λ is maximal, ℓ_1 does not intersect ℓ_ζ and, in fact, ℓ_1 must end at ζ.

Suppose that $c_\lambda \in C_\lambda \cap \Gamma_\alpha(\ell_\zeta) \setminus J_\lambda$. The forward cone $\Lambda_\alpha(c_\lambda)$ at c_λ has two edges which make an angle $\frac{\alpha}{2}$ at c_λ with $\ell(z_0, c_\lambda)$. One of these edges must cross ℓ_ζ since $c_\lambda \in \Gamma_\alpha(\ell_\zeta)$ and we may assume that it is the edge whose tangent is at angle $\frac{\alpha}{2}$ clockwise from the forward tangent to $\ell(z_0, c_\lambda)$ at c_λ. (If not, then we reverse the whole argument.) Denote this edge by ℓ_2. By the first paragraph of the proof, the edge ℓ_2 must intersect $\ell(z_0, b_\lambda)$ at a point w in $\{z : g(z, z_0) < \lambda\}$, and since ℓ_2 crosses ℓ_ζ while ℓ_1 does not, Lemma 2.6 implies that the forward angle of intersection, denoted by γ, of ℓ_2 and $\ell(z_0, b_\lambda)$ at w is larger than $\frac{\alpha}{2}$. Now we see that $\ell(z_0, b_\lambda)$ and $\ell(z_0, c_\lambda)$ form the edges of forward cones centered on ℓ_2 with angles γ and $\frac{\alpha}{2}$ respectively from ℓ_2, but which intersect at z_0. This contradicts Lemma 2.6, so there can be no such point c_λ. □

Lemma 4.5. *Let $0 < \alpha < \pi$ be fixed. There is a constant $c > 0$ only depending on α such that for any ℓ_ζ, $z \in \ell_\zeta$ and $w \in \gamma(z, \zeta)$,*

$$\frac{1}{c}\operatorname{dist}(w, \partial\Omega) \leq \operatorname{dist}(z, \partial\Omega) \leq c \operatorname{dist}(w, \partial\Omega).$$

Proof: We may again assume that Ω is a Jordan domain. Let $L = L(z_0, z)$ denote the complete Green line passing through z_0 and z. We may assume below that $L(z_0, z)$ contains a regular Green line in both directions from z_0. Suppose that $z_1, z_2 \in \gamma(z, \zeta)$ and that

$$\operatorname{dist}(z_2, \partial\Omega) \ll \operatorname{dist}(z_1, \partial\Omega).$$

We may as well take $\operatorname{dist}(z_1, \partial\Omega) = 1$ and $\operatorname{dist}(z_2, \partial\Omega) = \epsilon$. Let t_1 and t_2 denote respectively the trajectories of $\nabla g(\cdot, z_1)$ which make an angle of α and $\frac{\pi}{2}$ with $L(z_0, z_1)$ and lie on the same side of $L(z_0, z_1)$ in Ω as z_2. Let t_3 denote the trajectory of $\nabla g(\cdot, z_2)$ which ends at the boundary point ζ. Trajectories of $\nabla g(\cdot, z_2)$ between t_3 and $\gamma(z, \zeta)$ end on a set $A \subset \partial\Omega$ of ω_{z_2} measure $\geq \frac{\pi - \alpha}{4\pi}$. By Lemma 2.3, more than half of the ω_{z_2} measure of A is contained in a disk D of radius $K\epsilon$ centered at z_2. With sufficiently large K, the same disk contains ω_{z_2} measure at least $\frac{1}{8}$ corresponding to endpoints of trajectories of $\nabla g(\cdot, z_2)$ which lie on the opposite side of $L(z_0, z_2)$ and which point into the interior of $\{w : g(w, z_0) > g(z, z_0)\}$ from z_2. This forces t_1 and t_2 to pass through D and any curve in Ω connecting z_1 to the segment of $\partial\Omega$ determined by the endpoints of t_1 and t_2 must also pass through D. By the Beurling projection theorem and the maximum principle, we would then have

$$\frac{\pi-\alpha}{4\pi} < C\sqrt{K\epsilon}$$

so that ϵ can not be arbitrarily small. □

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