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OPTIMALITY OF EMBEDDINGS OF BESSEL-POTENTIAL-TYPE SPACES INTO GENERALIZED HÖLDER SPACES

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Abstract

We establish the sharpness of embedding theorems for Bessel-potential spaces modelled upon Lorentz-Karamata spaces and we prove the non-compactness of such embeddings. Target spaces in our embeddings are generalized Hölder spaces. As consequences of our results, we get continuous envelopes of Bessel-potential spaces modelled upon Lorentz-Karamata spaces.

1. Introduction

In a series of recent papers [**7**]–[**10**] a systematic research of embeddings of Bessel potential spaces modelled upon generalized Lorentz-Zygmund (GLZ) spaces was carried out. For a survey of these results we refer to [**20**]. The authors of those papers established embeddings of such spaces either into GLZ spaces or into Hölder-type spaces $C^{0,\lambda(\cdot)}(\overline{\Omega})$ and showed that their results are sharp (within the given scale of target spaces) and fail to be compact. They also clarified the role of the logarithmic terms involved in the quasi-norms of the spaces mentioned. This role proved to be important especially in limiting cases. In particular, they obtained refinements of the Sobolev embedding theorems, Trudinger's limiting embedding as well as embeddings of Sobolev spaces into $\lambda(\cdot)$-Hölder continuous functions including the result of Brézis and

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Wainger about almost Lipschitz continuity of elements of the (fractional) Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$ (cf. [**4**]).

Although GLZ-spaces form an important scale of spaces containing, for example, Zygmund classes $L^p(\log L)^\alpha$, Orlicz spaces of multiple exponential type, Lorentz spaces $L_{p,q}$, Lebesgue spaces L_p, etc., GLZ-spaces are a particular case of more general spaces, namely the Lorentz-Karamata (LK) spaces.

The embeddings mentioned above were extended in [**17**]–[**18**] to the case when Bessel-potential spaces are modelled upon LK-spaces. Since Neves considered more general targets (besides LK-spaces and Hölder-type spaces also generalized Hölder spaces), in several cases he obtained improvements of embeddings from [**7**]–[**10**]. On the other hand, there is a problem to prove the sharpness and the non-compactness of these embeddings. This problem was solved in [**12**] for embeddings with Lorentz-Karamata spaces as target spaces. The main aim of this paper is to establish the sharpness and the non-compactness when the target spaces are generalized Hölder spaces. Moreover, we also extend the results of [**18**] since our definition of LK-spaces (see Section 2) is more general than that given in [**18**]. As in [**13**], we do not assume any symmetry of slowly varying functions involved in the quasi-norms of LK-spaces. We also improve (cf. Remark 3.1 below) embeddings of Bessel spaces modelled upon LK-spaces into spaces of $\lambda(\cdot)$-Hölder continuous functions in the sublimiting case proved in [**18**] since here we consider embeddings into the scale of spaces which can be more finely tuned, namely into the scale of generalized Hölder spaces $\Lambda^{\lambda(\cdot)}_{\infty,r}$. As a consequence of our embedding results, we get continuity envelopes of Bessel-potential spaces modelled upon LK-spaces. For basic facts about these notions we refer to [**14**] and [**22**].

Our method of proving the sharpness and the non-compactness of the given embeddings is based on those of [**8**] and [**10**]. In contrast to [**22**], we do not use atomic decompositions.

The paper is organised as follows. Section 2 contains notation and basic definitions, while the main results are stated in Section 3. After some preliminary in the next section, the final Section 5 gives the proofs of the promised theorems.

2. Notation and basic definitions

As usual, $\mathbb{R}^n$ denotes Euclidean n-dimensional space. Let μ_n be the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and let Ω be a μ_n-measurable subset of $\mathbb{R}^n$. We denote by χ_Ω the characteristic function of Ω and write

$|\Omega|_n = \mu_n(\Omega)$. The family of all extended scalar-valued (real or complex) μ_n-measurable functions on Ω will be denoted by $\mathcal{M}(\Omega)$, and $\mathcal{M}^+(\Omega)$ will stand for the subset of $\mathcal{M}(\Omega)$ consisting of all those functions which are non-negative a.e. By $\mathcal{W}(\Omega)$ (or by $\mathcal{W}(a,b)$) we mean the class of weighted functions on Ω (or on (a,b)) consisting of all measurable functions which are positive a.e. on Ω (or on (a,b)). Let $f \in \mathcal{M}(\Omega)$. The *non-increasing rearrangement* of f is the function f^* defined on $[0,+\infty)$ by $f^*(t) = \inf\{\lambda \geq 0 : |\{x \in \Omega : |f(x)| > \lambda\}|_n \leq t\}$ for all $t \geq 0$. We shall also use the maximal function f^{**} of f^* defined by $f^{**}(t) = t^{-1}\int_0^t f^*(\tau)\,d\tau$, $t > 0$. Clearly, $f^*(t) \leq f^{**}(t)$, $t > 0$, and we also have the inequality

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t) \quad \text{for all} \quad t > 0, \tag{2.1}$$

cf. [**2**, p. 55]. For general facts about (rearrangement-invariant) Banach function spaces we refer to [**2**, Chapter 1, Chapter 2].

Now let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. We denote by $\boldsymbol{\ell}^{\boldsymbol{\alpha}}$ the function defined by $\boldsymbol{\ell}^{\boldsymbol{\alpha}}(t) = \prod_{i=1}^m \ell_i^{\alpha_i}(t)$ for all $t \in (0,+\infty)$, where $\ell_1, \ldots, \ell_m$ are positive functions defined on $(0,+\infty)$ by $\ell_1(t) = 1 + |\log t|$, and, if $m \geq 2$, $\ell_i(t) = 1 + \log \ell_{i-1}(t)$, $i \in \{2, \ldots, m\}$.

For two non-negative expressions (*i.e.* functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \precsim \mathcal{B}$ means that $\mathcal{A} \leq c\mathcal{B}$, for some positive constant c independent of the variables in the expressions $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{B} \precsim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. We adopt the convention that $a/(+\infty) = 0$ and $a/0 = +\infty$ for all $a > 0$. If $p \in [1,+\infty]$, the conjugate number p' is given by $1/p + 1/p' = 1$.

Following [**13**], we say that a positive and Lebesgue-measurable function b is *slowly varying* on $(0,+\infty)$, and write $b \in SV(0,+\infty)$, if, for each $\epsilon > 0$, $t^\epsilon b(t)$ is equivalent to a non-decreasing function on $(0,+\infty)$ and $t^{-\epsilon} b(t)$ is equivalent to a non-increasing function on $(0,+\infty)$.

Properties and examples of slowly varying functions can be found in [**23**, Chapter V, p. 186], [**3**], [**11**], [**15**], [**17**] and [**13**]. The following functions are slowly varying on $(0,+\infty)$:

(i) $b(t) = \boldsymbol{\ell}^{\boldsymbol{\alpha}}(t)$, $\boldsymbol{\alpha} \in \mathbb{R}^m$;

(ii) $b(t) = \boldsymbol{\ell}^{\boldsymbol{\alpha}}(t)\chi_{(0,1)}(t) + \boldsymbol{\ell}^{\boldsymbol{\beta}}(t)\chi_{[1,+\infty)}(t)$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$;

(iii) $b(t) = \exp(|\log t|^\alpha)$, $0 < \alpha < 1$;

(iv) $b_m(t) = \exp(\ell_m^\alpha(t))$, $0 < \alpha < 1$ and $m \in \mathbb{N}$.

Note that if $m \geq 2$, we may consider $\alpha = 1$ in the last example. In such a case $b_m \approx \ell_{m-1}$.

It can be shown (cf. [**13**]) that any $b \in SV(0,+\infty)$ is equivalent to a $\widetilde{b} \in SV(0,+\infty)$ which is continuous in $(0,+\infty)$. Consequently, without loss of generality, we shall assume that all slowly varying functions in question are continuous functions in $(0,+\infty)$.

Let $p, q \in (0,+\infty]$ and $b \in SV(0,+\infty)$. The *Lorentz-Karamata* (LK) space $L_{p,q;b}(\Omega)$ is defined to be the set of all functions $f \in \mathcal{M}(\Omega)$ such that

$$\|f\|_{p,q;b;\Omega} := \|t^{1/p-1/q}\, b(t)\, f^*(t)\|_{q;(0,+\infty)} \tag{2.2}$$

is finite. Here $\|.\|_{q;(a,b)}$ stands for the usual L_q (quasi-)norm over an interval $(a,b) \subseteq \mathbb{R}$.

When $0 < p < +\infty$, the Lorentz-Karamata space $L_{p,q;b}(\Omega)$ contains the characteristic function of every measurable subset of Ω with finite measure and hence, by linearity, every μ_n-simple function f satisfying $|\operatorname{supp} f|_n < +\infty$. When $p = +\infty$, the Lorentz-Karamata space $L_{p,q;b}(\Omega)$ is different from the trivial space if, and only if, $\|t^{1/p-1/q} b(t)\|_{q;(0,1)} < +\infty$.

If $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $b = \boldsymbol{\ell}^{\boldsymbol{\alpha}}$, then $L_{p,q;b}(\Omega)$ is precisely the generalized Lorentz-Zygmund (GLZ) space $L_{p,q;\boldsymbol{\alpha}}(\Omega)$ introduced in [**9**] and endowed with the (quasi-)norm $\|f\|_{p,q;\boldsymbol{\alpha};\Omega}$. When $\boldsymbol{\alpha} = (0, \ldots, 0)$, we obtain the Lorentz space $L_{p,q}(\Omega)$ endowed with the (quasi-)norm $\|.\|_{p,q;\Omega}$, which is just the Lebesgue space $L_p(\Omega)$ equipped with the (quasi-)norm $\|.\|_{p;\Omega}$ when $p = q$; if $p = q$ and $m = 1$, we obtain the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-)norm $\|\cdot\|_{p;\alpha_1;\Omega}$.

The *Bessel kernel* g_σ, $\sigma > 0$, is defined as that function on $\mathbb{R}^n$ whose Fourier transform is $\widehat{g_\sigma}(\xi) = (2\pi)^{-n/2}(1+|\xi|^2)^{-\sigma/2}$, $\xi \in \mathbb{R}^n$, where the Fourier transform $\hat{f}$ of a function f is given by $\hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$. It is known that g_σ is a positive, integrable function which is analytic except at the origin.

Let $\sigma > 0$, $p \in (1,+\infty)$, $q \in [1,+\infty]$, and $b \in SV(0,+\infty)$. The *Lorentz-Karamata-Bessel* potential space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ is defined to be

$$\{u : u = g_\sigma * f,\ f \in L_{p,q;b}(\mathbb{R}^n)\}$$

and is equipped with the (quasi-)norm $\|u\|_{\sigma;p,q;b} := \|f\|_{p,q;b}$.

For $\sigma = 0$, we put

$$g_0 * f = f \quad \text{and} \quad H^\sigma L_{p,q;b}(\mathbb{R}^n) = L_{p,q;b}(\mathbb{R}^n). \tag{2.3}$$

When $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $b = \boldsymbol{\ell}^{\boldsymbol{\alpha}}$, we obtain the logarithmic Bessel potential space $H^\sigma L_{p,q;\boldsymbol{\alpha}}(\mathbb{R}^n)$, endowed with the (quasi-)norm $\|u\|_{\sigma;p,q;\boldsymbol{\alpha}}$ and considered in [**9**]. Note that if $\boldsymbol{\alpha} = (0, \ldots, 0)$, $H^\sigma L_{p,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$ is simply the (fractional) Sobolev space $H^\sigma_p(\mathbb{R}^n)$ of the order σ.

When $k \in \mathbb{N}$, $p, q \in (1, +\infty)$ and $b \in SV(0, +\infty)$, then

$$H^k L_{p,q;b}(\mathbb{R}^n) = \{u : D^\alpha u \in L_{p,q;b}(\mathbb{R}^n), \text{ if } |\alpha| \le k\},$$

and

$$\|u\|_{k;p,q;b} \approx \sum_{|\alpha| \le k} \|D^\alpha u\|_{p,q;b}, \quad u \in H^k L_{p,q;b}(\mathbb{R}^n),$$

according to Lemma 4.5 below and [**18**, Theorem 5.3].

Let Ω be a domain in $\mathbb{R}^n$. The space of all scalar-valued (real or complex), bounded and continuous functions on Ω is denoted by $C_B(\Omega)$ and it is equipped with the $L_\infty(\Omega)$ norm. For each $h \in \mathbb{R}^n$, let $\Omega_h = \{x \in \Omega : x + h \in \Omega\}$ and let Δ_h be the difference operator defined on scalar functions f on Ω by $(\Delta_h f)(x) = f(x+h) - f(x)$ for all $x \in \Omega_h$. The *modulus of smoothness* of a function f in $C_B(\Omega)$ is defined by

$$\omega(f, t) := \sup_{|h| \le t} \|\Delta_h f | L_\infty(\Omega_h)\| \quad \text{for all} \quad t \ge 0.$$

If

$$\widetilde{\omega}(f, t) := \omega(f, t)/t \quad \text{for each} \quad t > 0,$$

then $\widetilde{\omega}(f, .)$ is equivalent to a non-increasing function on $(0, +\infty)$. We refer to [**2**, pp. 331–333] and to [**5**, pp. 40–50] for more details.

Let $q \in (0, +\infty]$ and let $\mathcal{L}_q$ be the class of all continuous functions λ: $(0, 1] \to (0, +\infty)$ which are increasing on some interval $(0, \delta)$, with $\delta = \delta_\lambda \in (0, 1]$, and satisfy

$$\lim_{t \to 0_+} \lambda(t) = 0$$

and

$$\left\| t^{-1/q} \frac{t}{\lambda(t)} \right\|_{q;(0,\delta)} < +\infty. \tag{2.4}$$

When $q = +\infty$, we simply write $\mathcal{L}$ instead of $\mathcal{L}_q$.

Let $q \in (0, +\infty]$, $\lambda \in \mathcal{L}_q$ and let Ω be a domain in $\mathbb{R}^n$. The *generalized Hölder space* $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\Omega})$ consists of all those functions $f \in C_B(\Omega)$ for which the norm

$$\|f | \Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\Omega})\| := \|f | L_\infty(\Omega)\| + \left\| t^{-1/q} \frac{\omega(f, t)}{\lambda(t)} \right\|_{q;(0,1)}$$

is finite. The space $\Lambda^{\lambda(\cdot)}_{\infty,\infty}(\overline{\Omega})$ coincides (cf. [**16**, Proposition 3.5]) with the space $C^{0,\lambda(\cdot)}(\overline{\Omega})$ defined by

$$\|f | C^{0,\lambda(\cdot)}(\overline{\Omega})\| := \sup_{x \in \Omega} |f(x)| + \sup_{\substack{x, y \in \Omega \\ 0 < |x - y| \le 1}} \frac{|f(x) - f(y)|}{\lambda(|x - y|)} < +\infty.$$

If $\lambda(t) = t$, $t \in (0,1]$, and $\Omega = \mathbb{R}^n$, then $\Lambda^{\lambda(\cdot)}_{\infty,\infty}(\overline{\Omega})$ coincides with the space $\mathrm{Lip}(\mathbb{R}^n)$ of the Lipschitz functions. Note also that if (2.4) does not hold, then $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\Omega})$ consists only of constant functions on Ω.

Remark 2.1. If $\Omega = \mathbb{R}^n$, then the space $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\Omega})$ is a particular case of the Besov-Hölder-Lipschitz space $\Lambda^{\lambda}_{p,q}(\mathbb{R}^n)$ with $p = \infty$ from [**16**]. If, moreover, $\lambda \in (0,1]$, b is a slowly varying function on $[1,+\infty)$ (for definition see [**18**]) and the function $\lambda(t) := t^{\lambda} b(1/t)$, $t \in (0,1]$, then the space $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\Omega})$ coincides with the Besov-Lipschitz-Karamata space $\Lambda^{\lambda,b}_{p,q}(\mathbb{R}^n)$ with $p = \infty$ from [**18**].

On the other hand, if Ω is a domain in $\mathbb{R}^n$, $\lambda \in (0,1]$ and $\lambda(t) := t^{\lambda}$, $t \in (0,1]$, then the space $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\Omega})$ coincides with the generalized space of Hölder continuous functions $C^{0,\lambda,q}(\overline{\Omega})$ introduced in [**1**, p. 232].

For $\rho \in (0,+\infty)$ and $x \in \mathbb{R}^n$, $B_n(x,\rho)$ stands for the open ball in $\mathbb{R}^n$ of radius ρ and centre x, whilst $\overline{B_n}(x,\rho)$ means its closure in $\mathbb{R}^n$. By ω_n we denote the volume of the unit ball in $\mathbb{R}^n$.

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

3. Statement of the results

In this section we present embeddings of Bessel-potential-type spaces into generalized Hölder spaces, which extend and improve those of [**10**] and [**18**]. Our main results state that such embeddings are sharp and fail to be compact.

Part (i) of the following theorem improves and extends [**10**, Theorem 3.2] and [**18**, Theorem 5.10] and discusses embeddings of Bessel potential spaces modelled upon Lorentz-Karamata spaces into generalized Hölder spaces in the sublimiting case. Parts (ii)–(iii) of this theorem imply that the embedding of part (i) is sharp while part (iv) shows that such an embedding fails to be compact.

Theorem 3.1. *Let $\sigma \in [1, n+1)$, $\max\{1, n/\sigma\} < p < n/(\sigma-1)$, $q \in (1,+\infty)$, $r \in [q,+\infty]$ and let $b \in SV(0,+\infty)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty domain. Let $\lambda\colon (0,1] \to (0,+\infty)$ be defined by*

$$\lambda(t) = t^{\sigma - n/p}[b(t^n)]^{-1}, \quad t \in (0,1]. \tag{3.1}$$

(Note that $\lambda \in \mathcal{L}_r$ for any $r \in [1,+\infty]$.)

(i) Then

$$H^{\sigma} L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{3.2}$$

(ii) If a function $\mu \in \mathcal{L}_r$ *satisfies*

$$\lim_{t\to 0_+} \frac{\frac{t}{\lambda(t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} = 0, \tag{3.3}$$

then the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{3.4}$$

does not hold.

(iii) Let $\overline{q} \in (0, q)$*. Then the embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,\overline{q}}(\overline{\Omega}). \tag{3.5}$$

fails.

(iv) The embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$$

is not compact.

Remark 3.1. (i) If $r = +\infty$, then (3.2) yields

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow C^{\lambda(\cdot)}(\overline{\mathbb{R}^n}),$$

cf. [**18**, Theorem 5.10].

(ii) As

$$\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,s}(\overline{\mathbb{R}^n}) \quad \text{if} \quad 0 < r < s \le +\infty, \tag{3.6}$$

among embeddings (3.2) the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \tag{3.7}$$

is optimal. (Note that embedding (3.6) can be proved analogously as [**12**, (3.6)] if one replaces the role of $f^*(t)$ by the role of $\widetilde{\omega}(f,t)$.)

(iii) By part (i) of Theorem 3.1, embedding (3.7) is continuous and, by part (iv) of Theorem 3.1, this embedding is not compact. Moreover, part (iv) of Theorem 3.1 also shows that we cannot arrive to a compact embedding if we replace the target space $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n})$ in (3.7) by a larger space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$ with $r > q$.

(iv) Put $X = H^\sigma L_{p,q;b}(\mathbb{R}^n)$ and $r = +\infty$. By Theorem 3.1 (i),

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\lambda(t)/t} \precsim \|f\|_X \quad \text{for all} \quad f \in X,$$

and, by Theorem 3.1 (i) and (ii) (cf. also part (v) of this remark), the inequality

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\mu(t)/t} \precsim \|f\|_X$$

does not hold for all $f \in X$ if $\mu \in \mathcal{L}$ satisfies

$$\lim_{t\to 0_+} \frac{\frac{t}{\lambda(t)}}{\frac{t}{\mu(t)}} = \lim_{t\to 0_+} \frac{\mu(t)}{\lambda(t)} = 0.$$

If we use an analogue of terminology from [**22**], this means that the function $\frac{\lambda(t)}{t} = t^{\sigma-n/p-1}[b(t^n)]^{-1}$, $t \in (0,1]$, is the continuous envelope function of the space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$. Using also part (iii) of Theorem 3.1, we can see that the couple

$$(t^{\sigma-n/p-1}[b(t^n)]^{-1}, q)$$

is the continuous envelope of the space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$.

(v) Let $r \in [q,+\infty]$. Using (3.1), we obtain

$$\left\|\tau^{-1/r}\frac{\tau}{\lambda(\tau)}\right\|_{r;(0,t)} \approx \frac{t}{\lambda(t)} \quad \text{for all} \quad t \in (0,1).$$

This implies that

$$\lim_{t\to 0_+} \frac{\frac{t}{\lambda(t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} \approx \lim_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\lambda(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}. \tag{3.8}$$

On the other hand, the estimate

$$\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)} \geq \frac{1}{\mu(t)}\|\tau^{1-1/r}\|_{r;(0,t)} \approx \frac{t}{\mu(t)} \tag{3.9}$$

(which holds for all t from an interval $(0,\delta)$ since $\mu \in \mathcal{L}_r$ and so μ is increasing in some interval $(0,\delta) \subset (0,1)$), shows that condition (3.3) is satisfied if

$$\lim_{t\to 0_+} \frac{\mu(t)}{\lambda(t)} = 0. \tag{3.10}$$

(vi) Let $r = +\infty$ and let the function $t \mapsto t/\mu(t)$ be equivalent to a non-decreasing function on some interval $(0,\delta) \subset (0,1)$. Then, for all $t \in (0,\delta)$,

$$\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)} = \left\|\frac{\tau}{\mu(\tau)}\right\|_{\infty;(0,t)} \approx \frac{t}{\mu(t)}. \tag{3.11}$$

Applying this estimate in (3.8), we can see that (3.3) is now equivalent to (3.10).

The next result is an analogue of Theorem 3.1 and concerns the limiting case when $p = n/(\sigma - 1)$. Part (i) of this theorem is an extention of [**10**, Theorem 3.3].

Theorem 3.2. *Let* $\sigma \in (1, n+1)$, $p = n/(\sigma - 1)$, $q \in (1, +\infty)$, $r \in [q, +\infty]$ *and let* $b \in SV(0, +\infty)$ *be such that*

$$\|t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} = +\infty. \tag{3.12}$$

Suppose that $\Omega \subset \mathbb{R}^n$ *is a nonempty domain and that* $\lambda_r \in \mathcal{L}_r$ *is defined by*

$$\lambda_r(t) = t[b(t^n)]^{q'/r} \left(\int_{t^n}^{2} \tau^{-1}[b(\tau)]^{-q'}\, d\tau \right)^{1/q'+1/r}, \quad t \in (0,1]. \tag{3.13}$$

(i) Then

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{3.14}$$

(ii) If a function $\mu \in \mathcal{L}_r$ *satisfies*

$$\lim_{t \to 0_+} \frac{\left\| \tau^{-1/r} \frac{\tau}{\lambda_r(\tau)} \right\|_{r;(0,t)}}{\left\| \tau^{-1/r} \frac{\tau}{\mu(\tau)} \right\|_{r;(0,t)}} = 0, \tag{3.15}$$

then the embedding

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(.)}_{\infty,r}(\overline{\Omega}) \tag{3.16}$$

does not hold.

(iii) Let $\overline{q} \in (0, q)$. *Then the embedding*

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{\overline{q}}(.)}_{\infty,\overline{q}}(\overline{\Omega}) \tag{3.17}$$

fails, where $\lambda_{\overline{q}}$ *is again defined by* (3.13) *with* r *replaced by* $\overline{q}$.

(iv) The embedding

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\Omega})$$

is not compact.

Remark 3.2. (i) Part (i) of Theorem 3.2 holds without assumption (3.12). However, if $\|t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} < +\infty$, then

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}(\mathbb{R}^n),$$

cf. [**18**, Theorem 5.12].

(ii) The target spaces in (3.14) form a scale $\{\Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\mathbb{R}^n})\}_{r=q}^{+\infty}$ whose endpoint spaces with $r = +\infty$ and $r = q$ are of particular interest. The former endpoint space $\Lambda^{\lambda_\infty(.)}_{\infty,\infty}(\overline{\mathbb{R}^n})$ corresponds to the target space in the Brézis-Wainger-type embedding while the latter endpoint space $\Lambda^{\lambda_q(.)}_{\infty,q}(\overline{\mathbb{R}^n})$ corresponds to the target space in the Triebel-type embedding. Since the spaces $\{\Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\mathbb{R}^n})\}_{r=q}^{+\infty}$ satisfy

$$\Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_s(.)}_{\infty,s}(\overline{\mathbb{R}^n}) \quad \text{if} \quad q \le r \le s \le +\infty, \tag{3.18}$$

the embedding (3.14) with $r = q$, that is,

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_q(.)}_{\infty,q}(\overline{\mathbb{R}^n}) \tag{3.19}$$

is optimal. (The proof of (3.18) is analogous to the proof of [**12**, (3.14)] if one replaces the role of $f^*(t)$ by the role of $\widetilde{\omega}(f,t)$.)

(iii) By part (i) of Theorem 3.2, embedding (3.19) is continuous and, by part (iv) of Theorem 3.2, this embedding is not compact. Moreover, part (iv) of Theorem 3.2 also shows that we cannot arrive to a compact embedding if we replace the target space $\Lambda^{\lambda_q(.)}_{\infty,q}(\overline{\mathbb{R}^n})$ in (3.19) by a larger space $\Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\Omega})$ with $r > q$.

(iv) Put $X = H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n)$ and $r = +\infty$. By Theorem 3.2 (i),

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\lambda_\infty(t)/t} \precsim \|f\|_X \quad \text{for all} \quad f \in X,$$

and, by Theorem 3.2 (i) and (ii) (cf. also part (v) of this remark), the inequality

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\mu(t)/t} \precsim \|f\|_X$$

does not hold for all $f \in X$ if $\mu \in \mathcal{L}$ satisfies

$$\lim_{t\to 0_+} \frac{\frac{t}{\lambda_\infty(t)}}{\frac{t}{\mu(t)}} = \lim_{t\to 0_+} \frac{\mu(t)}{\lambda_\infty(t)} = 0.$$

If we use an analogue of terminology from [**22**], this means that the function $\frac{\lambda_\infty(t)}{t} = \left(\int_{t^n}^2 \tau^{-1}[b(\tau)]^{-q'}\, d\tau\right)^{1/q'}$, $t \in (0,1]$, is the continuous envelope function of the space $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n)$. Using also part (iii) of Theorem 3.2, we can see that the couple

$$\left(\left(\int_{t^n}^2 \tau^{-1}[b(\tau)]^{-q'}\,d\tau\right)^{1/q'}, q\right)$$

is the continuous envelope of the space $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n)$.

(v) Let $r \in [q, +\infty]$. Using (3.13) and (3.12), we arrive at

$$\left\|\tau^{-1/r}\frac{\tau}{\lambda_r(\tau)}\right\|_{r;(0,t)} \approx \frac{t}{\lambda_\infty(t)} \quad \text{for all} \quad t \in (0,1). \tag{3.20}$$

This implies that

$$\lim_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\lambda_r(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} \approx \lim_{t\to 0_+} \frac{\frac{t}{\lambda_\infty(t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}. \tag{3.21}$$

Together with estimate (3.9), this shows that condition (3.15) is satisfied if

$$\lim_{t\to 0_+} \frac{\mu(t)}{\lambda_\infty(t)} = 0. \tag{3.22}$$

(vi) Let $r = +\infty$ and let the function $t \mapsto t/\mu(t)$ be equivalent to a non-decreasing function on some interval $(0,\delta) \subset (0,1)$. Then, by applying estimate (3.11) in (3.21), we can see that (3.15) is now equivalent to (3.22).

(vii) Let $r \in [q, +\infty)$. Since any function $\rho \in \mathcal{L}_r$ satisfies $\left\|\tau^{-1/r}\frac{\tau}{\rho(\tau)}\right\|_{r;(0,\delta)} < +\infty$ (cf. (2.4)), we have $\left\|\tau^{-1/r}\frac{\tau}{\rho(\tau)}\right\|_{r;(0,t)} \to 0$ as $t \to 0_+$. In particular, this holds with $\rho = \mu$ and $\rho = \lambda_r$. Thus, L'Hospital's rule gives

$$\lim_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\lambda_r(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} = \lim_{t\to 0_+} \frac{\mu(t)}{\lambda_r(t)} \tag{3.23}$$

provided that the last limit exists.

(viii) Let $q \in (1, +\infty)$, $r \in [q, +\infty]$, $b \in SV(0, +\infty)$, $b \not\equiv 0$ and let (3.12) hold. Then (3.15) is satisfied when

$$\lim_{t\to 0_+} \frac{\mu(t)}{\lambda_s(t)} = 0 \quad \text{for some} \quad s \in [r, +\infty]. \tag{3.24}$$

Indeed, if $s = +\infty$, then the result follows from part (v). If $s < +\infty$, then the assertion is a consequence of (3.23), the identity

$$\lambda_r(t) = \lambda_s(t) \left(\frac{\int_{t^n}^2 \tau^{-1}[b(\tau)]^{-q'}\, d\tau}{[b(t^n)]^{-q'}} \right)^{1/r-1/s}, \quad t \in (0,1], \tag{3.25}$$

(which follows from (3.13)) and the fact that the function

$$t \mapsto \frac{[b(t)]^{-q'}}{\int_t^2 \tau^{-1}[b(\tau)]^{-q'}\, d\tau} \tag{3.26}$$

is bounded for $t \in (0,1)$, for any $b \in SV(0,+\infty)$, $b \not\equiv 0$, $b \not\equiv +\infty$.

4. Preliminaries

We shall need weighted Hardy inequalities, where the weights are slowly varying functions.

Lemma 4.1 ([**12**, Lemma 4.1])**.** *Let $1 \le q \le r \le +\infty$, $\nu \in \mathbb{R}\backslash\{0\}$ and let $b, \tilde{b} \in SV(0,+\infty)$.*

(i) The inequality

$$\left\| t^{\nu-1/r}\tilde{b}(t) \int_0^t g(u)\, du \right\|_{r;(0,+\infty)} \precsim \left\| t^{\nu+1/q'} b(t)\, g(t) \right\|_{q;(0,+\infty)} \tag{4.1}$$

holds for all $g \in \mathcal{M}^+(0,+\infty)$ if, and only if,

$$\nu < 0 \quad \textit{and} \quad \tilde{b} \precsim b \quad \textit{on} \quad (0,+\infty). \tag{4.2}$$

(ii) The inequality

$$\left\| t^{\nu-1/r}\, \tilde{b}(t) \int_t^{+\infty} g(u)\, du \right\|_{r;(0,+\infty)} \precsim \left\| t^{\nu+1/q'} b(t)\, g(t) \right\|_{q;(0,+\infty)} \tag{4.3}$$

holds for all $g \in \mathcal{M}^+(0,+\infty)$ if, and only if,

$$\nu > 0 \quad \textit{and} \quad \tilde{b} \precsim b \quad \textit{on} \quad (0,+\infty). \tag{4.4}$$

Throughout this section we shall assume that $\mathcal{G}$ is a function on $(0,1]$ with the following properties:

(4.5) $\mathcal{G}$ is positive and continuous on $(0,1]$;

(4.6) $\mathcal{G}$ is non-increasing on $(0,s_0]$, where $s_0 \in (0,1]$ is a fixed number;

(4.7) $\mathcal{G}(t/2) \precsim \mathcal{G}(t), \quad t \in (0,1]$.

Let $\varphi \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\int_{\mathbb{R}} \varphi(t)\,dt = 1$ and $\operatorname{supp}\varphi = [-1,1]$. Then the function φ_ε, with $\varepsilon > 0$, defined by $\varphi_\varepsilon(t) := \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$ for all $t \in \mathbb{R}$, satisfies

$$\varphi_\varepsilon \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\varphi_\varepsilon = [-\varepsilon,\varepsilon] \quad \text{and} \quad \int_{\mathbb{R}} \varphi_\varepsilon(t)\,dt = 1. \tag{4.8}$$

We now use φ to assign to the function $\mathcal{G}$ a family of functions $\{\mathcal{G}_s\}$ as in [**10**]. Let us extend $\mathcal{G}$ by zero outside the interval $(0,1]$, and for each $s \in (0,1)$ define the function $\mathcal{G}_s$ by

$$\mathcal{G}_s(t) := (\chi_{[s,+\infty)}\,\psi\,\mathcal{G}) * \varphi_{\frac{s}{4}}(t), \quad t \in \mathbb{R}, \tag{4.9}$$

with $\psi \in C_0^\infty(\mathbb{R})$ defined by $\psi = \chi_{[-2+\frac{1}{16},\frac{3}{4}-\frac{1}{16}]} * \varphi_{\frac{1}{16}}$.

Some properties of $\mathcal{G}_s$, $s \in \left(0,\frac{1}{4}\right)$, are summarised in the next lemma due to Edmunds, Gurka and Opic [**10**, Lemma 4.1].

Lemma 4.2. *If* $s \in \left(0,\frac{1}{4}\right)$ *and the functions* $\mathcal{G}_s$ *are defined by* (4.9) *(with* $\mathcal{G}$ *satisfying* (4.5)–(4.7)*), then*

$$\mathcal{G}_s \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\mathcal{G}_s \subset \left[\frac{s}{2},1\right] \quad \textit{and} \quad \mathcal{G}_s \geq 0. \tag{4.10}$$

Moreover, there are positive constants C_1, C_2 *and* C_3 *(independent of* s *and* t*) such that*

$$\mathcal{G}_s(t) \leq C_1\,\mathcal{G}(t)\,\chi_{[\frac{s}{2},1]}(t), \qquad t \in (0,1], \tag{4.11}$$

$$\left|\frac{d}{dt}\mathcal{G}_s(t)\right| \leq C_2\,s^{-1}\,\mathcal{G}(t)\,\chi_{[\frac{s}{2},1]}(t), \quad t \in (0,1], \tag{4.12}$$

$$\mathcal{G}_s(t) \geq C_3\mathcal{G}(t), \qquad t \in \left[2s,\frac{1}{2}\right]. \tag{4.13}$$

In addition, if

$$\mathcal{G} \in C^1(0,1) \quad \textit{and} \quad \left|\frac{d}{dt}\mathcal{G}(t)\right| \precsim t^{-1}\mathcal{G}(t), \quad t \in (0,1), \tag{4.14}$$

then there is a positive constant C_4 *(independent of* s *and* t*) such that*

$$\left|\frac{d}{dt}\mathcal{G}_s(t)\right| \leq C_4\,t^{-1}\mathcal{G}(t), \quad t \in [2s,1]. \tag{4.15}$$

Now, as in [**10**], we use the family $\{\mathcal{G}_s\}$ to define another family $\{h_s\}$, $h_s\colon \mathbb{R}^n \to \mathbb{R}$, which are important to prove our main results. For any $s \in \left(0, \frac{1}{4}\right)$, let a_s be a positive number and let $\mathcal{G}_s$ be the function given by (4.9); we define the function h_s by

$$h_s(x) := a_s\, \mathcal{G}_s(|x|) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{4.16}$$

It follows from (4.10) that

$$h_s \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp} h_s \subset \overline{B_n}(0,1) \backslash \overline{B_n}(0,s/2). \tag{4.17}$$

Let $\sigma \in [1, n+1)$ and $s \in \left(0, \frac{1}{4}\right)$. To prove Theorems 3.1 and 3.2, we define functions u_s as in [**10**] by

$$u_s(x) := x_1 (g_{\sigma-1} * h_s)(x), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{4.18}$$

with h_s from (4.16). Some properties of these functions are summarised in the following lemma. Parts (i) and (ii) are an extention of [**10**, Lemma 4.7 (i) and (ii)] and part (iii) is due to Edmunds, Gurka and Opic [**10**, Lemma 4.7 (iii)].

Lemma 4.3. *Let $\sigma \in [1, n+1)$, $p, q \in (1, +\infty)$ and $b \in SV(0, +\infty)$.*

(i) Suppose (in addition to (4.5)–(4.7)*) that the function $\mathcal{G}$ satisfies* (4.14). *Then $u_s \in H^\sigma L_{p,q;b}(\mathbb{R}^n)$, $s \in \left(0, \frac{1}{4}\right)$, and there exists a positive constant c such that, for all $s \in \left(0, \frac{1}{4}\right)$,*

$$\|u_s\|_{\sigma;p,q;b} \le c\, a_s \left(V_1(s) + V_2(s)\right),$$

where V_1 and V_2 are defined by

$$V_1(s) = \left(\int_s^1 \left[\mathcal{G}(t) t^{n/p}\, b(t^n) \right]^q \frac{dt}{t} \right)^{1/q} \quad \text{and} \quad V_2(s) = \mathcal{G}(s) s^{n/p} b(s^n). \tag{4.19}$$

(ii) If $\sigma \in (1, n+1)$, then there exists a positive constant c such that for every $s \in \left(0, \frac{1}{4}\right)$ and $x = (t, 0, \dots, 0) \in \mathbb{R}^n$, $t \in \left[2s, \frac{1}{2}\right]$,

$$|u_s(x) - u_s(0)| \ge c\, t\, a_s \int_t^{1/2} \tau^{\sigma-2} \mathcal{G}(\tau)\, d\tau.$$

(iii) Let $\sigma \in (1, n+1)$, $S \in \left(0, \frac{1}{4}\right)$. Suppose that the numbers a_s from (4.16) *are bounded,* i.e.,

$$a_s \le c \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right) \quad \text{with some} \quad c \in (0, +\infty). \tag{4.20}$$

Moreover, assume (in addition to (4.5)–(4.7)*) that the function* $\mathcal{G}$ *and the numbers* a_s *satisfy*

$$a_s \int_{2s}^{S/2} t^{\sigma-2}\,\mathcal{G}(t)\,dt \to +\infty \quad as \quad s \to 0_+. \tag{4.21}$$

Then there exist $\varepsilon = \varepsilon(\sigma) \in \big(0, \frac{1}{2}\big)$, $s_1 = s_1(S) \in \big(0, \frac{S}{4}\big)$ *and a positive constant* c *(independent of* S *and* s_1*) such that*

$$|[u_s(x) - u_S(x)] - [u_s(0) - u_S(0)]| \geq c\, s\, a_s \int_{2s}^{S/2} t^{\sigma-2}\mathcal{G}(t)\,dt$$

for every $s \in (0, s_1)$ *and* $x = (\varepsilon\, s, 0, \ldots, 0) \in \mathbb{R}^n$.

We just need to prove parts (i) and (ii), because part (iii) is proved in [**10**]. To prove part (i) of Lemma 4.3, we use some auxiliary results.

Lemma 4.4. *Let* T *be a quasi-linear operator such that, for all* $q \in (1, +\infty)$,

$$T\colon L_q(\mathbb{R}^n) \to L_q(\mathbb{R}^n) \tag{4.22}$$

is bounded. Let $1 < p < +\infty$, $1 \leq r \leq +\infty$ *and let* $b \in SV(0, +\infty)$. *Then*

$$T\colon L_{p,r;b}(\mathbb{R}^n) \to L_{p,r;b}(\mathbb{R}^n) \tag{4.23}$$

is bounded.

Proof: The proof is analogous to the proofs of [**9**, Corollary 3.15] and [**18**, Corollary 3.4]. ☐

The next lemma extends [**21**, Chapter V, Lemma 3], [**9**, Lemma 4.1] and [**18**, Lemma 5.2].

Lemma 4.5. *Let* $\sigma \in [1, +\infty)$, $p \in (1, +\infty)$, $q \in (1, +\infty)$ *and* $b \in SV(0, +\infty)$. *Then* $f \in H^\sigma L_{p,q;b}(\mathbb{R}^n)$ *if, and only if,* $f \in H^{\sigma-1} L_{p,q;b}(\mathbb{R}^n)$ *and the distributional derivatives* $\frac{\partial f}{\partial x_j}$ *belong to* $H^{\sigma-1} L_{p,q;b}(\mathbb{R}^n)$ $(j = 1, \ldots, n)$. *Moreover, the (quasi-)norms* $\|f\|_{\sigma;p,q;b}$ *and* $\|f\|_{\sigma-1;p,q;b} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{\sigma-1;p,q;b}$ *are equivalent on* $H^\sigma L_{p,q;b}(\mathbb{R}^n)$.

Proof: See [**21**, Chapter V, Lemma 3], [**9**, Lemma 4.1] and [**18**, Lemma 5.2]. ☐

Proof of Lemma 4.3: (i) Taking into account [**12**, Lemma 4.3], Lemmas 4.4 and 4.5, the proof is similar to the proof of [**10**, Lemma 4.7 (i)].

(ii) Let $s \in \left(0, \frac{1}{4}\right)$. Since, by (4.17), $\operatorname{supp} h_s \subset \overline{B_n}(0,1)$,

$$(g_{\sigma-1} * h_s)(x) \approx (I_{\sigma-1} * h_s)(x) \quad \text{for all} \quad x \in B_n(0,1).$$

Therefore, for $x = (t, 0, \dots, 0)$, $t \in \left[2s, \frac{1}{2}\right]$, we have

$$\begin{aligned} |u_s(x) - u_s(0)| &= u_s(x) = t\,(g_{\sigma-1} * h_s)(x) \\ &\approx t(I_{\sigma-1} * h_s)(x) \gtrsim t \int_{|y|>t} \frac{h_s(y)}{|y|^{n-\sigma+1}}\,dy, \end{aligned} \tag{4.24}$$

see details in [**6**, (3.12)]. Using spherical coordinates and (4.16), we obtain

$$\begin{aligned} \int_{|y|>t} \frac{h_s(y)}{|y|^{n-\sigma+1}}\,dy &= \int_t^{+\infty} \int_{\{|y|=\rho\}} \frac{a_s \mathcal{G}_s(|y|)}{|y|^{n-\sigma+1}}\,d\vartheta\,d\rho \\ &= a_s \int_t^{+\infty} \frac{\mathcal{G}_s(\rho)}{\rho^{n-\sigma+1}} \omega_n n \rho^{n-1}\,d\rho \\ &\gtrsim a_s \int_t^{1/2} \rho^{\sigma-2} \mathcal{G}_s(\rho)\,d\rho. \end{aligned} \tag{4.25}$$

Estimates (4.24), (4.25) and (4.13) imply that

$$|u_s(x) - u_s(0)| \geq c\,t\,a_s \int_t^{1/2} \rho^{\sigma-2} \mathcal{G}(\rho)\,d\rho,$$

which yields the result of part (ii). □

5. Proof of the main results

Proof of Theorem 3.1:

Step 1: Proof of part (i). Since the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ (cf. [**18**, Lemma 5.1]), it is enough to prove (cf. [**18**, Proposition 5.6]) that

$$\|u\|\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \precsim \|u\|_{\sigma;p,q;b} \quad \text{for all} \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Let $u \in \mathcal{S}(\mathbb{R}^n) \subset H^\sigma L_{p,q;b}(\mathbb{R}^n)$. Then Lemma 4.5 shows that $\frac{\partial u}{\partial x_i} \in H^{\sigma-1} L_{p,q;b}(\mathbb{R}^n)$, for $i = 1, \dots, n$. Now, by [**12**, Theorem 3.1 (i)], with σ–1 instead of σ, we have

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{p_{\sigma-1},q;b} \precsim \left\|\frac{\partial u}{\partial x_i}\right\|_{\sigma-1;p,q;b}, \quad i=1,\dots,n,$$

where $1/p_{\sigma-1} = 1/p - (\sigma-1)/n$. Hence, again by Lemma 4.5,

$$\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_{\sigma-1},q;b} \precsim \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{\sigma-1;p,q;b} \precsim \|u\|_{\sigma;p,q;b}. \tag{5.1}$$

Using (3.1), the estimate (cf. [**14**, Proposition 5.12 (i)])

$$\omega(u,t) \precsim \int_0^t |\nabla u|^*(\sigma^n)\,d\sigma, \quad t>0, \tag{5.2}$$

where $|\nabla u|$ denotes the Euclidean norm of the gradient of u, Lemma 4.1 (i) (with $\nu = n/p - \sigma < 0$) and the change of variables, we obtain

$$\begin{aligned}\left\|t^{-1/r}\frac{\omega(u,t)}{\lambda(t)}\right\|_{r;(0,1)} &\precsim \left\|t^{n/p-\sigma-1/r}b(t^n)\int_0^t |\nabla u|^*(\tau^n)\,d\tau\right\|_{r;(0,1)}\\ &\precsim \|t^{n/p-\sigma+1/q'}b(t^n)|\nabla u|^*(t^n)\|_{q;(0,+\infty)}\\ &\approx \|t^{1/p-(\sigma-1)/n-1/q}b(t)|\nabla u|^*(t)\|_{q;(0,+\infty)}.\end{aligned}$$

Furthermore, the estimate $|\nabla u|^*(t) \le |\nabla u|^{**}(t)$, (2.1), Lemma 4.1 (i) (with $\nu = 1/p_{\sigma-1} - 1 < 0$) and (5.1) imply that

$$\begin{aligned}&\|t^{1/p-(\sigma-1)/n-1/q}b(t)|\nabla u|^*(t)\|_{q;(0,+\infty)}\\ &\qquad\precsim \|t^{1/p_{\sigma-1}-1/q}b(t)|\nabla u|^{**}(t)\|_{q;(0,+\infty)}\\ &\qquad\precsim \sum_{i=1}^{n}\|t^{1/p_{\sigma-1}-1/q}b(t)\left(\frac{\partial u}{\partial x_i}\right)^{**}(t)\|_{q;(0,+\infty)}\\ &\qquad\approx \sum_{i=1}^{n}\|t^{1/p_{\sigma-1}-1/q}b(t)\left(\frac{\partial u}{\partial x_i}\right)^{*}(t)\|_{q;(0,+\infty)}\\ &\qquad\precsim \|u\|_{\sigma;p,q;b}.\end{aligned}$$

Consequently,

$$\left\|t^{-1/r}\frac{\omega(u,t)}{\lambda(t)}\right\|_{r;(0,1)} \precsim \|u\|_{\sigma;p,q;b}. \tag{5.3}$$

As in [**18**, Proposition 5.6], we also have $\|u\|_\infty \precsim \|u\|_{\sigma;p,q;b}$. This and (5.3) yield

$$\|u|\Lambda^{\lambda(\cdot)}_{\infty,r}\| \precsim \|u\|_{\sigma;p,q;b} \quad \text{for all} \quad u \in \mathcal{S}(\mathbb{R}^n).$$

The proof of part (i) is complete.

Step 2. We shall assume without loss of generality that $B_n(0,1) \subset \Omega$. Let $s \in \left(0, \frac{1}{4}\right)$ and $\gamma < 0$. Define the function $\mathcal{G}$ by

$$\mathcal{G}(t) = \int_t^{+\infty} \tau^{\gamma - n/p - 1} \, [b(\tau^n)]^{-1} \, d\tau \approx t^{\gamma - n/p} \, [b(t^n)]^{-1}, \quad t \in (0,1], \tag{5.4}$$

and put

$$a_s = s^{-\gamma}. \tag{5.5}$$

The function $\mathcal{G}$ satisfies (4.5)–(4.7). Because

$$|\mathcal{G}'(t)| = t^{\gamma - n/p - 1}[b(t^n)]^{-1} \approx \frac{\mathcal{G}(t)}{t}, \quad t \in (0,1),$$

the function $\mathcal{G}$ satisfies (4.14) as well. Let us consider the functions u_s, $s \in \left(0, \frac{1}{4}\right)$, defined by (4.18). By Lemma 4.3 (i), for all $s \in \left(0, \frac{1}{4}\right)$,

$$\begin{aligned} \|u_s\|_{\sigma;p,q;b} &\precsim a_s(V_1(s) + V_2(s)) \approx s^{-\gamma} \left(\left(\int_s^1 t^{\gamma q - 1} \, dt \right)^{1/q} + s^{\gamma} \right) \\ &\approx s^{-\gamma} s^{\gamma} = 1. \end{aligned} \tag{5.6}$$

We shall consider two cases:

- If $\sigma = 1$, then (4.18), (4.16) and (2.3) imply that

$$u_s(x) = a_s \, x_1 \, \mathcal{G}_s(|x|), \quad x \in \mathbb{R}^n, \quad s \in \left(0, \frac{1}{4}\right). \tag{5.7}$$

Thus, if we put $x = (2s, 0, \ldots, 0)$ for each $s \in \left(0, \frac{1}{4}\right)$, we obtain from (4.13), (4.7), (5.4) and (5.5) that

$$|u_s(x) - u_s(0)| = u_s(x) \geq C_3 a_s \, 2s \mathcal{G}(2s) \succsim s \, a_s \, \mathcal{G}(s) \approx s^{1 - n/p} [b(s^n)]^{-1}. \tag{5.8}$$

Moreover, if we take $S \in \left(0, \frac{1}{4}\right)$, $s \in \left(0, \frac{S}{4}\right)$, then $|x| = 2s < \frac{S}{2}$, and so $u_S(x) = 0$ by (4.10) and (5.7). Thus, for all $s \in \left(0, \frac{S}{4}\right)$ and $x = (2s, 0, \ldots, 0)$, (5.8) yields

$$|[u_s(x) - u_S(x)] - [u_s(0) - u_S(0)]| = u_s(x) \geq c_1 \, s^{1 - n/p} [b(s^n)]^{-1}, \tag{5.9}$$

with a positive constant c_1 independent of S and s.

• If $\sigma \in (1, n+1)$, then, by Lemma 4.3 (ii), there exists a positive constant c such that

$$
\begin{aligned}
|u_s(x) - u_s(0)| &\geq 2c\, s^{1-\gamma} \int_{2s}^{1/2} t^{\sigma-2+\gamma-n/p}[b(t^n)]^{-1}\, dt \\
&\gtrsim s^{\sigma-n/p}[b(s^n)]^{-1}
\end{aligned}
\tag{5.10}
$$

for every $s \in \left(0, \frac{1}{8}\right)$ and $x = (2s, 0, \ldots, 0)$. Furthermore, if we take $S \in \left(0, \frac{1}{4}\right)$, we can see that the conditions (4.20) and (4.21) also hold. Indeed, $a_s = s^{-\gamma} \precsim 1$ for all $s \in \left(0, \frac{1}{4}\right)$ because $\gamma < 0$. Moreover, since $\sigma - n/p - 1 < 0$ and $\gamma < 0$, we have, for all sufficiently small s,

$$
\begin{aligned}
a_s \int_{2s}^{S/2} t^{\sigma-2}\, \mathcal{G}(t)\, dt &\approx a_s \int_{2s}^{S/2} t^{\sigma-2+\gamma-n/p}[b(t^n)]^{-1}\, dt \\
&\approx s^{-\gamma+\sigma-1+\gamma-n/p}[b(s^n)]^{-1} \\
&\approx s^{\sigma-1-n/p}[b(s^n)]^{-1},
\end{aligned}
$$

which tends to $+\infty$ as $s \to 0_+$. Hence, by Lemma 4.3 (iii), there exist $\varepsilon = \varepsilon(\sigma) \in \left(0, \frac{1}{2}\right)$, $s_1 = s_1(S) \in \left(0, \frac{S}{8}\right)$ and a positive constant c (independent of S and s_1) such that, for every $s \in (0, s_1)$ and $x = (\varepsilon s, 0, \ldots, 0)$,

$$
\begin{aligned}
|[u_s(x) - u_S(x)] - [u_s(0) - u_S(0)]| &\geq c\, s^{1-\gamma} \int_{2s}^{S/2} t^{\sigma-2}\mathcal{G}(t)\, dt \\
&\geq c_1 s^{\sigma-n/p}[b(s^n)]^{-1},
\end{aligned}
\tag{5.11}
$$

with a positive constant c_1 independent of S and s_1.

STEP 3. Let λ be the function defined by (3.1). Since $b \in SV(0, +\infty)$, we have, for any fixed $k \in (0, +\infty)$,

$$
\lambda(kt) \approx \lambda(t), \quad t \in (0, 1]. \tag{5.12}
$$

Let us assume that (3.3) and (3.4) hold. Then, by (5.6), (5.8) or (5.10) (with $x = (2s, 0, \ldots, 0)$) and (5.12), we obtain

$$
\begin{aligned}
1 &\gtrsim \|u_s\|_{\sigma;p,q;b} \gtrsim \|u_s|\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})\| \geq \left\| t^{-1/r}\frac{\omega(u_s,t)}{\mu(t)}\right\|_{r;(0,1)} \\
&\geq \left\| t^{-1/r}\frac{\omega(u_s,t)}{\mu(t)}\right\|_{r;(0,2s)} \gtrsim \frac{\omega(u_s,2s)}{2s}\left\| t^{-1/r}\frac{t}{\mu(t)}\right\|_{r;(0,2s)} \\
&\geq \frac{|u_s(x)-u_s(0)|}{2s}\left\| t^{-1/r}\frac{t}{\mu(t)}\right\|_{r;(0,2s)} \\
&\gtrsim \frac{s^{\sigma-n/p}[b(s^n)]^{-1}}{2s}\left\| t^{-1/r}\frac{t}{\mu(t)}\right\|_{r;(0,2s)} \approx \frac{\|t^{-1/r}\frac{t}{\mu(t)}\|_{r;(0,2s)}}{\frac{2s}{\lambda(2s)}}
\end{aligned}
$$

for all $s \in \left(0, \frac{1}{8}\right)$, which contradicts assumption (3.3). The proof of part (ii) is complete.

Step 4. Take $S \in \left(0, \frac{1}{4}\right)$ fixed. Let λ be the function defined by (3.1). Then, (5.9) or (5.11) (with $x = (ks, 0, \ldots, 0)$, where $k = 2$ or $k = \varepsilon$ if $\sigma = 1$ or $\sigma \in (1, n+1)$, respectively) and (3.1) yield, for every sufficiently small positive s,

$$
\begin{aligned}
\|(u_s - u_S)|\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})\| &\geq \left\| t^{-1/r}\frac{\omega(u_s-u_S,t)}{\lambda(t)}\right\|_{r;(0,1)} \\
&\geq \left\| t^{-1/r}\frac{\omega(u_s-u_S,t)}{\lambda(t)}\right\|_{r;(0,ks)} \\
&\gtrsim \frac{\omega(u_s-u_S,ks)}{ks}\left\| t^{-1/r}\frac{t}{\lambda(t)}\right\|_{r;(0,ks)} \\
&\geq \frac{|[u_s(x)-u_S(x)]-[u_s(0)-u_S(0)]|}{ks} \\
&\qquad\qquad \times \left\| t^{-1/r}\frac{t}{\lambda(t)}\right\|_{r;(0,ks)} \\
&\geq c_1\frac{s^{\sigma-n/p}[b(s^n)]^{-1}}{ks}(ks)^{n/p-\sigma+1}b((ks)^n) \geq c_2,
\end{aligned} \tag{5.13}
$$

with c_2 a positive constant independent of s and S. Therefore, if we consider the sequence $\{u_{1/k}\}_{k=k_0}^{+\infty}$, with k_0 sufficiently large, then, by (5.6), this sequence is bounded in $H^\sigma L_{p,q;b}(\mathbb{R}^n)$. However, by (5.13), it has no Cauchy subsequence in $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$. The proof of part (iv) is complete.

Step 5. Let $\overline{q} \in (0, q)$. Let now $\mathcal{G}$ be the function defined by

$$\begin{aligned}\mathcal{G}(t) &= \int_t^{+\infty} \tau^{-n/p-1} \ell_1^{-1/\overline{q}}(\tau)[b(\tau^n)]^{-1}\, d\tau \\ &\approx t^{-n/p} \ell_1^{-1/\overline{q}}(t)[b(t^n)]^{-1}, \quad t \in (0,1).\end{aligned} \tag{5.14}$$

Put

$$a_s = 1, \quad s \in \left(0, \frac{1}{4}\right). \tag{5.15}$$

The function $\mathcal{G}$ satisfies (4.5)–(4.7). Because

$$|\mathcal{G}'(t)| = t^{-n/p-1} \ell_1^{-1/\overline{q}}(t)[b(t^n)]^{-1} \approx \frac{\mathcal{G}(t)}{t}, \quad t \in (0,1),$$

the function $\mathcal{G}$ satisfies (4.14) as well. Let us define the functions u_s, $s \in \left(0, \frac{1}{4}\right)$, by (4.18). By Lemma 4.3 (i) and (5.15), for all $s \in \left(0, \frac{1}{4}\right)$,

$$\|u_s\|_{\sigma;p,q;b} \precsim a_s(V_1(s) + V_2(s)) = (V_1(s) + V_2(s)), \tag{5.16}$$

where

$$\begin{aligned}V_1(s) &= \left(\int_s^1 \left[\mathcal{G}(t) t^{n/p} b(t^n) \right]^q \frac{dt}{t} \right)^{1/q} \\ &\approx \left(\int_s^1 \ell_1^{-q/\overline{q}}(t) \frac{dt}{t} \right)^{1/q} < \left(\int_0^1 \ell_1^{-q/\overline{q}}(t) \frac{dt}{t} \right)^{1/q} \approx 1,\end{aligned}$$

because $\overline{q} < q$, and

$$V_2(s) = \mathcal{G}(s) s^{n/p} b(s^n) \approx \ell_1^{-1/\overline{q}}(t) \precsim 1.$$

Hence, the functions u_s given by (4.18) satisfy

$$\|u_s\|_{\sigma;p,q;b} \precsim 1 \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right), \tag{5.17}$$

which means that

$$u_s \in H^\sigma L_{p,q;b}(\mathbb{R}^n) \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right).$$

We shall again consider two cases:

• If $\sigma = 1$, then (4.18), (4.16) and (2.3) imply that

$$u_s(x) = a_s\, x_1\, \mathcal{G}_s(|x|), \quad x \in \mathbb{R}^n, \quad s \in \left(0, \frac{1}{4}\right). \tag{5.18}$$

Thus, if, for each $s \in \left(0, \frac{1}{4}\right)$, we put $x = (t, 0, \dots, 0)$, $t \in \left[2s, \frac{1}{2}\right]$, we obtain from (4.13), (5.14) and (5.15) that

$$\begin{aligned} \omega(u_s, t) &\geq |u_s(x) - u_s(0)| = u_s(x) \geq C_3 a_s\, t\mathcal{G}(t) \approx t\, \mathcal{G}(t) \\ &\approx t^{1-n/p} \ell_1^{-1/\overline{q}}(t)[b(t^n)]^{-1}. \end{aligned} \tag{5.19}$$

• Suppose now that $\sigma \in (1, n+1)$ and $s \in \left(0, \frac{1}{8}\right)$. Then, by Lemma 4.3 (ii), (5.14) and (5.15), there exists a positive constant c such that, for every $x = (t, 0, \dots, 0)$ with $t \in \left[2s, \frac{1}{4}\right]$,

$$\begin{aligned} \omega(u_s, t) &\geq |u_s(x) - u_s(0)| = u_s(x) \\ &\geq c\, t \int_t^{1/2} \tau^{\sigma-2-n/p} \ell_1^{-1/\overline{q}}(\tau)[b(\tau^n)]^{-1}\, d\tau \\ &\gtrsim t^{\sigma-n/p} \ell_1^{-1/\overline{q}}(t)[b(t^n)]^{-1}. \end{aligned} \tag{5.20}$$

Let us assume that (3.5) holds. Then by (3.1), (5.17), either (5.19) or (5.20), we obtain, for all sufficiently small s,

$$\begin{aligned} 1 &\gtrsim \|u_s\|_{\sigma;p,q;b} \gtrsim \|u_s|\Lambda^{\lambda(\cdot)}_{\infty,\overline{q}}(\overline{\Omega})\| \\ &\geq \left\| t^{-1/\overline{q}} \frac{\omega(u_s, t)}{\lambda(t)} \right\|_{\overline{q};(0,1)} \geq \left\| t^{-1/\overline{q}} \frac{\omega(u_s, t)}{\lambda(t)} \right\|_{\overline{q};(2s,1/4)} \\ &\gtrsim \left\| t^{-1/\overline{q}} \ell_1^{-1/\overline{q}}(t) \right\|_{\overline{q};(2s,1/4)} = \left(\ell_2(2s) - \ell_2(1/4)\right)^{1/\overline{q}}. \end{aligned}$$

Since the last expression tends to $+\infty$ as $s \to 0_+$, we can see that (3.5) cannot hold. The proof of part (iii) is complete. □

Proof of Theorem 3.2:

Step 1. The proof of part (i) can be seen in [**19**, Theorem 5.7].

Step 2. We shall assume, without loss of generality, that $B_n(0,1) \subset \Omega$. Let $\beta \in (-q', +\infty)$ and let the function $\mathcal{G}$ be defined by

$$\begin{aligned} \mathcal{G}(t) &= \int_t^2 \tau^{-\sigma}[b(\tau^n)]^{-q'} \left(\int_{\tau^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{1/q'+\beta/q'} d\tau \\ &\approx t^{1-\sigma}[b(t^n)]^{-q'} \left(\int_{t^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{1/q'+\beta/q'}, \quad t \in (0,1], \end{aligned} \tag{5.21}$$

and let the numbers a_s, $s \in \left(0, \frac{1}{4}\right)$, be given by

$$a_s = \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{-\beta/q'-1}. \tag{5.22}$$

The function $\mathcal{G}$ satisfies (4.5)–(4.7). As

$$|\mathcal{G}'(t)| = t^{-\sigma}[b(t^n)]^{-q'} \left(\int_{t^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{1/q'+\beta/q'} \approx \frac{\mathcal{G}(t)}{t}, \quad t \in (0,1),$$

the function $\mathcal{G}$ satisfies (4.14) as well.

We again consider the functions u_s, $s \in \left(0, \frac{1}{4}\right)$, defined by (4.18). By Lemma 4.3 (i), the identity $\sigma - 1 = n/p$, the inequality $\beta > -q'$ and (5.21), we obtain

$$\|u_s\|_{\sigma;p,q;b} \precsim a_s(V_1(s) + V_2(s)) \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right), \tag{5.23}$$

where

$$\begin{aligned} V_1(s) &= \left(\int_s^1 \left[\mathcal{G}(t) t^{\sigma-1} b(t^n)\right]^q \frac{dt}{t} \right)^{1/q} \\ &\approx \left(\int_s^1 [b(t^n)]^{-q'} \left(\int_{t^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{q(1/q'+\beta/q')} \frac{dt}{t} \right)^{1/q} \\ &\approx \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{1+\beta/q'} \end{aligned}$$

and

$$V_2(s) \approx [b(s^n)]^{1-q'} \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi \right)^{1/q'+\beta/q'}.$$

Now, for all $s \in \left(0, \frac{1}{4}\right)$,

$$a_s\, V_1(s) \approx \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi\right)^{1+\beta/q'-1-\beta/q'} = 1 \tag{5.24}$$

and

$$\begin{aligned} a_s\, V_2(s) &\approx [b(s^n)]^{1-q'} \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi\right)^{-1/q} \\ &\precsim [b(s^n)]^{1-q'} \left(s^{n/2}[b(s^n)]^{-q'}\right)^{-1/q} \left(\int_{s^n}^2 \xi^{-1}\xi^{-1/2}\, d\xi\right)^{-1/q} \\ &\approx [b(s^n)]^{1-q'+q'/q} = 1. \end{aligned} \tag{5.25}$$

Thus, by (5.23), (5.24) and (5.25),

$$\|u_s\|_{\sigma;p,q;b} \precsim 1 \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right). \tag{5.26}$$

Step 3. Let $k \in (0, +\infty)$ be fixed. Since $b \in SV(0, +\infty)$, by [**13**, Proposition 2.2 (iii)], we have

$$b(kt) \approx b(t) \quad \text{for all} \quad t \in (0, +\infty). \tag{5.27}$$

By Lemma 4.3 (ii), there exists a positive constant c such that

$$\begin{aligned} |u_s(x) - u_s(0)| &\ge 2c\, s\, a_s \int_{2s}^{1/2} t^{\sigma-2}\mathcal{G}(t)\, dt \\ &\succsim s\, a_s \int_{2s}^{1/2} [b(t^n)]^{-q'} \left(\int_{t^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi\right)^{1/q'+\beta/q'} \frac{dt}{t} \\ &\approx s\, a_s \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi\right)^{1/q'+\beta/q'+1} \\ &\approx s \left(\int_{s^n}^2 \xi^{-1}[b(\xi)]^{-q'}\, d\xi\right)^{1/q'} \end{aligned} \tag{5.28}$$

for every $s \in \left(0, \frac{1}{8}\right)$ and $x = (2s, 0, \ldots, 0)$.

Let us assume that (3.15) and (3.16) hold and let λ_r be the function defined by (3.13). Then, by (5.26), (3.16) and (5.28), with $x =$

$(2s, 0, \dots, 0)$, we obtain

$$
\begin{aligned}
1 &\gtrsim \|u_s\|_{\sigma;p,q;b} \gtrsim \|u_s|\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})\| \geq \left\| t^{-1/r} \frac{\omega(u_s,t)}{\mu(t)} \right\|_{r;(0,1)} \\
&\geq \left\| t^{-1/r} \frac{\omega(u_s,t)}{\mu(t)} \right\|_{r;(0,2s)} \gtrsim \frac{\omega(u_s,2s)}{2s} \left\| t^{-1/r} \frac{t}{\mu(t)} \right\|_{r;(0,2s)} \\
&\geq \frac{|u_s(x)-u_s(0)|}{2s} \left\| t^{-1/r} \frac{t}{\mu(t)} \right\|_{r;(0,2s)} \\
&\gtrsim \left(\int_{s^n}^2 \xi^{-1} [b(\xi)]^{-q'}\, d\xi \right)^{1/q'} \left\| t^{-1/r} \frac{t}{\mu(t)} \right\|_{r;(0,2s)}
\end{aligned}
\tag{5.29}
$$

for all $s \in \left(0, \frac{1}{8}\right)$. On the other hand, by (3.20), we have

$$
\left\| t^{-1/r} \frac{t}{\lambda_r(t)} \right\|_{r;(0,s)} = \left(\int_{s^n}^2 \xi^{-1} [b(\xi)]^{-q'}\, d\xi \right)^{-1/q'} \tag{5.30}
$$

for all $s \in \left(0, \frac{1}{8}\right)$. Consequently, (5.29), (5.30), (5.27) and a change of variables imply that

$$
1 \gtrsim \frac{\left\| t^{-1/r} \frac{t}{\mu(t)} \right\|_{r;(0,2s)}}{\left\| t^{-1/r} \frac{t}{\lambda_r(t)} \right\|_{r;(0,s)}} \approx \frac{\left\| t^{-1/r} \frac{t}{\mu(t)} \right\|_{r;(0,2s)}}{\left\| t^{-1/r} \frac{t}{\lambda_r(t)} \right\|_{r;(0,2s)}},
$$

which contradicts the assumption (3.15). The proof of part (ii) is complete.

STEP 4. Take $S \in \left(0, \frac{1}{4}\right)$. We can see that (4.20) holds because

$$
\begin{aligned}
a_s &= \left(\int_{s^n}^2 \xi^{-1} [b(\xi)]^{-q'}\, d\xi \right)^{-1-\beta/q'} \\
&\leq \left(\int_{1}^2 \xi^{-1} [b(\xi)]^{-q'}\, d\xi \right)^{-1-\beta/q'} \lesssim (b(2))^{\beta+q'} \approx 1
\end{aligned}
$$

for all $s \in \left(0, \frac{1}{4}\right)$. Moreover, condition (4.21) also holds. Indeed, for all $s \in \left(0, \frac{S}{8}\right)$,

$$\begin{aligned} a_s \int_{2s}^{S/2} t^{\sigma-2}\,\mathcal{G}(t)\,dt &\approx a_s \int_{2s}^{S/2} [b(t^n)]^{-q'} \left(\int_{t^n}^{2} \xi^{-1}[b(\xi)]^{-q'}\,d\xi\right)^{1/q'+\beta/q'} \frac{dt}{t} \\ &\approx a_s \left(\int_{s^n}^{2} \xi^{-1}[b(\xi)]^{-q'}\,d\xi\right)^{1/q'+\beta/q'+1} \\ &\approx \left(\int_{s^n}^{2} \xi^{-1}[b(\xi)]^{-q'}\,d\xi\right)^{1/q'}, \end{aligned}$$

which tends to $+\infty$ as $s \to 0_+$ in view of assumption (3.12). Hence, by Lemma 4.3 (iii), there exist $\varepsilon = \varepsilon(\sigma) \in (0, \frac{1}{2})$, $s_1 = s_1(S) \in (0, \frac{S}{4})$ and a positive constant c (independent of S and s_1) such that, for every $s \in \left(0, \frac{s_1}{2}\right)$ and $x = (\varepsilon s, 0, \ldots, 0)$,

$$\begin{aligned} |[u_s(x)-u_S(x)]-[u_s(0)-u_S(0)]| &\geq c\, s\, a_s \int_{2s}^{S/2} t^{\sigma-2}\mathcal{G}(t)\,dt \\ &\geq c_1 s \left(\int_{s^n}^{2} \xi^{-1}[b(\xi)]^{-q'}\,d\xi\right)^{1/q'}, \end{aligned} \tag{5.31}$$

with a positive constant c_1 independent of S and s_1. Let λ_r be the function defined by (3.13). Then, for every sufficiently small positive s and $x = (\varepsilon s, 0, \ldots, 0)$, (5.31), (5.30), (5.27) and a change of variables give

$$\begin{aligned} \|(u_s-u_S)|\Lambda_{\infty,r}^{\lambda_r(\cdot)}(\overline{\Omega})\| &\geq \left\| t^{-1/r}\frac{\omega(u_s-u_S,t)}{\lambda_r(t)}\right\|_{r;(0,1)} \\ &\geq \left\| t^{-1/r}\frac{\omega(u_s-u_S,t)}{\lambda_r(t)}\right\|_{r;(0,s\varepsilon)} \\ &\gtrsim \frac{\omega(u_s-u_S,s\varepsilon)}{s\varepsilon}\left\| t^{-1/r}\frac{t}{\lambda_r(t)}\right\|_{r;(0,s\varepsilon)} \\ &\geq \frac{|[u_s(x)-u_S(x)]-[u_s(0)-u_S(0)]|}{s\varepsilon} \\ &\qquad \times \left\| t^{-1/r}\frac{t}{\lambda_r(t)}\right\|_{r;(0,s\varepsilon)} \\ &\gtrsim \left(\int_{s^n}^{2}\xi^{-1}[b(\xi)]^{-q'}\,d\xi\right)^{1/q'}\left\| t^{-1/r}\frac{t}{\lambda_r(t)}\right\|_{r;(0,s\varepsilon)} \\ &\approx 1. \end{aligned} \tag{5.32}$$

Therefore, if we consider the sequence $\{u_{1/k}\}_{k=k_0}^{+\infty}$, with k_0 sufficiently large, then, by (5.26), this sequence is bounded in $H^\sigma L_{p,q;b}(\mathbb{R}^n)$. However, by (5.32), it has no Cauchy subsequence in $\Lambda^{\lambda_r(.)}_{\infty,r}(\overline{\Omega})$. The proof of part (iv) is now complete.

STEP 5. Let $\overline{q} \in (0,q)$ and $\alpha \in (-1/\overline{q}, -1/q)$. Let now $\mathcal{G}$ be the function defined by

$$\begin{aligned}\mathcal{G}(t) = \int_t^2 \tau^{-n/p-1}[b(\tau^n)]^{-q'} \left(\int_{\tau^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi}\right)^{-\frac{1}{q}} \\ \times \left[\ell_1\left(\int_{\tau^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi}\right)\right]^\alpha d\tau \\ \approx t^{-n/p}[b(t^n)]^{-q'} \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi}\right)^{-\frac{1}{q}} \\ \times \left[\ell_1\left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi}\right)\right]^\alpha\end{aligned} \tag{5.33}$$

for all $t \in (0,1)$, and put

$$a_s = 1, \quad s \in \left(0, \frac{1}{4}\right). \tag{5.34}$$

The function $\mathcal{G}$ satisfies (4.5)–(4.7). Because

$$\begin{aligned}|\mathcal{G}'(t)| = t^{-n/p-1}[b(t^n)]^{-q'} \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi}\right)^{-\frac{1}{q}} \left[\ell_1\left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi}\right)\right]^\alpha \\ \approx \frac{\mathcal{G}(t)}{t}\end{aligned}$$

for all $t \in (0,1)$, the function $\mathcal{G}$ satisfies (4.14) as well. Define the functions u_s, $s \in \left(0, \frac{1}{4}\right)$, by (4.18). By Lemma 4.3 (i) and (5.34), for all $s \in \left(0, \frac{1}{4}\right)$,

$$\|u_s\|_{\sigma;p,q;b} \precsim a_s(V_1(s) + V_2(s)) = (V_1(s) + V_2(s)), \tag{5.35}$$

where

$$
\begin{aligned}
V_1(s) &= \left(\int_s^1 \left[\mathcal{G}(t) t^{n/p} b(t^n) \right]^q \frac{dt}{t} \right)^{1/q} \\
&< \left\| t^{-1/q} [b(t^n)]^{-q'/q} \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{-\frac{1}{q}} \left[\ell_1 \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha} \right\|_{q;(0,1)} \\
&\approx 1,
\end{aligned}
$$

because $\alpha q + 1 < 0$, and

$$
\begin{aligned}
V_2(s) &= \mathcal{G}(s) s^{n/p} b(s^n) \\
&\approx [b(s^n)]^{-q'/q} \left(\int_{s^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{-\frac{1}{q}} \left[\ell_1 \left(\int_{s^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha} \\
&\approx s^{1/2} [b(s^n)]^{-q'/q} \left(\int_{s^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{-\frac{1}{q}} \\
&\qquad \times \left[\ell_1 \left(\int_{s^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha} \left(\int_s^1 t^{-q/2} \frac{dt}{t} \right)^{1/q} \\
&\precsim \left(\int_s^1 t^{q/2} [b(t^n)]^{-q'} \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{-1} \right. \\
&\qquad \left. \times \left[\ell_1 \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha q} t^{-q/2} \frac{dt}{t} \right)^{1/q} \\
&= V_1(s) \precsim 1.
\end{aligned}
$$

Hence, the functions u_s given by (4.18) satisfy

$$
\|u_s\|_{\sigma;p,q;b} \precsim 1 \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right), \tag{5.36}
$$

which means that

$$
u_s \in H^\sigma L_{p,q;b}(\mathbb{R}^n) \quad \text{for all} \quad s \in \left(0, \frac{1}{4}\right).
$$

Let $s \in \left(0, \frac{1}{8}\right)$. Then, by Lemma 4.3 (ii), (5.33) and (5.34), there exists a positive constant c such that, for every $x = (t, 0, \dots, 0)$ with $t \in \left[2s, \frac{1}{4}\right]$,

(5.37)

$$\omega(u_s, t) \geq |u_s(x) - u_s(0)| = u_s(x) \geq c\, t \int_t^{1/2} \tau^{\sigma-2} \mathcal{G}(\tau)\, d\tau$$

$$\approx t \int_t^{1/2} [b(\tau^n)]^{-q'} \left(\int_{\tau^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{-\frac{1}{q}} \left[\ell_1 \left(\int_{\tau^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha} \frac{d\tau}{\tau}$$

$$\approx t \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{\frac{1}{q'}} \left[\ell_1 \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha}.$$

Let us assume that (3.17) holds. Then, by (3.13) with $r = \overline{q}$, (5.36) and (5.37), we obtain, for all sufficiently small s,

$$1 \gtrsim \|u_s\|_{\sigma;p,q;b} \gtrsim \|u_s | \Lambda_{\infty,\overline{q}}^{\lambda_{\overline{q}}(.)}(\overline{\Omega})\|$$

$$\geq \left\| t^{-1/\overline{q}} \frac{\omega(u_s, t)}{\lambda_{\overline{q}}(t)} \right\|_{\overline{q};(0,1)} \geq \left\| t^{-1/\overline{q}} \frac{\omega(u_s, t)}{\lambda_{\overline{q}}(t)} \right\|_{\overline{q};(2s,1/4)}$$

$$\approx \left\| t^{-1/\overline{q}} [b(t^n)]^{-q'/\overline{q}} \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right)^{-\frac{1}{\overline{q}}} \left[\ell_1 \left(\int_{t^n}^2 [b(\xi)]^{-q'} \frac{d\xi}{\xi} \right) \right]^{\alpha} \right\|_{\overline{q};(2s,1/4)}.$$

However, the last expression tends to $+\infty$ as $s \to 0_+$ because $\alpha\overline{q} + 1 > 0$. Therefore, the embedding (3.17) cannot hold. The proof of part (iii) is complete. ☐

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