ESTIMATES OF GREEN FUNCTIONS AND HARMONIC MEASURES FOR ELLIPTIC OPERATORS WITH SINGULAR DRIFT TERMS

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Abstract

In this paper, we prove the existence and uniqueness of the continuous Green function G for the elliptic operator $L = \operatorname{div}(A(x)\nabla_x) + B(x) \cdot \nabla_x$ with singular drift term B on a $C^{1,1}$ bounded domain D in $\mathbb{R}^n$, $n \geq 3$, and its comparability to the Green function G_0 of $L_0 = \operatorname{div}(A(x)\nabla_x)$. Basing on this result we establish the equivalence of the L-harmonic measure and the surface measure on ∂D. These results extend some first ones proved for elliptic operators with less singular drift terms.

1. Introduction

In this paper we will interest with the following elliptic operator

$$L = \operatorname{div}(A(x)\nabla_x) + B(x) \cdot \nabla_x$$

on a $C^{1,1}$ bounded domain D in $\mathbb{R}^n$, $n \geq 3$. Our aim is to investigate the existence and uniqueness of the continuous L-Green function G and its comparability to the Green function G_0 of $L_0 = \operatorname{div}(A(x)\nabla_x)$, where A is a uniformly elliptic matrix with Lipschitz-continuous coefficients and the drift term B is in a class $\mathcal{K}_{\mathrm{loc}}(D)$ generalizing the well known Kato class K_{n+1} used in [**9**]. Basing on the comparability of the Green functions we will be able to estimate the L-Poisson kernel which allow us to prove that the L-harmonic measure and the surface measure are mutually absolutely continuous on ∂D. Consequently, the operators L and L_0 have the same potential theory on D. Before describing the body of this paper, we first recall that the same results are studied in [**4**] and [**15**] for more general elliptic operators but with coefficients Hölder continuous

up to the boundary of D. Subsequently, Cranston and Zhao [9] studied the operator $L = -\Delta_x + B(x) \cdot \nabla_x$ and proved, by a probabilistic approach, the same results when $|B|^2 \in K_n$ and $|B| \in K_{n+1}$ (see Theorem 3.13 in [9]). In particular our results extend this theorem to a large class of elliptic operators with more singular drift terms. Recently, Ancona [5] studied the equivalence of Green functions for more general elliptic operators on manifolds or domains. However, in the Euclidian case (see Theorems 9.1 and 9.1' in [5]), his results do not cover the ones proved here. In fact, we will provide (see Proposition 4.4) examples of vector fields in the class $\mathcal{K}_{\mathrm{loc}}(D)$ which do not satisfy the conditions required by Ancona [5]. For other results concerning stability of Green function under perturbations we refer the reader to [1], [6], [8], [12], [13], [16]–[25], [27] and [28].

Our paper is organized as follows. In Section 2, we give some notations and we state some known results. In Section 3, we prove a $3G_0$-type inequality which is the key in proving our main result in the next. In Section 4, we introduce the class $\mathcal{K}_{\mathrm{loc}}(D)$ and we study some of its properties. In particular we prove that $\mathcal{K}_{\mathrm{loc}}(D)$ is more general than the Kato class K_{n+1}. In Section 5, we prove the existence and uniqueness of the continuous L-Green function G when the vector B is in the class $\mathcal{K}_{\mathrm{loc}}(D)$. In Section 6, we apply the previous results to prove the equivalence of the L-harmonic measure and the surface measure on ∂D.

Throughout the paper the letter C denotes a generic positive constant which may vary in value from line to line.

2. Notations and known results

Throughout this paper D denotes a $C^{1,1}$ bounded domain in $\mathbb{R}^n$, $n \geq 3$. This means that for all $z \in \partial D$, there exists $r_0 > 0$ such that $B(z, r_0) \cap \partial D$ is the graph of a $C^{1,1}$-function. As already mentioned, we will consider the elliptic operator

$$L = L_0 + B(x) \cdot \nabla_x,$$

where $L_0 = \mathrm{div}(A(x)\nabla_x)$, on D. The matrix $A(x) = (a_{ij}(x))_{1 \leq i,j \leq n}$ is assumed to be uniformly elliptic i.e. $\mu^{-1}\|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \mu\|\xi\|^2$, for all $x \in D$, $\xi \in \mathbb{R}^n$ and some $\mu \geq 1$ with μ-Lipschitz continuous coefficients and the vector $B = B(x) \in L^1_{\mathrm{loc}}(D)$. We will denote by G_0 the L_0-Green function on D. For $x \in D$, let $d(x) = d(x, \partial D)$ denotes the distance from x to the boundary of D. We recall the following useful estimates proved in [11], [14], [26] and [27].

Theorem 2.1. *There exists a constant $C = C(n, \mu, D)$ such that, for all $x, y \in D$,*

1) $$\frac{1}{C} N(x, y) \leq G_0(x, y) \leq C N(x, y),$$

where

$$N(x, y) = \min\left(1, \frac{d(x)}{|x - y|}\right) \min\left(1, \frac{d(y)}{|x - y|}\right) \frac{1}{|x - y|^{n-2}},$$

2) $$|\nabla_x G_0(x, y)| \leq C \min\left(1, \frac{d(y)}{|x - y|}\right) \frac{1}{|x - y|^{n-1}}.$$

The dependence of the constant C on D is only on the diameter of D and the curvature of ∂D.

3. A $3G_0$-type inequality

In this section, we prove the following fundamental inequality which is the key in proving our main result in the next. This inequality is reminiscent of the $3G_0$-inequality proved in [**22**] and [**24**] on $C^{1,1}$-domains and in [**23**] on Lipschitz domains which played an important role to study Schrödinger equations.

For $x, z \in D$, let

$$\Gamma(x, z) = \min\left(1, \frac{d(z)}{|x - z|}\right) \frac{1}{|x - z|^{n-1}}.$$

Theorem 3.1. *There exists a constant $C_0 = C_0(n, \mu, D) > 0$ such that for all $x, y, z \in D$, we have*

$$\frac{G_0(x, z)|\nabla_z G_0(z, y)|}{G_0(x, y)} \leq C_0\Big(\Gamma(x, z) + \Gamma(y, z)\Big).$$

Proof: From Theorem 2.1, we have

$$(3.1) \quad \frac{G_0(x, z)|\nabla_z G_0(z, y)|}{G_0(x, y)} \leq C\omega(x, z, y)\frac{1}{|z - y|}\left(\frac{|x - y|}{|x - z||z - y|}\right)^{n-2},$$

where

$$\omega(x, z, y) = \frac{\min\left(1, \frac{d(x)}{|x-z|}\right) \min\left(1, \frac{d(z)}{|x-z|}\right) \min\left(1, \frac{d(y)}{|z-y|}\right)}{\min\left(1, \frac{d(x)}{|x-y|}\right) \min\left(1, \frac{d(y)}{|x-y|}\right)}.$$

Case 1: $|x - z| \geq |z - y|$.

We have $|x - y| \leq |x - z| + |z - y| \leq 2|x - z|$ and so

$$(3.2) \qquad \left(\frac{|x - y|}{|x - z||z - y|} \right)^{n-2} \leq \frac{2^{n-2}}{|z - y|^{n-2}}.$$

Moreover, by using the inequalities $\frac{ab}{a+b} \leq \min(a, b) \leq 2\frac{ab}{a+b}$, for all $a, b > 0$ and $|d(z) - d(y)| \leq |z - y|$, we have

$$\omega(x, z, y) \leq 2 \frac{\min\left(1, \frac{d(z)}{|x-z|}\right)}{\min\left(1, \frac{d(y)}{|x-y|}\right)} \min\left(1, \frac{d(y)}{|z - y|}\right)$$

$$(3.3) \qquad\qquad \leq 8 \frac{d(z)}{d(y)} \min\left(1, \frac{d(y)}{|z - y|}\right)$$

$$\qquad\qquad \leq 16 \min\left(1, \frac{d(z)}{|z - y|}\right).$$

Combining (3.1)–(3.3), we obtain

$$\frac{G_0(x, z)|\nabla_z G_0(z, y)|}{G_0(x, y)} \leq C\Gamma(y, z).$$

Case 2: $|z - y| \geq |x - z|$.

We have $|x - y| \leq |x - z| + |z - y| \leq 2|z - y|$ and so

$$(3.4) \qquad \left(\frac{|x - y|}{|x - z||z - y|} \right)^{n-2} \leq \frac{2^{n-2}}{|x - z|^{n-2}}.$$

Moreover, we have

$$\omega(x, z, y) \leq 2 \frac{\min\left(1, \frac{d(x)}{|x-z|}\right)}{\min\left(1, \frac{d(x)}{|x-y|}\right)} \min\left(1, \frac{d(z)}{|x - z|}\right)$$

$$(3.5) \qquad\qquad \leq 8 \frac{d(x) + |x - y|}{d(x) + |x - z|} \min\left(1, \frac{d(z)}{|x - z|}\right)$$

$$\qquad\qquad \leq 8 \left(1 + 2\frac{|z - y|}{|x - z|}\right) \min\left(1, \frac{d(z)}{|x - z|}\right)$$

$$\qquad\qquad \leq 24 \frac{|z - y|}{|x - z|} \min\left(1, \frac{d(z)}{|x - z|}\right).$$

Combining (3.1)–(3.5), we obtain

$$\frac{G_0(x,z)|\nabla_z G_0(z,y)|}{G_0(x,y)} \le C\Gamma(x,z). \qquad \square$$

4. The class $\mathcal{K}_{\text{loc}}(D)$

In order to introduce the class $\mathcal{K}_{\text{loc}}(D)$, we shall first recall the well known Kato classes K_n and K_n^{loc} (see [2]).

Definition 4.1. Let V be a Borel measurable function on D. We say that V is in the local Kato class K_n^{loc} if it satisfies

$$\lim_{r\to 0} \sup_{x\in E} \int_{D\cap(|x-z|<r)} \frac{|V(z)|}{|x-z|^{n-2}}\, dz = 0,$$

for every compact subset $E \subset D$. We have $K_n^{\text{loc}} \subset L_{\text{loc}}^1(D)$.

We say that V is in the Kato class K_n if in the above definition $\sup_{x\in E}$ is replaced by $\sup_{x\in D}$. Clearly, if $V \in K_n$, then by a compactness argument (since $\overline{D}$ is bounded), we have

$$\sup_{x\in \overline{D}} \int_D \frac{|V(z)|}{|x-z|^{n-2}}\, dz < +\infty.$$

In particular, $K_n \subset L^1(D)$.

The Kato class K_n^{loc} introduced in [2] by Aizenman and Simon is considered as the biggest possible space of potentials $V = V(x)$ so that the Harnack inequalities and the continuity of solutions for the Schrödinger operator $-\Delta + V(x)$ hold. It is also used later by several authors to study the potential theory of the Schrödinger operator (see [27], [8], [7], [28] and the references therein). For the operators with drift terms $L = -\Delta + B(x)\cdot\nabla_x$, Cranston and Zhao [9] proved that when $|B| \in K_{n+1}$ and $|B|^2 \in K_n$, the Green functions and harmonic measures for L and Δ are equivalent.

Definition 4.2. Let B be a Borel measurable vector field on D. We say that B is in the class $\mathcal{K}_{\text{loc}}(D)$ if $B \in K_{n+1}^{\text{loc}}$ and satisfies

$$\|B\| \equiv \sup_{x\in D} \int_D \Gamma(x,z)|B(z)|\, dz < +\infty.$$

Obviously, $K_{n+1} \subset \mathcal{K}_{\text{loc}}(D) \subset K_{n+1}^{\text{loc}}$. In what follows we study some properties of the class $\mathcal{K}_{\text{loc}}(D)$. In particular we prove that $\mathcal{K}_{\text{loc}}(D)$ contains more singular vector fields than the Kato class K_{n+1} used by Cranston and Zhao in [9].

Proposition 4.3. *Let $D = B(0,1)$ be the unit ball. Then, for all $1 < \lambda < 2$, all $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$, the radial vector field on D,*

$$B_\lambda(r) = \frac{1}{(1-r)\left(1 + \text{Log}\,\frac{1}{1-r}\right)^\lambda}\vec{u} \in \mathcal{K}_{\text{loc}}(D) \setminus K_{n+1}.$$

Proof: Consider a spherical coordinate system $(r, \theta_1, \ldots, \theta_{n-1})$ such that $x = (|x|, 0, \ldots, 0)$ and set $|x| = r_0$, we have

$$\sup_{x \in \overline{D}} \int_D \frac{|B_\lambda(z)|}{|x - z|^{n-1}}\,dz$$

$$= C \sup_{0 \leq r_0 \leq 1} \int_0^1 r^{n-1}|B_\lambda(r)| \int_0^\pi \frac{(\sin\theta_1)^{n-2}}{(r^2 + r_0^2 - 2rr_0\cos\theta_1)^{\frac{n-1}{2}}}\,d\theta_1\,dr.$$

By making the change of variables $t = \tan\frac{\theta_1}{2}$ and letting $\alpha = \frac{r+r_0}{|r-r_0|}$, we obtain

(4.1) $$\sup_{x \in \overline{D}} \int_D \frac{|B_\lambda(z)|}{|x - z|^{n-1}}\,dz = C \sup_{0 \leq r_0 \leq 1} \int_0^1 \frac{r^{n-1}|B_\lambda(r)|}{|r - r_0|^{n-1}}\varphi(\alpha)\,dr,$$

where

$$\varphi(\alpha) = \int_0^\infty \frac{t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}(1+\alpha^2 t^2)^{\frac{n-1}{2}}}\,dt.$$

Moreover, we have

$$\varphi(\alpha) = \int_0^\infty \frac{t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}(1+\alpha^2 t^2)^{\frac{n-1}{2}}}\,dt$$

(4.2)

$$\geq \frac{C}{\alpha^{n-1}} \int_{1/\alpha}^1 \frac{1}{t}\,dt = \frac{C}{\alpha^{n-1}}\,\text{Log}\,\alpha.$$

Combining (4.1) and (4.2), we obtain

$$\sup_{x \in \overline{D}} \int_D \frac{|B_\lambda(z)|}{|x - z|^{n-1}}\,dz \geq C \sup_{0 \leq r_0 \leq 1} \int_0^1 \frac{r^{n-1}|B_\lambda(r)|}{(r + r_0)^{n-1}}\,\text{Log}\,\frac{r + r_0}{|r - r_0|}\,dr$$

$$\geq C \int_0^1 r^{n-1}|B_\lambda(r)|\,\text{Log}\,\frac{1}{1 - r}\,dr = +\infty.$$

Now, we prove that B_λ is in the class $\mathcal{K}_{\text{loc}}(D)$. Clearly B_λ is locally bounded and so $B_\lambda \in K_{n+1}^{\text{loc}}$. We will show that $\|B_\lambda\| < +\infty$. By using

the same spherical coordinate system as above, we obtain

$$\|B_\lambda\| \equiv \sup_{x \in D} \int_D \Gamma(x, z)|B_\lambda(z)| \, dz$$

(4.3)

$$= C \sup_{0 \le r_0 < 1} \int_0^1 \frac{r^{n-1}|B_\lambda(r)|}{|r - r_0|^{n-1}} \psi(\alpha) \, dr,$$

where

$$\psi(\alpha) = \int_0^\infty \min\left(1, \frac{(1-r)(1+t^2)^{1/2}}{|r-r_0|(1+\alpha^2 t^2)^{1/2}}\right) \frac{t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}(1+\alpha^2 t^2)^{\frac{n-1}{2}}} \, dt.$$

By recalling that, for $a > 0$, $(1 \vee a)^2 \le 1 + a^2 \le 2(1 \vee a)^2$, it follows that

(4.4)

$$\psi(\alpha) = \int_0^\infty \min\left(1, \frac{(1-r)(1+t^2)^{1/2}}{|r-r_0|(1+\alpha^2 t^2)^{1/2}}\right) \frac{t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}(1+\alpha^2 t^2)^{\frac{n-1}{2}}} \, dt$$

$$\le C \left[\min\left(1, \frac{1-r}{|r-r_0|}\right) \int_0^{1/\alpha} t^{n-2} \, dt \right.$$

$$+ \frac{1}{\alpha^{n-1}} \int_{1/\alpha}^1 \min\left(1, \frac{1-r}{(r+r_0)t}\right) \frac{1}{t} \, dt$$

$$\left. + \min\left(1, \frac{1-r}{r+r_0}\right) \frac{1}{\alpha^{n-1}} \int_1^\infty \frac{1}{t^n} \, dt \right]$$

$$\le \frac{C}{\alpha^{n-1}} \left[\min\left(1, \frac{1-r}{|r-r_0|}\right) + \int_{1/\alpha}^1 \min\left(1, \frac{1-r}{(r+r_0)t}\right) \frac{1}{t} \, dt \right].$$

On the other hand

$$\int_{1/\alpha}^1 \min\left(1, \frac{1-r}{(r+r_0)t}\right) \frac{1}{t} \, dt \le 2 \int_{1/\alpha}^1 \frac{1-r}{(1-r+(r+r_0)t)t} \, dt$$

$$= 2 \int_{\frac{|r-r_0|}{1-r}}^{\frac{r+r_0}{1-r}} \frac{1}{(s+1)s} \, ds$$

(4.5)

$$= 2 \operatorname{Log}\left[\left(\frac{r+r_0}{1+r_0}\right)\left(1 + \frac{1-r}{|r-r_0|}\right)\right]$$

$$\le 2 \operatorname{Log}\left(1 + \frac{1-r}{|r-r_0|}\right).$$

Combining (4.4) and (4.5) and recalling that $\min(1, t) \leq \mathrm{Log}(1 + t)$, for all $t \geq 0$, it follows that

$$(4.6) \qquad \psi(\alpha) \leq \frac{C}{\alpha^{n-1}} \mathrm{Log}\left(1 + \frac{1-r}{|r - r_0|}\right).$$

Combining (4.3) and (4.6), we obtain

$$
(4.7)
\begin{aligned}
\|B_\lambda\| &\leq C \sup_{0 \leq r_0 \leq 1} \int_0^1 \frac{r^{n-1}|B_\lambda(r)|}{(r + r_0)^{n-1}} \mathrm{Log}\left(1 + \frac{1-r}{|r - r_0|}\right) dr \\
&\leq C \sup_{0 \leq r_0 \leq 1} \int_0^1 |B_\lambda(r)| \mathrm{Log}\left(1 + \frac{1-r}{|r - r_0|}\right) dr.
\end{aligned}
$$

For $r_0 \in [0, 1]$, set

$$
\begin{aligned}
F(r_0) &= \int_0^1 |B_\lambda(r)| \mathrm{Log}\left(1 + \frac{1-r}{|r - r_0|}\right) dr \\
&= \int_0^1 \frac{\mathrm{Log}\left(1 + \frac{1-r}{|r-r_0|}\right)}{(1 - r)\left(1 + \mathrm{Log}\frac{1}{1-r}\right)^\lambda} dr.
\end{aligned}
$$

By making the change of variables $s = \frac{1-r_0}{1-r}$, we have

$$
F(r_0) = \int_{1-r_0}^\infty \frac{\mathrm{Log}\left(1 + \frac{1}{|s-1|}\right)}{s\left(1 + \mathrm{Log}\frac{s}{1-r_0}\right)^\lambda} ds.
$$

For $r_0 \in \left[0, \frac{1}{2}\right]$, we have

$$(4.8) \qquad F(r_0) \leq \int_{1/2}^\infty \frac{\mathrm{Log}\left(1 + \frac{1}{|s-1|}\right)}{s} ds = C < +\infty.$$

For $r_0 \in \left[\frac{1}{2}, 1\right]$, we have

$$
(4.9)
\begin{aligned}
F(r_0) &= \int_{1-r_0}^\infty \frac{\mathrm{Log}\left(1 + \frac{1}{|s-1|}\right)}{s\left(1 + \mathrm{Log}\, s\right)^\lambda} ds \\
&= \int_{1-r_0}^{1/2} \cdots ds + \int_{1/2}^\infty \cdots ds \equiv I_1 + I_2.
\end{aligned}
$$

We have

$$I_1 = \int_{1-r_0}^{1/2} \frac{\text{Log}\left(1 + \frac{1}{|s-1|}\right)}{s\left(1 + \text{Log}\frac{s}{1-r_0}\right)^{\lambda}}\, ds$$

(4.10)
$$\leq \text{Log}\, 3 \int_{1-r_0}^{1/2} \frac{1}{s\left(1 + \text{Log}\frac{s}{1-r_0}\right)^{\lambda}}\, ds$$

$$\leq \text{Log}\, 3 \int_{1}^{\infty} \frac{1}{t\left(1 + \text{Log}\, t\right)^{\lambda}}\, dt = C < +\infty,$$

and

$$I_2 = \int_{1/2}^{\infty} \frac{\text{Log}\left(1 + \frac{1}{|s-1|}\right)}{s\left(1 + \text{Log}\frac{s}{1-r_0}\right)^{\lambda}}\, ds$$

(4.11)
$$\leq \int_{1/2}^{\infty} \frac{\text{Log}\left(1 + \frac{1}{|s-1|}\right)}{s}\, ds = C < +\infty.$$

Combining (4.7)–(4.11), we get

$$\|B_{\lambda}\| \leq C < +\infty. \qquad \qquad \square$$

In what follows we will provide examples of vector fields in the class K_{n+1} which do not satisfy the conditions required by Ancona in [**5**]. To this end we first recall (see Theorem 9.1' in [**5**]) that for our operator the results of Ancona are applied to vectors B in $L^p_{\text{loc}}(D)$, for $p > n$ satisfying:

$$(d(x))^{1-\frac{n}{p}}\|B\|_{L^p\left(B\left(x, \frac{d(x)}{2}\right)\right)} \leq \varphi(d(x)),$$

for all $x \in D$ with φ is a nondecreasing function on $(0, +\infty)$ and satisfies $\int_0^1 \frac{\varphi(s)}{s}\, ds < \infty$.

We have the following:

Proposition 4.4. *Let* $D = B(0,1)$ *be the unit ball in* $\mathbb{R}^n$. *For all* $\lambda > 1/n$, *the radial vector field* B_{λ} *defined on* D *by*

$$B_{\lambda}(r) = \begin{cases} \left(\dfrac{1}{2} - r\right)^{-\frac{1}{n}}\left|\text{Log}\left(\dfrac{1}{2} - r\right)\right|^{-\lambda} & \text{if } 0 \leq r < \dfrac{1}{2} \\ 0 & \text{if } \dfrac{1}{2} \leq r < 1 \end{cases}$$

is in K_{n+1} *and does not belong to any* $L^p_{\text{loc}}(D)$ *with* $p > n$.

Proof: Clearly, for $p > n$,

$$\int_0^{\frac{1}{2}} |B_\lambda(r)|^p \, dr = \int_0^{\frac{1}{2}} \frac{dr}{\left(\frac{1}{2} - r\right)^{\frac{p}{n}} \left|\text{Log}\left(\frac{1}{2} - r\right)\right|^{p\lambda}} = +\infty$$

and so B_λ does not belong to $L^p_{\text{loc}}(D)$.

On the other hand, we have

$$\int_0^1 |B_\lambda(r)|^n \, dr = \int_0^{\frac{1}{2}} \frac{dr}{\left(\frac{1}{2} - r\right) \left|\text{Log}\left(\frac{1}{2} - r\right)\right|^{n\lambda}} < +\infty$$

and so $B_\lambda \in L^n(D)$.

Hence, for a given $\varepsilon > 0$, there is $\alpha = \alpha(\varepsilon) > 0$ such that

$$\left(\int_{\frac{1}{2}-\alpha}^{\frac{1}{2}} |B_\lambda(r)|^n \, dr\right)^{\frac{1}{n}} < \varepsilon.$$

Thus,

(4.12)

$$\int_{D \cap (|x-z| \leq r)} \frac{|B_\lambda(z)|}{|x-z|^{n-1}} \, dz = \int_{D \cap (|x-z| \leq r) \cap \left(\frac{1}{2}-\alpha \leq |z| < \frac{1}{2}\right)} \frac{|B_\lambda(z)|}{|x-z|^{n-1}} \, dz$$

$$+ \int_{D \cap (|x-z| \leq r) \cap \left(|z| \leq \frac{1}{2}-\alpha\right)} \frac{|B_\lambda(z)|}{|x-z|^{n-1}} \, dz$$

$$\equiv I_1(x) + I_2(x).$$

By the Hölder inequality, we have

(4.13) $$I_1(x) \leq \omega_n^{1-\frac{1}{n}} \left(\int_{\frac{1}{2}-\alpha}^{\frac{1}{2}} |B_\lambda(r)|^n \, dr\right)^{\frac{1}{n}} \leq \omega_n^{1-\frac{1}{n}} \varepsilon,$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$.

Moreover

(4.14) $$I_2(x) \leq \frac{\alpha^{-\frac{1}{n}}}{(\text{Log}\, 2)^\lambda} \int_{|x-z|<r} \frac{dz}{|x-z|^{n-1}} = \frac{\alpha^{-\frac{1}{n}}}{(\text{Log}\, 2)^\lambda} \omega_n r.$$

Combining (4.12)–(4.14), we obtain, for r sufficiently small,

$$\sup_{x \in D} \int_{D \cap (|x-z| \leq r)} \frac{|B_\lambda(z)|}{|x-z|^{n-1}} \, dz \leq 2\omega_n^{1-\frac{1}{n}} \varepsilon,$$

and so $B_\lambda \in K_{n+1}$. $\square$

5. The L-Green function G

In this section we prove the existence of the L-Green function G on D and its comparability to G_0, when the drift term B is in the class $\mathcal{K}_{\mathrm{loc}}(D)$. Following the classical potential theory a Borel measurable function $G\colon D \times D \to \,]0, +\infty]$ is called a Green function for the operator L on D if it satisfies for all $y \in D$, $LG(\cdot, y) = -\varepsilon_y$ in the distributional sense, where ε_y is the Dirac measure at y and $G(\cdot, y)$ vanishes on ∂D. We say that G is *continuous* if it is continuous outside the diagonal. Our main result is the following:

Theorem 5.1. *Let $B \in \mathcal{K}_{\mathrm{loc}}(D)$ with $\|B\| < 1/4C_0$. Then, there exists a unique continuous L-Green function G on D comparable to G_0, i.e. there is a constant $C = C(n, \mu, D, \|B\|) > 0$ such that*

$$\frac{1}{C}G_0 \leq G \leq CG_0.$$

Proof: Let B as in the theorem. From Theorem 3.1, we have, for $x, y \in D$,

(5.1)
$$\int_D G_0(x, z)|B(z)||\nabla_z G_0|(z, y)\, dz$$
$$\leq C_0 \int_D (\Gamma(x, z) + \Gamma(y, z))|B(z)|\, dz\, G_0(x, y)$$
$$\leq 2C_0\|B\|G_0(x, y).$$

Since $G_0 < \infty$ outside the diagonal then for all $x \neq y \in D$, let

$$G_0 * (B\nabla G_0)(x, y) = \int_D G_0(x, z)B(z)\nabla_z G_0(z, y)\, dz.$$

In view of (5.1), we may define, for any $m \in \mathbb{N}$, $G_0 * (B\nabla G_0)^{*m}$ outside the diagonal by $G_0 * (B\nabla G_0)^{*m} = (G_0 * (B\nabla G_0)^{*m-1}) * (B\nabla G_0)$ with $G_0 * (B\nabla G_0)^{*0} = G_0$.

By iteration we obtain, for all $m \in \mathbb{N}$,

(5.2)
$$G_0 * (|B\nabla G_0|)^{*m}(x, y) \leq (2C_0\|B\|)^m G_0(x, y).$$

Since $2C_0\|B\| < 1$, then the series

$$\sum_{m=0}^{+\infty} G_0 * (|B\nabla G_0|)^{*m}(x, y)$$

is well defined and converges for $x \neq y \in D$.

Let us define G by

$$G(x,y) = \begin{cases} \sum_{m=0}^{+\infty} G_0 * (B\nabla G_0)^{*m}(x,y) & \text{if } x \neq y \\ +\infty & \text{if } x = y. \end{cases}$$

We will prove that G is continuous. To this end we will show that, for any $m \geq 1$, $p_m(x,y) \equiv G_0 * (|B\nabla G_0|)^{*m}(x,y)$ is continuous outside the diagonal.

Let $(x_0, y_0) \in D \times D$ with $x_0 \neq y_0$. Put $\rho = |x_0 - y_0| > 0$, $E_1 = \overline{B}(x_0, \rho/4)$ and $E_2 = \overline{B}(y_0, \rho/4)$. Since $B \in \mathcal{K}_{\text{loc}}(D)$, then for any $\varepsilon > 0$, there is $r \in (0, \rho/4)$ such that, for $i = 1, 2$,

$$\sup_{x \in E_i} \int_{D \cap (|x-z|<r)} \frac{|B(z)|}{|x-z|^{n-1}} \, dz < \varepsilon.$$

For $(x,y) \in B(x_0, r/2) \times B(y_0, r/2)$, we have

$$p_1(x,y) \equiv \int_D G_0(x,z)|B(z)||\nabla_z G_0|(z,y) \, dz$$

(5.3)
$$= \int_{D \cap (|x-z|<r)} \cdots \, dz + \int_{D \cap (|y-z|<r)} \cdots \, dz$$

$$+ \int_{D \cap (|x-z|\geq r) \cap (|y-z|\geq r)} \cdots \, dz$$

$$\equiv p_{11}(x,y) + p_{12}(x,y) + p_{13}(x,y).$$

By Theorem 2.1, we have

$$p_{11}(x,y) \equiv \int_{D \cap (|x-z|<r)} G_0(x,z)|B(z)||\nabla_z G_0|(z,y) \, dz$$

(5.4)
$$\leq \frac{C}{\rho^{n-1}} \int_{D \cap (|x-z|<r)} G_0(x,z)|B(z)| \, dz$$

$$\leq \frac{C}{\rho^{n-1}} \int_{D \cap (|x-z|<r)} \frac{|B(z)|}{|x-z|^{n-1}} \, dz \leq \frac{C}{\rho^{n-1}} \varepsilon.$$

In the same way, we have

$$p_{12}(x,y) \equiv \int_{D \cap (|y-z|<r)} G_0(x,z)|B(z)||\nabla_z G_0|(z,y)\, dz$$

(5.5)
$$\leq \frac{C}{\rho^{n-2}} \int_{D \cap (|y-z|<r)} |B(z)||\nabla_z G_0|(z,y)\, dz$$

$$\leq \frac{C}{\rho^{n-2}} \int_{D \cap (|y-z|<r)} \frac{|B(z)|}{|x-z|^{n-1}}\, dz \leq \frac{C}{\rho^{n-2}} \varepsilon.$$

On the other hand, for $|x-z| \geq r$ and $|y-z| \geq r$, we have

$$G_0(x,z)|B(z)||\nabla_z G_0|(z,y) \leq \frac{C}{r^{n-1}}|B(z)|\Gamma(x,z),$$

and

$$|x_0 - z| \leq |x-z| + |x-x_0| \leq \frac{3}{2}|x-z|$$

which imply

$$G_0(x,z)|B(z)||\nabla_z G_0|(z,y) \leq \frac{C}{r^{n-1}}|B(z)|\Gamma(x_0,z).$$

Since

$$\int_D |B(z)|\Gamma(x_0,z)\, dz \leq \|B\| < \infty,$$

then by the dominated convergence theorem we deduce that

(5.6)
$$\lim_{(x,y) \to (x_0,y_0)} p_{13}(x,y) = p_{13}(x_0,y_0).$$

Combining (5.3)–(5.6), we then deduce that p_1 is continuous in (x_0, y_0).

Clearly, by using the equality $p_m(x,y) = p_{m-1} * |B\nabla G_0|(x,y)$ and the same arguments as above, we prove the continuity of p_m outside the diagonal. Therefore by (5.2), it follows that G is continuous.

Moreover, since $G_0(\cdot, y)$ vanishes on ∂D, then by (5.2), $G(\cdot, y)$ satisfies the same property, and, for all $y \in D$, we have

$$G(\cdot, y) = G_0(\cdot, y) + \int_D G(\cdot, z)B(z)\nabla_z G_0(z,y)\, dz \quad \text{on} \quad D \setminus \{y\}$$

which implies

$$LG(\cdot, y) = -\varepsilon_y$$

in the distributional sense. Thus G is a continuous Green function for L on D. From (5.2) we also have, for all $x \neq y \in D$,

$$|G(x,y) - G_0(x,y)| \leq \sum_{m=1}^{+\infty} G_0 * (|B\nabla G_0|)^{*m}(x,y)$$

$$\leq \sum_{m=1}^{+\infty} (2C_0\|B\|)^m G_0(x,y)$$

$$= \frac{2C_0\|B\|}{1 - 2C_0\|B\|} G_0(x,y).$$

By recalling that $G = G_0 = +\infty$ on the diagonal and $4C_0\|B\| < 1$, we obtain

$$\left(\frac{1 - 4C_0\|B\|}{1 - 2C_0\|B\|} \right) G_0 \leq G \leq \left(\frac{1}{1 - 2C_0\|B\|} \right) G_0.$$

Conversely, assume that there exists a continuous Green function $\tilde{G}$ for L on D which is comparable to G_0. By definition, for all $y \in D$, $L\tilde{G}(\cdot, y) = -\varepsilon_y$ and $\tilde{G}(\cdot, y) = 0$ on ∂D. This implies

$$\tilde{G}(\cdot, y) = G_0(\cdot, y) + \int_D \tilde{G}(\cdot, z)B(z)\nabla_z G_0(z, y)\, dz \quad \text{a.e. on } D.$$

Since $\tilde{G}$ and G_0 are continuous, it follows that

$$\tilde{G}(\cdot, y) = G_0(\cdot, y) + \int_D \tilde{G}(\cdot, z)B(z)\nabla_z G_0(z, y)\, dz \quad \text{on} \quad D \setminus \{y\}.$$

By iteration we obtain, for all $p \in \mathbb{N}$

$$\tilde{G}(\cdot, y) = \sum_{m=0}^{p} G_0 * (B\nabla G_0)^{*m}(\cdot, y) + \tilde{G} * (B\nabla G_0)^{*(p+1)}(\cdot, y) \quad \text{on} \quad D\setminus\{y\}.$$

Since $\tilde{G}$ is comparable to G_0, then by (5.2), we have

$$\tilde{G} * (|B\nabla G_0|)^{*(p+1)}(\cdot, y) \leq (2C_0\|B\|)^{p+1} G_0(\cdot, y).$$

By recalling that $2C_0\|B\| < 1$ and letting $p \to \infty$, we obtain

$$\tilde{G}(\cdot, y) = \sum_{m=0}^{+\infty} G_0 * (B\nabla G_0)^{*m}(\cdot, y) \quad \text{on} \quad D \setminus \{y\}.$$

By the comparability of $\tilde{G}$ to G_0, we also have $\tilde{G} = +\infty$ on the diagonal. Thus $\tilde{G} = G$ which completes the proof. $\square$

Remark 5.2. By the Hölder inequality, it is easy to see that the class $\mathcal{K}_{\mathrm{loc}}(D)$ contains all the spaces $L^p(D)$ with $p > n$ and so in contrast to Theorem 9.1' in Ancona's paper [**5**], our result (Theorem 5.1) provide a constant C which does not depend on the parameter p.

6. Estimates of the *L*-harmonic measure

In [**10**], Dahlberg proved that the Laplace harmonic measure and the surface measure are equivalent on the boundary of a Lipschitz domain. This result was later extended by Ancona [**4**] to a class of uniformly elliptic operators with bounded Hölder continuous coefficients. Subsequently, Cranston and Zhao [**9**] studied the problem for the operator $\frac{1}{2}\Delta + B(x) \cdot \nabla_x$ with B and $|B|^2$ are in the Kato classes K_{n+1} and K_n, respectively. In [**5**], Ancona also deduced the equivalence of the harmonic measure and the surface measure (see Theorems 9.3 and 9.4). In this section, we will investigate the same problem for the operator $L = \mathrm{div}(A(x)\nabla_x) + B(x) \cdot \nabla_x$ with the drift term B in $\mathcal{K}_{\mathrm{loc}}(D)$. We have shown in Section 5 that when $B \in \mathcal{K}_{\mathrm{loc}}(D)$ with $\|B\| \leq 1/4C_0$, the operator L admits a continuous Green function and so it has positive continuous solutions. By the standard theory [**3**], [**8**], we have the existence and uniqueness of the L-Martin Poisson kernel and the unique solvability of the Dirichlet problem with continuous boundary data. In what follows, we estimate the L-Poisson kernel and the L-harmonic measure.

Theorem 6.1. *Let* $B \in \mathcal{K}_{\mathrm{loc}}(D)$ *with* $\|B\| \leq 1/4C_0$. *Then, the L-Poisson kernel P on D satisfies the following estimates: there exists a constant* $C = C(n, \mu, D, \|B\|) > 0$, *such that*

$$\frac{1}{C}\frac{d(x)}{|x - Q|^n} \leq P(x,Q) \leq C\frac{d(x)}{|x - Q|^n},$$

for all $x \in D$ *and* $Q \in \partial D$.

Proof: Since D is of $C^{1,1}$-boundary, then by the divergence theorem, we have

$$P(x,Q) = \frac{\partial G(x,Q)}{\partial N_Q},$$

where $N_Q = A(Q)n_Q$ with n_Q is the unit inner normal to ∂D at Q.

We write $N_Q = T_Q + a(Q)n_Q$, where T_Q is a tangential vector field to ∂D at Q.

Therefore

$$\frac{\partial G(x,Q)}{\partial N_Q} = \frac{\partial G(x,Q)}{\partial T_Q} + a(Q)\frac{\partial G(x,Q)}{\partial n_Q}.$$

Since $G(x, \cdot) \equiv 0$ on ∂D, then

$$\frac{\partial G(x, Q)}{\partial T_Q} \equiv \frac{\partial}{\partial T_Q}(G(x, \cdot))(Q) = 0$$

and so

$$\frac{\partial G(x, Q)}{\partial N_Q} = a(Q)\frac{\partial G(x, Q)}{\partial n_Q}$$

(6.1)

$$\equiv a(Q) \lim_{r \to 0^+} \frac{G(x, Q + rn_Q)}{r}.$$

On the other hand, by recalling that, for $r > 0$ small, $d(Q + rn_Q) = r$, it follows from the comparability of G to G_0 and the estimates in Theorem 2.1, that

(6.2) $\qquad \dfrac{1}{C}\dfrac{d(x)}{|x - Q - rn_Q|^n} \leq \dfrac{G(x, Q + rn_Q)}{r} \leq C\dfrac{d(x)}{|x - Q - rn_Q|^n}.$

By noting that $1/\mu \leq \langle A(Q)n_Q, n_Q \rangle \leq \mu$, combining (6.1), (6.2) and letting r to 0 we obtain the estimates stated in the theorem. $\qquad\square$

For $x \in D$, let m_x denotes the L-harmonic measure at x on ∂D and σ the surface measure on ∂D. We have the following:

Corollary 6.2. *Let $B \in \mathcal{K}_{\mathrm{loc}}(D)$ with $\|B\| \leq 1/4C_0$. Then, the measures m_x and σ, are equivalent on ∂D with densities in $L^\infty(\partial D)$.*

Proof: For any Borel subset $E \subset \partial D$, we have

$$m_x(E) = \int_E P(x, Q)\, d\sigma(Q),$$

and so by Theorem 6.1, it follows that

$$\frac{1}{C}\frac{d(x)}{d(D)^n}\sigma(E) \leq m_x(E) \leq \frac{C}{d(x)^{n-1}}\sigma(E). \qquad\qquad\square$$

Remarks.

1. Since the matrix $A = A(x) \in C^{0,1}(D)$, we can also write L in the non-divergence form

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + (B + B')(x) \cdot \nabla_x,$$

where $B' = (B'_1, \ldots, B'_n)$ with $B'_j = \sum_{i=1}^{n} \frac{\partial a_{ij}}{\partial x_i}(x) \in L^\infty(D)$.

From this observation we see that our results are valid for the operators in the non-divergence form as well.

2. Although the adjoint operator $L^* = \mathrm{div}(A(x)\nabla_x) - \mathrm{div}(B(x)\cdot)$ has a different structure from L, by the equality $G^*(x,y) = G(y,x)$, where G^* is the L^*-Green function on D, we deduce that our results are also true for L^*.

3. By Theorem 3.3 in [**11**] and the arguments used to prove the lower bound (Theorem 1 in [**27**]), the estimates stated in Theorem 2.1 are also valid on domains satisfying the uniformly exterior ball condition. So, our results are also true for this class of domains which contains the class of $C^{1,1}$-domains (see [**7**, p. 179]).

4. As is shown in [**8**] and [**23**], the comparability of the Green functions yields some potential theoretic results for L which are known to hold for the Laplace operator Δ. For instance we deduce interior and boundary Harnack principles for nonnegative L-solutions, Hölder continuity of bounded solutions, integral representations of nonnegative L-solutions, etc, ...

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