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S_4-SYMMETRY ON THE TITS CONSTRUCTION OF EXCEPTIONAL LIE ALGEBRAS AND SUPERALGEBRAS

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Abstract

The classical Tits construction provides models of the exceptional simple Lie algebras in terms of a unital composition algebra and a degree three simple Jordan algebra. A couple of actions of the symmetric group S_4 on this construction are given. By means of these actions, the models provided by the Tits construction are related to models of the exceptional Lie algebras obtained from two different types of structurable algebras. Some models of exceptional Lie superalgebras are discussed too.

Introduction

In a previous paper [**EO07**], the authors have studied those Lie algebras with an action of the symmetric group of degree 4, denoted by S_4, by automorphisms. Under some conditions, these Lie algebras are coordinatized by the structurable algebras introduced by Allison [**All78**].

The purpose of this paper is to show how a structurable algebra of an admissible triple appears naturally when considering an action by automorphisms of the symmetric group S_4 on the classical Tits construction of the exceptional Lie algebras [**Tit66**]. This can be extended to the superalgebra setting. A different S_4 action will be considered too, related to the structurable algebras consisting of a tensor product of two composition algebras. This provides connections of the Tits construction to other models of the exceptional Lie algebras.

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The paper is structured as follows. The first section will be devoted to show how the symmetric group S_4 acts by automorphisms of the split Cayley algebra. Sections 2 and 3 will review, respectively, the classical Tits construction [**Tit66**] of the exceptional Lie algebras, and the structurable algebras of admissible triples attached to separable Jordan algebras of degree 3. Then Section 4 will show how to extend the action of S_4 on the Cayley algebra to an action by automorphisms on the Tits Construction. The associated coordinate algebra will be shown to be isomorphic to the structurable algebra attached to the Jordan algebra used in the construction. The proof involves many computations, but the isomorphism given is quite natural. Section 5 will extend the results of the previous section to the superalgebra setting. Here some structurable superalgebras appear as coordinate superalgebras of the exceptional Lie superalgebras $G(3)$ and $F(4)$. Section 6 will deal with a different action of S_4 on the Tits construction. This time an action of S_4 by automorphisms is given on a central simple degree 3 Jordan algebra, and this action is extended to an action on the Lie algebras. The associated structurable coordinate algebra is shown to be isomorphic to the tensor product of two unital composition algebras: the one used 'on the left' in the Tits construction, and the one that coordinatizes the Jordan algebra involved.

All these previous results suggest a characterization of those Lie algebras endowed with an action of S_4 by automorphisms in such a way that the coordinate algebra is unital, in terms of the existence of a subalgebra isomorphic to the three dimensional orthogonal Lie algebra $\mathfrak{so}_3$, with the property that, as a module for this subalgebra, the Lie algebra is a sum of copies of the adjoint module, of its natural 5 dimensional irreducible module, and of the trivial module. This characterization is proved in Section 7.

In what follows, all the algebras and superalgebras considered will be defined over a ground field k of characteristic $\neq 2, 3$.

1. Composition algebras

Let C be a unital composition algebra over k. Thus C is a finite dimensional k-algebra with a nondegenerate quadratic form $n\colon C \to k$ such that $n(ab) = n(a)n(b)$ for any $a, b \in C$. Then each element $a \in c$ satisfies the degree 2 equation (see [**ZSSS82**, Chapter 2, Lemma 2]):

$$a^2 - t(a)a + n(a)1 = 0$$

where $t(a) = n(a,1)$ $\big(= n(a+1) - n(a) - 1\big)$ is called the trace. The subspace of trace zero elements will be denoted by C^0.

Moreover, for any $a, b \in C$, the linear map $D_{a,b}\colon C \to C$ given by

$$D_{a,b}(c) = [[a,b],c] + 3(a,c,b) \tag{1.1}$$

where $[a,b] = ab - ba$ is the commutator, and $(a,c,b) = (ac)b - a(cb)$ the associator, is the inner derivation determined by the elements a, b (see [**Sch95**, Chapter III, §8]). These derivations span the whole Lie algebra of derivations $\mathfrak{Der} C$ of C. Besides, they satisfy

$$D_{a,b} = -D_{b,a}, \quad D_{ab,c} + D_{bc,a} + D_{ca,b} = 0, \tag{1.2}$$

for any $a, b, c \in C$.

The dimension of C is restricted to 1, 2, 4 (quaternion algebras) or 8 (Cayley algebras), and for dimensions 2, 4 or 8 there is a unique unital composition algebra with zero divisors. These are called split. The unique split Cayley algebra has a basis (see [**ZSSS82**, Chapter 2]) $\{e_1, e_2, u_0, u_1, u_2, v_0, v_1, v_2\}$ with multiplication given by:

(1.3)

$$\begin{aligned}
&e_l^2 = e_l,\ l = 1,2,\ e_1e_2 = 0 = e_2e_1,\\
&e_1u_i = u_i = u_ie_2,\ e_2v_i = v_i = v_ie_1,\ i = 0,1,2,\\
&e_2u_i = 0 = u_ie_1,\ e_1v_i = 0 = v_ie_2,\ i = 0,1,2,\\
&u_iu_{i+1} = v_{i+2} = -u_{i+1}u_i,\ v_iv_{i+1} = u_{i+2} = -v_{i+1}v_i,\ \text{indices modulo 3},\\
&u_i^2 = 0 = v_i^2,\ i = 0,1,2,\\
&u_iv_j = -\delta_{ij}e_1,\ v_iu_j = -\delta_{ij}e_2,\ i,j = 0,1,2.
\end{aligned}$$

It follows that $n(e_l) = n(u_i) = n(v_i) = 0$, $l = 1,2$, $i = 0,1,2$, while $n(e_1, e_2) = 1$, $n(u_i, v_j) = \delta_{ij}$, $i,j = 0,1,2$, and the unity element is $1 = e_1 + e_2$.

The unique split quaternion algebra is, up to isomorphism, the subalgebra spanned by $\{e_1, e_2, u_1, v_1\}$, which in turn is isomorphic to the associative algebra of order 2 matrices over k: $\mathrm{Mat}_2(k)$. The unique split composition algebra of dimension 2 is the subalgebra $ke_1 + ke_2$, which is isomorphic to $k \times k$.

The symmetric group of degree 4, denoted by S_4, is generated by the permutations

$$\tau_1 = (12)(34), \quad \tau_2 = (23)(14), \quad \varphi = (123), \quad \tau = (12), \tag{1.4}$$

which satisfy the relations:

$$\tau_1\tau_2=\tau_2\tau_1,\quad \varphi\tau_1=\tau_2\varphi,\quad \varphi\tau_2=\tau_1\tau_2\varphi,$$
$$\tau_1\tau=\tau\tau_1,\quad \tau_2\tau=\tau\tau_2\tau_1,\quad \tau\varphi=\varphi^2\tau.$$

The subgroup generated by τ_1 and τ_2 is Klein's 4-group V, the one generated by τ_1, τ_2 and φ is the alternating group A_4.

Let C be the split Cayley algebra and take a basis as in (1.3). The symmetric group S_4 embeds in the automorphism group of C as follows (the automorphisms of C will be denoted by the same Greek letters):

$$\text{(1.5)}\quad \begin{cases} e_1 \text{ and } e_2 \text{ are fixed by any element of } S_4,\\ \tau_1(u_0)=u_0,\ \tau_2(u_0)=-u_0,\ \tau_1(v_0)=v_0,\ \tau_2(v_0)=-v_0,\\ \tau_1(u_1)=-u_1,\ \tau_2(u_1)=u_1,\ \tau_1(v_1)=-v_1,\ \tau_2(v_1)=v_1,\\ \tau_1(u_2)=-u_2,\ \tau_2(u_2)=-u_2,\ \tau_1(v_2)=-v_2,\ \tau_2(v_2)=-v_2,\\ \varphi(u_i)=u_{i+1},\ \varphi(v_i)=v_{i+1},\ \text{indices modulo 3},\\ \tau(u_0)=-u_0,\ \tau(u_1)=-u_2,\ \tau(u_2)=-u_1,\\ \tau(v_0)=-v_0,\ \tau(v_1)=-v_2,\ \tau(v_2)=-v_1. \end{cases}$$

The action of Klein's 4 group V gives a grading of C over $\mathbb{Z}_2\times\mathbb{Z}_2$:

$$\text{(1.6)}\qquad C=C_{(\bar 0,\bar 0)}\oplus C_{(\bar 1,\bar 0)}\oplus C_{(\bar 0,\bar 1)}\oplus C_{(\bar 1,\bar 1)},$$

where

$$\text{(1.7)}\qquad \begin{matrix} C_{(\bar 0,\bar 0)}=ke_1+ke_2, & C_{(\bar 1,\bar 0)}=ku_0+kv_0,\\ C_{(\bar 0,\bar 1)}=ku_1+kv_1, & C_{(\bar 1,\bar 1)}=ku_2+kv_2. \end{matrix}$$

Any automorphism ψ of C induces an automorphism of $\mathfrak{der}C$: $d\mapsto \psi d\psi^{-1}$. Note that one has $D_{1,a}=0$ for any $a\in C$, so that

$$\mathfrak{der}C=D_{C,C}=D_{C^0,C^0}=D_{U,V}+D_{U,U}+D_{V,V}+D_{e_1-e_2,U}+D_{e_2-e_1,V},$$

where U (respectively V) denotes here the span of the u_i's (resp. v_i's). But, because of (1.2)

$$D_{U,U}=D_{V^2,U}\subseteq D_{V,UV+VU}\subseteq D_{ke_1+ke_2,V}=D_{e_2-e_1,V},$$

and, similarly, $D_{V,V}\subseteq D_{e_1-e_2,U}$. Since the decomposition $C=(ke_1+ke_2)\oplus U\oplus V$ is a grading of C over $\mathbb{Z}_3$, which induces a grading of $\mathfrak{der}C$, it follows that

$$\mathfrak{der}C=D_{U,V}\oplus D_{e_1-e_2,U}\oplus D_{e_2-e_1,V}$$

is the associated grading of $\mathfrak{der}C$ over $\mathbb{Z}_3$.

The action of Klein's 4 group on $\mathfrak{Der} C$ produces an associated grading on $\mathfrak{Der} C$:

$$\mathfrak{Der} C = (\mathfrak{Der} C)_{(\bar 0,\bar 0)} \oplus (\mathfrak{Der} C)_{(\bar 1,\bar 0)} \oplus (\mathfrak{Der} C)_{(\bar 0,\bar 1)} \oplus (\mathfrak{Der} C)_{(\bar 1,\bar 1)}. \tag{1.8}$$

Let us compute $(\mathfrak{Der} C)_{(\bar 1,\bar 0)} = D_{C_{(\bar 0,\bar 0)},C_{(\bar 1,\bar 0)}} + D_{C_{(\bar 0,\bar 1)},C_{(\bar 1,\bar 1)}}$. Because of (1.2)

$$\begin{aligned} D_{u_1,u_2} &= D_{v_2v_0,u_2} = -D_{v_0u_2,v_2} - D_{u_2v_2,v_0} \\ &= D_{e_1,v_0} = -D_{\frac{1}{2}-e_1,v_0} = -\frac{1}{2}D_{e_2-e_1,v_0}, \end{aligned} \tag{1.9}$$

and, with the same argument, $D_{v_1,v_2} = -\frac{1}{2}D_{e_1-e_2,u_0}$. Therefore,

$$(\mathfrak{Der} C)_{(\bar 1,\bar 0)} = \operatorname{span}\{D_{e_1-e_2,u_0}, D_{e_2-e_1,v_0}, D_{u_1,v_2}, D_{v_1,u_2}\}. \tag{1.10}$$

Using the multiplication table in (1.3), the next equations follow:

$$\begin{aligned} &D_{u_i,v_j}(e_l) = [[u_i,v_j],e_l] + 3(u_i,e_l,v_j) = 0, \\ &D_{e_1-e_2,u_i}(e_1) = [[e_1-e_2,u_i],e_1] + 3(e_1-e_2,e_1,u_i) = 2[u_i,e_1] = -2u_i, \\ &D_{u_i,v_{i+1}}(u_i) = 3(u_i,u_i,v_{i+1}) = 0, \\ &D_{u_i,v_{i+1}}(v_i) = 3(u_i,v_i,v_{i+1}) = 3v_{i+1}, \end{aligned}$$

for any $i,j = 0,1,2$ (indices modulo 3), and by symmetry $e_1 \leftrightarrow e_2$, $u_i \leftrightarrow v_i$, one has also

$$D_{v_i,u_{i+1}}(v_i) = 0, \quad D_{v_i,u_{i+1}}(u_i) = 3u_{i+1}.$$

From here it follows that the elements in (1.10) are linearly independent, so that $\dim(\mathfrak{Der} C)_{(\bar 1,\bar 0)} = 4$.

2. Tits construction

Some results in [**BZ96**, Sections 3 and 4] (see also [**Tit66**] and [**BE03**]) will be reviewed in this section.

Let C be a unital composition algebra over the ground field k with norm n and trace t. Let J be a unital Jordan algebra with a *normalized trace* $t_J\colon J \to k$. That is, t_J is a linear map such that $t_J(1) = 1$ and $t_J\big((xy)z\big) = t_J\big(x(yz)\big)$ for any $x,y,z \in J$. Then $J = k1 \oplus J^0$, where $J^0 = \{x \in J : t_J(x) = 0\}$. For $x,y \in J^0$,

$$xy = t_J(xy)1 + x * y, \tag{2.1}$$

where $x * y = xy - t_J(xy)1$ gives a commutative multiplication on J^0. For $x,y \in J$, the linear map $d_{x,y}\colon J \to J$ defined by

$$d_{x,y}(z) = x(yz) - y(xz), \tag{2.2}$$

is the inner derivation of J determined by the elements x and y. Since $d_{1,x} = 0$ for any x, it is enough to deal with the inner derivations $d_{x,y}$, with $x, y \in J^0$.

Given C and J as before, consider the space

$$\mathcal{T}(C,J) = \mathfrak{der} C \oplus \left(C^0 \otimes J^0\right) \oplus d_{J,J} \tag{2.3}$$

(unadorned tensor products are always considered over k), with the anticommutative multiplication $[.,.]$ specified by:

(2.4)
- $\mathfrak{der} C$ and $d_{J,J}$ are Lie subalgebras,
- $[\mathfrak{der} C, d_{J,J}] = 0$,
- $[D, a \otimes x] = D(a) \otimes x$, $[d, a \otimes x] = a \otimes d(x)$,
- $[a \otimes x, b \otimes y] = t_J(xy) D_{a,b} + \left([a,b] \otimes x * y\right) + 2t(ab) d_{x,y}$,

for all $D \in \mathfrak{der} C$, $d \in d_{J,J}$, $a, b \in C^0$, and $x, y \in J^0$. Here the bracket $[.,.]$ follows the conventions in [**BE03**, (1.4)].

The conditions for $\mathcal{T}(C,J)$ to be a Lie algebra are the following:

$$\text{(2.5)}\quad \begin{aligned} &\text{(i)} \quad \sum_{\circlearrowleft} t\left([a_1, a_2] a_3\right) d_{(x_1 * x_2), x_3} = 0,\\ &\text{(ii)} \quad \sum_{\circlearrowleft} t_J\left((x_1 * x_2) x_3\right) D_{[a_1,a_2],a_3} = 0,\\ &\text{(iii)} \quad \sum_{\circlearrowleft} \Big(D_{a_1,a_2}(a_3) \otimes t_J\left(x_1 x_2\right) x_3 + [[a_1,a_2],a_3] \otimes (x_1 * x_2) * x_3 \\ &\qquad\qquad + 2t(a_1 a_2) a_3 \otimes d_{x_1,x_2}(x_3) \Big) = 0 \end{aligned}$$

for any $a_1, a_2, a_3 \in C^0$ and any $x_1, x_2, x_3 \in J^0$. The notation "$\sum_{\circlearrowleft}$" indicates summation over the cyclic permutation of the variables.

These conditions appear in [**BE03**, Proposition 1.5], but there they are stated in the more general setting of superalgebras, a setting we will deal with later on. In particular, these conditions are fulfilled if J is a separable Jordan algebra of degree three over k and $t_J = \frac{1}{3} T$, where T denotes the generic trace of J (see for instance [**Jac68**]).

3. The algebra $\left(\mathcal{A}(J), -\right)$

Let J be a unital Jordan algebra over k with a normalized trace t_J as in the previous section. For any $x, y \in J$, consider the new commutative product on J defined by

$$x \times y = 2xy - 3t_J(x) y - 3t_J(y) x + \left(9 t_J(x) t_J(y) - 3 t_J(xy)\right) 1, \tag{3.1}$$

for any $x, y \in J$. Note that for any $x, y \in J^0$ the following holds:

(3.2a) $\quad 1 \times 1 = 2,$

(3.2b) $\quad 1 \times x = -x,$

(3.2c) $\quad x \times y = 2xy - 3t_J(xy)1 = 2x * y - t_J(xy)1.$

In case J is a separable Jordan algebra of degree 3 with generic trace T and generic norm N, and with $t_J = \frac{1}{3}T$, this is the cross product considered in [**All78**, p. 148], corresponding to the admissible triple (T, N, N) on the pair (J, J). Now, as in [**All78**], consider the space

$$\mathcal{A}(J) = \left\{ \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} : \alpha, \beta \in k,\ x, y \in J \right\},$$

with multiplication given by

$$\begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \begin{pmatrix} \alpha' & x' \\ y' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + 3t_J(xy') & \alpha x' + \beta' x + y \times y' \\ \alpha' y + \beta y' + x \times x' & \beta\beta' + 3t_J(yx') \end{pmatrix}$$

for any $\alpha, \beta, \alpha', \beta' \in k$, and $x, y, x', y' \in J$. It follows that the map:

$$\begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \xrightarrow{-} \begin{pmatrix} \beta & x \\ y & \alpha \end{pmatrix}$$

is an involution (involutive antiautomorphism) of $\mathcal{A}(J)$. Besides, $\mathcal{A}(J)$ is unital with $1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and it is shown to be a structurable algebra in [**All78**, p. 148]. An easy proof is given here in Corollary 4.5.

Recall [**All78**] that a unital algebra with involution $(B, -)$ is said to be *structurable* if $[T_z, V_{x,y}] = V_{T_z x, y} - V_{x, T_{\bar{z}} y}$ for any $x, y, z \in B$, where $V_{x,y} z = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$ and $T_x = V_{x,1}$. These algebras coordinatize some 5-graded Lie algebras.

Remark 3.3. Let A be a commutative algebra endowed with a cubic form $N \colon A \to k$ such that $(x^2)^2 = N(x)x$ for any $x \in A$. These algebras have been called *admissible cubic algebras* in [**EO00**], where the relationships of these algebras to Jordan algebras have been studied. Then there is a symmetric associative bilinear form $\langle .|. \rangle$ on A, called the *trace form*, such that $N(x) = \langle x | x^2 \rangle$ for any $x \in A$. If $N \neq 0$, then $\langle .|. \rangle$ is uniquely determined. Then the trilinear form given by $N(x, y, z) = 6\langle x | yz \rangle$ satisfies $N(x, x, x) = 6N(x)$ for any x, and hence $\bigl(3\langle .|. \rangle, N, N\bigr)$ is an admissible triple on (A, A) in the sense of [**All78**, p. 148], with associated cross product $x \times y = 2xy$ for any $x, y \in A$, so $x^\sharp = \frac{1}{2} x \times x = x^2$. Then, as before, the linear space

$$\mathcal{A}(A) = \left\{ \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} : \alpha, \beta \in k,\ x, y \in A \right\},$$

with multiplication given by

$$\begin{pmatrix}\alpha & x\\ y & \beta\end{pmatrix}\begin{pmatrix}\alpha' & x'\\ y' & \beta'\end{pmatrix}=\begin{pmatrix}\alpha\alpha'+3\langle x|y'\rangle & \alpha x'+\beta' x+2yy'\\ \alpha' y+\beta y'+2xx' & \beta\beta'+3\langle y|x'\rangle\end{pmatrix} \tag{3.4}$$

for any $\alpha, \beta, \alpha', \beta' \in k$, and $x, y, x', y' \in A$, is a structurable algebra, whose involution is given by

$$\begin{pmatrix}\alpha & x\\ y & \beta\end{pmatrix}\xrightarrow{-}\begin{pmatrix}\beta & x\\ y & \alpha\end{pmatrix}.$$

4. An S_4-action on the Tits construction

Let C be the split Cayley algebra over k and let J be any unital Jordan algebra with a normalized trace t_J such that $\mathcal{T} = \mathcal{T}(C, J)$ is a Lie algebra. Then the action of the symmetric group S_4 as automorphisms of C in (1.5) extends to an action by automorphisms on $\mathcal{T} = \mathfrak{der}C \oplus \big(C^0 \otimes J^0\big) \oplus d_{J,J}$ given by:

$$\psi\big(D + (a \otimes x) + d\big) = \psi D\psi^{-1} + (\psi(a) \otimes x) + d, \tag{4.1}$$

for any $\psi \in S_4$, $D \in \mathfrak{der}C$, $d \in d_{J,J}$, $a \in C^0$ and $x \in J^0$.

As in (1.6), the action of Klein's 4-group induces a grading over $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\mathcal{T} = \mathcal{T}(C, J) = \mathcal{T}_{(\bar 0,\bar 0)} \oplus \mathcal{T}_{(\bar 1,\bar 0)} \oplus \mathcal{T}_{(\bar 0,\bar 1)} \oplus \mathcal{T}_{(\bar 1,\bar 1)}. \tag{4.2}$$

Under these circumstances (see [**EO07**, Section 2]), the subspace $\mathcal{T}_{(\bar 1,\bar 0)} = \{X \in \mathcal{T} : \tau_1(X) = X,\ \tau_2(X) = -X\}$ becomes an algebra with involution by means of:

$$\begin{aligned} X \cdot Y &= -\tau\big([\varphi(X), \varphi^2(Y)]\big),\\ \bar X &= -\tau(X), \end{aligned} \tag{4.3}$$

for any $X, Y \in \mathcal{T}_{(\bar 1,\bar 0)}$. Here $\varphi = (123)$ and $\tau = (12)$ as in (1.4).

The purpose of this section is to prove the following result:

Theorem 4.4. *Let C be the split Cayley algebra over k and let J be a unital Jordan algebra with a normalized trace t_J over k such that the Tits algebra $\mathcal{T} = \mathcal{T}(C, J)$ is a Lie algebra. Then the algebra with involution $\big(\mathcal{T}_{(\bar 1,\bar 0)}, \cdot, -\big)$ is isomorphic to the algebra $\big(\mathcal{A}(J), -\big)$ defined in Section 3.*

Before going into the proof of this result, let us note the next well-known consequence (see [**All78**, p. 148]), which follows immediately from [**EO07**, Theorem 2.9] and the fact that the algebra $\mathcal{A}(J)$ is unital:

Corollary 4.5. *Under the conditions of Theorem 4.4, the algebra with involution $\big(\mathcal{A}(J), -\big)$ is a structurable algebra.*

To prove Theorem 4.4, first note that, because of (4.1), (1.7), and (1.10), one has

$$\begin{aligned}\mathcal{T}_{(\bar 1,\bar 0)} &= \bigl\{X\in\mathcal{T} : \tau_1(X)=X,\ \tau_2(X)=-X\bigr\}\\ &= (\mathfrak{der}C)_{(\bar 1,\bar 0)}\oplus\bigl(C_{(\bar 1,\bar 0)}\otimes J^0\bigr)\\ &= \operatorname{span}\{D_{u_1,v_2},D_{v_1,u_2},D_{e_1-e_2,u_0},D_{e_2-e_1,v_0}\}\oplus\bigl(u_0\otimes J^0\bigr)\oplus\bigl(v_0\otimes J^0\bigr).\end{aligned}$$

Consider the bijective linear map

$$\begin{aligned}\Phi\colon \mathcal{T}_{(\bar 1,\bar 0)} &\longrightarrow \mathcal{A}(J)\\ D_{v_1,u_2} &\longmapsto \begin{pmatrix}3&0\\0&0\end{pmatrix}\\ D_{u_1,v_2} &\longmapsto \begin{pmatrix}0&0\\0&3\end{pmatrix}\\ D_{e_1-e_2,u_0} &\longmapsto \begin{pmatrix}0&2\\0&0\end{pmatrix}\\ D_{e_2-e_1,v_0} &\longmapsto \begin{pmatrix}0&0\\2&0\end{pmatrix}\\ u_0\otimes x &\longmapsto \begin{pmatrix}0&x\\0&0\end{pmatrix}\\ v_0\otimes x &\longmapsto \begin{pmatrix}0&0\\x&0\end{pmatrix}\end{aligned}$$

where $x\in J^0$. To prove Theorem 4.4 it is enough to prove that Φ is a homomorphism of algebras with involution. Note that there is the order 2 automorphism of C given by $e_1\leftrightarrow e_2$, $u_i\leftrightarrow v_i$, which extends to an order 2 automorphism ϵ of $\mathcal{T}(C,J)$. On the other hand, there is the natural order 2 automorphism of $\mathcal{A}(J)$ given by $\left(\begin{smallmatrix}\alpha&x\\y&\beta\end{smallmatrix}\right)\overset{\delta}{\leftrightarrow}\left(\begin{smallmatrix}\beta&y\\x&\alpha\end{smallmatrix}\right)$ which satisfies $\Phi\epsilon=\delta\Phi$. This simplifies the number of computations to be done. Thus, it is enough to check that $\Phi(X\cdot Y)=\Phi(X)\Phi(Y)$ for the following pairs X, Y in $\mathcal{T}_{(\bar 1,\bar 0)}$:

(i) $X=Y=D_{v_1,u_2}$: Here

$$\begin{aligned}D_{v_1,u_2}\cdot D_{v_1,u_2} &= -\tau\Bigl([\varphi(D_{v_1,u_2}),\varphi^2(D_{v_1,u_2})]\Bigr)\\ &= -\tau\Bigl([D_{v_2,u_0},D_{v_0,u_1}]\Bigr)\\ &= -\tau\Bigl(D_{D_{v_2,u_0}(v_0),u_1}+D_{v_0,D_{v_2,u_0}(u_1)}\Bigr).\end{aligned}$$

But $D_{v_2,u_0}(v_0)=3(v_2,v_0,u_0)=-3v_2$ and $D_{v_2,u_0}(u_1)=3(v_2,u_1,u_0)=0$, so that

$$\begin{aligned} D_{v_1,u_2}\cdot D_{v_1,u_2} &= -\tau\bigl(-3D_{v_2,u_1}\bigr) = -\tau\bigl(3D_{u_1,v_2}\bigr) \\ &= -3D_{\tau(u_1),\tau(v_2)} = -3D_{u_2,v_1} = 3D_{v_1,u_2}. \end{aligned}$$

Therefore,

$$\Phi\bigl(D_{v_1,u_2}\cdot D_{v_1,u_2}\bigr) = 3\Phi\bigl(D_{v_1,u_2}\bigr) = \left(\begin{smallmatrix} 9 & 0\\ 0 & 0\end{smallmatrix}\right),$$

while

$$\Phi\bigl(D_{v_1,u_2}\bigr)\Phi\bigl(D_{v_1,u_2}\bigr) = \left(\begin{smallmatrix} 3 & 0\\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 0\\ 0 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} 9 & 0\\ 0 & 0\end{smallmatrix}\right).$$

(ii) $X = D_{v_1,u_2}$, $Y = D_{u_1,v_2}$: This is easier since

$$[\varphi(D_{v_1,u_2}),\varphi^2(D_{u_1,v_2})] = [D_{v_2,u_0}, D_{u_0,v_1}] = 0,$$

as $D_{v_2,u_0}(u_0) = 0 = D_{v_2,u_0}(v_1)$. Hence both $X\cdot Y$ and $\Phi(X)\Phi(Y)$ are 0.

(iii) $X = D_{v_1,u_2}$, $Y = \frac{1}{2}D_{e_1-e_2,u_0} = D_{e_1,u_0}$: Here

$$[\varphi(D_{v_1,u_2}),\varphi^2(D_{e_1,u_0})] = [D_{v_2,u_0}, D_{e_1,u_2}] = 3D_{e_1,u_0},$$

because $D_{v_2,u_0}(e_1) = 0$ and $D_{v_2,u_0}(u_2) = 3(v_2,u_2,u_0) = -3v_2v_1 = 3u_0$. Thus,

$$X\cdot Y = -\tau\bigl(3D_{e_1,u_0}\bigr) = -3D_{\tau(e_1),\tau(u_0)} = 3D_{e_1,u_0} = 3Y,$$

while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 3 & 0\\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 3\\ 0 & 0\end{smallmatrix}\right) = 3\Phi(Y).$$

(iv) $X = D_{v_1,u_2}$, $Y = \frac{1}{2}D_{e_2-e_1,v_0} = D_{e_2,v_0}$: In this case $\Phi(X)\Phi(Y) = 0$, but also

$$[\varphi(D_{v_1,u_2}),\varphi^2(D_{e_2,v_0})] = [D_{v_2,u_0}, D_{e_2,v_2}] = 0,$$

as $D_{v_2,u_0}(e_2) = 0 = D_{v_2,u_0}(v_2)$, so $X\cdot Y = 0$ too.

(v) $X = D_{v_1,u_2}$, $Y = u_0\otimes x$, with $x\in J^0$: Here

$$\begin{aligned} [\varphi(D_{v_1,u_2}),\varphi^2(u_0\otimes x)] &= [D_{v_2,u_0}, u_2\otimes x] = D_{v_2,u_0}(u_2)\otimes x \\ &= 3(v_2,u_2,u_0)\otimes x = 3u_0\otimes x. \end{aligned}$$

Hence, $X\cdot Y = -\tau(3u_0\otimes x) = 3u_0\otimes x = 3Y$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 3 & 0\\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & x\\ 0 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 3x\\ 0 & 0\end{smallmatrix}\right) = 3\Phi(Y).$$

(vi) $X = D_{v_1,u_2}$, $Y = v_0\otimes x$, with $x\in J^0$: Then

$$[\varphi(D_{v_1,u_2}),\varphi^2(v_0\otimes x)] = [D_{v_2,u_0}, v_2\otimes x] = D_{v_2,u_0}(v_2)\otimes x = 0,$$

and both $X\cdot Y$ and $\Phi(X)\Phi(Y) = 0$.

(vii) $X = D_{e_1,u_0}$, $Y = D_{v_1,u_2}$: $[\varphi(X), \varphi^2(Y)] = [D_{e_1,u_1}, D_{v_0,u_1}] = 0$, as $D_{e_1,u_1}(v_0) = 0 = D_{e_1,u_1}(u_1)$, so $X \cdot Y = 0$, but also

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0.$$

(viii) $X = D_{e_1,u_0}$, $Y = D_{u_1,v_2}$: One has

$$\begin{aligned}
[\varphi(X), \varphi^2(Y)] &= [D_{e_1,u_1}, D_{u_0,v_1}] \\
&= -D_{D_{u_0,v_1}(e_1),u_1} - D_{e_1,D_{u_0,v_1}(u_1)} \\
&= -D_{e_1,3(u_0,u_1,v_1)} = 3D_{e_1,u_0},
\end{aligned}$$

so that $X \cdot Y = -\tau(3D_{e_1,u_0}) = 3D_{e_1,u_0} = 3X$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 3 \\ 0 & 0 \end{smallmatrix}\right) = 3\Phi(X).$$

(ix) $X = D_{e_1,u_0}$, $Y = D_{e_1,u_0}$: Here

$$\begin{aligned}
[\varphi(X), \varphi^2(Y)] &= [D_{e_1,u_1}, D_{e_1,u_2}] \\
&= D_{D_{e_1,u_1}(e_1),u_2} + D_{e_1,D_{e_1,u_1}(u_2)} \\
&= D_{-u_1,u_2} + D_{e_1,-v_0} = -2D_{e_1,v_0} = 2D_{e_2,v_0}
\end{aligned}$$

(see (1.9)). Hence $X \cdot Y = -\tau\bigl(2D_{e_2,v_0}\bigr) = 2D_{e_2,v_0}$ and $\Phi(X \cdot Y) = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 1\times 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right),$$

because of (3.2a).

(x) $X = D_{e_1,u_0}$, $Y = D_{e_2,v_0}$: Then

$$\begin{aligned}
[\varphi(X), \varphi^2(Y)] &= [D_{e_1,u_1}, D_{e_2,v_2}] \\
&= D_{D_{e_1,u_1}(e_2),v_2} + D_{e_2,D_{e_1,u_1}(v_2)} \\
&= D_{u_1,v_2} + 0 = D_{u_1,v_2}.
\end{aligned}$$

Thus, $X \cdot Y = -\tau\bigl(D_{u_1,v_2}\bigr) = -D_{\tau(u_1),\tau(v_2)} = -D_{u_2,v_1} = D_{v_1,u_2}$, so $\Phi(X \cdot Y) = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

(xi) $X = D_{e_1,u_0}$, $Y = u_0 \otimes x$, $x \in J^0$: In this case

$$[\varphi(X), \varphi^2(Y)] = [D_{e_1,u_1}, u_2 \otimes x] = D_{e_1,u_1}(u_2) \otimes x = -v_0 \otimes x,$$

so $X \cdot Y = -\tau(-v_0 \otimes x) = -v_0 \otimes x$ and $\Phi(X \cdot Y) = \left(\begin{smallmatrix} 0 & 0 \\ -x & 0 \end{smallmatrix}\right)$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 1\times x & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ -x & 0 \end{smallmatrix}\right),$$

because of (3.2b).

(xii) $X = D_{e_1,u_0}$, $Y = v_0 \otimes x$, $x \in J^0$: For these X and Y

$$[\varphi(X), \varphi^2(Y)] = [D_{e_1,u_1}, v_2 \otimes x] = D_{e_1,u_1}(v_2) \otimes x = 0,$$

so $X \cdot Y = 0$. Here $\Phi(x)\Phi(Y) = 0$ too, since $t_J(x) = 0$.

(xiii) $X = u_0 \otimes x$, $Y = D_{v_1,u_2}$, $x \in J^0$: Here

$$[\varphi(X), \varphi^2(Y)] = [u_1 \otimes x, D_{v_0,u_1}] = -D_{v_0,u_1}(u_1) \otimes x = 0,$$

so $X \cdot Y = 0$ and also

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0.$$

(xiv) $X = u_0 \otimes x$, $Y = D_{u_1,v_2}$, $x \in J^0$: Now

$$\begin{aligned}[\varphi(X), \varphi^2(Y)] &= [u_1 \otimes x, D_{u_0,v_1}] = -D_{u_0,v_1}(u_1) \otimes x \\ &= -3(u_0, u_1, v_1) \otimes x = 3u_0 \otimes x,\end{aligned}$$

so $X \cdot Y = -\tau(3u_0 \otimes x) = 3u_0 \otimes x = 3X$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 3x \\ 0 & 0 \end{smallmatrix}\right) = 3\Phi(X).$$

(xv) $X = u_0 \otimes x$, $Y = D_{e_1,u_0}$, $x \in J^0$: Here

$$[\varphi(X), \varphi^2(Y)] = [u_1 \otimes x, D_{e_1,u_2}] = -D_{e_1,u_2}(u_1) \otimes x = -v_0 \otimes x,$$

so $X \cdot Y = -\tau(-v_0 \otimes x) = -v_0 \otimes x$ and $\Phi(X \cdot Y) = \left(\begin{smallmatrix} 0 & 0 \\ -x & 0 \end{smallmatrix}\right)$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ x \times 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ -x & 0 \end{smallmatrix}\right),$$

because of (3.2b).

(xvi) $X = u_0 \otimes x$, $Y = D_{e_2,v_0}$, $x \in J^0$: In this case

$$[\varphi(X), \varphi^2(Y)] = [u_1 \otimes x, D_{e_2,v_2}] = -D_{e_2,v_2}(u_1) \otimes x = 0,$$

so $\Phi(X \cdot Y) = 0$, while $\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = 0$, as $t_J(x) = 0$.

(xvii) $X = u_0 \otimes x$, $Y = u_0 \otimes y$, $x, y \in J^0$: Here

$$\begin{aligned}[\varphi(X), \varphi^2(Y)] &= [u_1 \otimes x, u_2 \otimes y] \\ &= t_J(xy)D_{u_1,u_2} + [u_1, u_2] \otimes x * y + 2t(u_1u_2)d_{x,y} \\ &= -t_J(xy)D_{e_2,v_0} + 2v_0 \otimes x * y\end{aligned}$$

(see (1.9)). Hence

$$X \cdot Y = -\tau\big(-t_J(xy)D_{e_2,v_0} + 2v_0 \otimes x * y\big) = -t_J(xy)D_{e_2,v_0} + 2v_0 \otimes x * y,$$

and

$$\Phi(X \cdot Y) = \left(\begin{smallmatrix} 0 & 0 \\ -t_J(xy) + 2x*y & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ x \times y & 0 \end{smallmatrix}\right),$$

using (3.2c), while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ x \times y & 0 \end{smallmatrix}\right).$$

(xviii) $X = u_0 \otimes x$, $Y = v_0 \otimes y$, $x, y \in J^0$: In this final case

$$\begin{aligned}[\varphi(X), \varphi^2(Y)] &= [u_1 \otimes x, v_2 \otimes y] \\ &= t_J(xy) D_{u_1,v_2} + [u_1, v_2] \otimes x * y \\ &= t_J(xy) D_{u_1,v_2},\end{aligned}$$

so

$$\begin{aligned}X \cdot Y &= -\tau\big(t_J(xy) D_{u_1,v_2}\big) = -t_J(xy) D_{\tau(u_1),\tau(v_2)} \\ &= -t_J(xy) D_{u_2,v_1} = t_J(xy) D_{v_1,u_2},\end{aligned}$$

and $\Phi(X \cdot Y) = \left(\begin{smallmatrix} 3t_J(xy) & 0 \\ 0 & 0 \end{smallmatrix}\right)$, while

$$\Phi(X)\Phi(Y) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3t_J(xy) & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

Therefore, Φ is an algebra isomorphism. Besides, it is clear that $\Phi(\bar{X}) \Big(= \Phi\big(-\tau(X)\big)\Big) = \overline{\Phi(X)}$ for any $X \in \mathcal{T}_{(\bar{1},\bar{0})}$, so that Φ is an isomorphism of algebras with involution. This finishes the proof of the theorem.

Remark 4.6. Given the split Cayley algebra C and the unital Jordan algebra J as above, the subspace $Q = \operatorname{span}\{1 = e_1 + e_2, u_0 + v_0, u_1 + v_1, u_2 + v_2\}$ is a quaternion subalgebra of C, which is invariant under the action of the symmetric group S_4. Then so is $\mathcal{T}(Q, J) = D_{Q,Q} \oplus \big(Q^0 \otimes J^0\big) \oplus d_{J,J}$, which is a subalgebra of $\mathcal{T}(C, J)$. Besides,

$$\mathcal{T}(Q, J)_{(\bar{1},\bar{0})} = k D_{u_1+v_1,u_2+v_2} \oplus \big((u_0 + v_0) \otimes J^0\big).$$

Note that $D_{u_1,u_2} = -\frac{1}{2} D_{e_2-e_1,v_0}$ and $D_{v_1,v_2} = -\frac{1}{2} D_{e_1-e_2,u_0}$ by (1.9), so that

$$D_{u_1+v_1,u_2+v_2} = D_{v_1,u_2} + D_{u_1,v_2} - \frac{1}{2} D_{e_1-e_2,u_0} - \frac{1}{2} D_{e_2-e_1,v_0}.$$

Therefore, under the isomorphism Φ, $\mathcal{T}(Q, J)_{(\bar{1},\bar{0})}$ maps onto the following commutative subalgebra of $\mathcal{A}(J)$:

$$S = \left\{ \begin{pmatrix} 3\alpha & -\alpha + x \\ -\alpha + x & 3\alpha \end{pmatrix} : \alpha \in k,\, x \in J^0 \right\}.$$

Moreover, the linear isomorphism

$$\begin{aligned} J &\longrightarrow S \\ \alpha 1 + x &\longmapsto \begin{pmatrix} \frac{3}{4}\alpha & -\frac{1}{4}\alpha + \frac{1}{2}x \\ -\frac{1}{4}\alpha + \frac{1}{2}x & \frac{3}{4}\alpha \end{pmatrix}\end{aligned}$$

for $\alpha \in k$ and $x \in J^0$, is easily checked, using (3.2), to be an algebra isomorphism. Hence, the structurable algebra attached to $\mathcal{T}(Q, J)$ is just, up to isomorphism, the Jordan algebra J itself.

Consider the linear isomorphism

$$\begin{aligned} \mathcal{T}(Q,J) &\to \big(Q^0\otimes J\big)\oplus d_{J,J}\\ D_{a,b} &\mapsto [a,b]\otimes 1\\ a\otimes x &\mapsto a\otimes x\\ d &\mapsto d \end{aligned}$$

for any $a,b\in Q^0$, $x\in J^0$ and $d\in d_{J,J}$. Since Q is associative, $D_{a,b}(c)=[[a,b],c]$ for any $a,b,c\in Q$ by (1.1), and hence the map $D_{a,b}\mapsto [a,b]$ is well defined. Note that for any $a,b\in Q^0$ and $x,y\in J^0$, one has

$$[a\otimes x,b\otimes y]=t_J(x,y)D_{a,b}+[a,b]\otimes x*y+2t(ab)d_{x,y},$$

which maps into

$$[a,b]\otimes(t_J(xy)+x*y)+2t(ab)d_{x,y}=([a,b]\otimes xy)+2t(ab)d_{x,y}.$$

Hence this linear isomorphism is an isomorphism of Lie algebras, where the Lie bracket on $(Q^0\otimes J)\oplus d_{J,J}$ is determined by $[a\otimes x,b\otimes y]=([a,b]\otimes xy)+2t(ab)d_{x,y}$ for any $a,b\in Q^0$ and $x,y\in J$. This bracket makes sense for any quaternion algebra and any Jordan algebra (not necessarily unital nor endowed with a normalized trace), as shown in [**Tit62**]. (See also Remark 7.13.)

5. Superalgebras

As considered in [**BZ96**], [**BE03**], the Jordan algebra J in Tits construction can be replaced by a Jordan superalgebra, as long as the superalgebra version of (2.5) is fulfilled (see [**BE03**, Proposition 1.5]). In this case $\mathcal{T}(C,J)$ becomes a Lie superalgebra. In particular [**BE03**, Theorem 2.5], this is always the case for the Jordan superalgebra D_2 and the Jordan superalgebra $J=J(V,\vartheta)$ of a nondegenerate supersymmetric superform ϑ on the superspace $V=V_{\bar 0}\oplus V_{\bar 1}$ with $V_{\bar 0}=0$ and $\dim V_{\bar 1}=2$. Both Jordan superalgebras are endowed with a normalized trace.

With C the split Cayley algebra over k, $\mathcal{T}(C,J(V,\vartheta))$ is the simple Lie superalgebra $G(3)$, while $\mathcal{T}(C,D_2)$ is the simple Lie superalgebra $F(4)$. Hence the symmetric group S_4 acts on the Lie superalgebras $G(3)$ and $F(4)$ by automorphisms. In both cases, the superalgebra with superinvolution $\big(\mathcal{A}(J),-\big)$ can be defined as in Section 3.

The arguments in the previous section are valid in the superalgebra setting, as long as the appropriate parity signs are inserted. Therefore, as a consequence of Theorem 4.4 and Corollary 4.5, we get:

Theorem 5.1. *Let C be the split Cayley algebra over k and let J be one of the Jordan superalgebras $J = D_2$ or $J = J(V,\vartheta)$. Let $\mathcal{T} = \mathcal{T}(C,J)$ be the Lie superalgebra constructed by the Tits construction. Then the algebra with involution $\big(\mathcal{T}_{(\bar{1},\bar{0})},\cdot,-\big)$ is isomorphic to the algebra $\big(\mathcal{A}(J),-\big)$.*

Corollary 5.2. *The superalgebras with superinvolution $\big(\mathcal{A}(J),-\big)$, for $J = D_2$ and $J = J(V,\vartheta)$, are structurable superalgebras.*

Remark 5.3. The simple Lie superalgebras $D(2,1;\alpha)$ ($\alpha \neq 0,-1$), can be constructed directly from the Jordan superalgebras of type D_α as $\mathcal{T}(Q,D_\alpha)$ as in Remark 4.6, even though D_α has a normalized trace only for $\alpha = 2$ or $\frac{1}{2}$ (see [**BE03**]). □

Remark 5.4. Consider the 'tiny' Kaplansky superalgebra K, with even part $K_{\bar{0}} = ke$, odd part $K_{\bar{1}} = kx + ky$, and supercommutative multiplication given by $e^2 = e$, $ex = \frac{1}{2}x$, $ey = \frac{1}{2}y$, and $xy = e$. K is a simple nonunital Jordan superalgebra. Then K is an admissible cubic superalgebra with $N(z) = \langle z|z^2\rangle$, where $\langle .|.\rangle$ is the supersymmetric bilinear form such that $\langle e|e\rangle = 1$ and $\langle x|y\rangle = 2$. Thus we obtain a structurable superalgebra $\mathcal{A}(K)$ as in Remark 3.3, with the product given by (3.4).

On the other hand, for the Jordan superalgebra $J = J(V,\vartheta)$, one has $J_{\bar{0}} = k1$, $J_{\bar{1}} = ku+kv$, with $uv = 1$. Then a straightforward computation shows that the linear map:

$$\mathcal{A}(K) \longrightarrow \mathcal{A}(J)$$

$$\begin{pmatrix} \alpha_1 & \gamma_1 e+\mu_1 x+\nu_1 y \\ \gamma_2 e+\mu_2 x+\nu_2 y & \alpha_2 \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_1 & \gamma_1 1-\mu_1 u+2\nu_1 v \\ \gamma_2 1+\mu_2 u-2\nu_2 v & \alpha_2 \end{pmatrix}$$

where $\alpha_i,\gamma_i,\mu_i,\nu_i \in k$ ($i = 1,2$), is an isomorphism of structurable algebras.

6. Another action of S_4 on the Tits construction

In this section, an action of the symmetric group S_4 by automorphisms on the Jordan algebra of hermitian 3×3 matrices over a unital composition algebra will be considered. This is extended naturally to an action of S_4 by automorphisms on the Tits construction, which gives rise to a structurable algebra. This latter algebra is isomorphic to the tensor product of the two composition algebras involved in the Tits construction. Therefore, this S_4-action clarifies the relationship between the Tits construction and the construction of the Lie algebras in the Magic Square by means of a couple of composition algebras (see [**BS03**], [**LM02**], or [**Eld04**]).

Let $\hat{C}$ be a unital composition algebra over our ground field k, and consider the Jordan algebra of 3×3 hermitian matrices over $\hat{C}$:

$$J = H_3(\hat{C}) = \begin{pmatrix} \alpha_1 & x_0 & \bar{x}_2 \\ \bar{x}_0 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_0 \end{pmatrix},$$

with $\alpha_i \in k$ and $x_i \in \hat{C}$, $i = 0, 1, 2$. The following notations will be used:

$$e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\iota_0(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \iota_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \quad \iota_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}.$$

J is a Jordan algebra with the product given by $X \circ Y = \frac{1}{2}(XY + YX)$, which satisfies:

$$e_i \circ e_j = \delta_{ij} e_i,$$

$$e_{i+1} \circ \iota_i(x) = e_{i+2} \circ \iota_i(x) = \frac{1}{2}\iota_i(x), \quad e_i \circ \iota_i(x) = 0,$$

$$\iota_i(x) \circ \iota_{i+1}(y) = \frac{1}{2}\iota_{i+2}(\overline{xy}),$$

$$\iota_i(x) \circ \iota_i(y) = \frac{1}{2}t(x\bar{y})(e_{i+1} + e_{i+2}),$$

where $x, y \in \hat{C}$, t denotes the trace in $\hat{C}$, $x \mapsto \bar{x}$ the canonical involution, and the indices are taken modulo 3.

The symmetric group S_4 embeds in the automorphism group of J as follows:

$$\tau_1 \colon e_i \mapsto e_i,\ \iota_0(x) \mapsto \iota_0(x),\ \iota_1(x) \mapsto -\iota_1(x),\ \iota_2(x) \mapsto -\iota_2(x),$$

$$\tau_2 \colon e_i \mapsto e_i,\ \iota_0(x) \mapsto -\iota_0(x),\ \iota_1(x) \mapsto \iota_1(x),\ \iota_2(x) \mapsto -\iota_2(x),$$

$$\varphi \colon e_i \mapsto e_{i+1},\ \iota_i(x) \mapsto \iota_{i+1}(x),$$

$$\tau \colon e_0 \mapsto e_0,\ e_1 \mapsto e_2,\ e_2 \mapsto e_1,\ \iota_0(x) \mapsto \iota_0(\bar{x}),\ \iota_1(x) \mapsto \iota_2(\bar{x}),\ \iota_2(x) \mapsto \iota_1(\bar{x}),$$

for any $i = 0, 1, 2$ (indices modulo 3) and $x \in \hat{C}$.

The arguments in [**Sch95**, Chapter IV, §9] show that there is the following grading over $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the Lie algebra of derivations of J:

$$\mathfrak{Der} J = d_{J,J} = \{d \in \mathfrak{Der} J : d(e_i) = 0\ (i = 0, 1, 2)\} \oplus \left(\oplus_{i=0}^{2} d_{e_{i+1} - e_{i+2}, \iota_i(\hat{C})}\right),$$

where $d_{x,y}$ is defined as in (2.2): $d_{x,y}(z) = x \circ (y \circ z) - y \circ (x \circ z)$. This is precisely the grading induced by the action of Klein's 4-group.

The action of S_4 on J extends to an action of S_4 by automorphisms of the Lie algebra $\mathcal{T}(C, J)$, where C is another unital composition algebra over k:

$$\psi\big(D + (a \otimes x) + d\big) = D + (a \otimes \psi(x)) + \psi d \psi^{-1}, \tag{6.1}$$

for any $\psi \in S_4$, $D \in \mathfrak{der} C$, $d \in d_{J,J}$, $a \in C^0$ and $x \in J^0$.

As in (1.6) and (4.2), the action of Klein's 4-group induces a grading over $\mathbb{Z}_2 \otimes \mathbb{Z}_2$:

$$\mathcal{T} = \mathcal{T}(C, J) = \mathcal{T}_{(\bar{0},\bar{0})} \oplus \mathcal{T}_{(\bar{1},\bar{0})} \oplus \mathcal{T}_{(\bar{0},\bar{1})} \oplus \mathcal{T}_{(\bar{1},\bar{1})}. \tag{6.2}$$

Again (see [**EO07**, Section 2]), the subspace $\mathcal{T}_{(\bar{1},\bar{0})} = \{X \in \mathcal{T} : \tau_1(X) = X,\ \tau_2(X) = -X\}$ becomes an algebra with involution by means of:

$$\begin{aligned} X \cdot Y &= -\tau\big([\varphi(X), \varphi^2(Y)]\big), \\ \bar{X} &= -\tau(X), \end{aligned} \tag{6.3}$$

for any $X, Y \in \mathcal{T}_{(\bar{1},\bar{0})}$, as in (4.3).

Theorem 6.4. *Let C and $\hat{C}$ be two unital composition algebras over k, whose traces will be both denoted by t, and let J be the Jordan algebra of 3×3 hermitian matrices over $\hat{C}$. Let $\mathcal{T} = \mathcal{T}(C, J)$ be the associated Tits algebra, with the action of the symmetric group S_4 given by* (6.1). *Then the algebra with involution $\big(\mathcal{T}_{(\bar{1},\bar{0})}, \cdot, -\big)$ is isomorphic to the structurable algebra $C \otimes \hat{C}$, with multiplication $(a \otimes x)(b \otimes y) = ab \otimes xy$ and involution $\overline{a \otimes x} = \bar{a} \otimes \bar{x}$, for any $a, b \in C$ and $x, y \in \hat{C}$.*

Proof: To begin with, the subspace $\mathcal{T}_{(\bar{1},\bar{0})}$ is

$$\mathcal{T}_{(\bar{1},\bar{0})} = \big(C^0 \otimes \iota_0(\hat{C})\big) \oplus d_{e_1 - e_2, \iota_0(\hat{C})}.$$

For ease of notation, write $d_i(x) = d_{e_{i+1} - e_{i+2}, \iota_i(x)}$ for any $i = 0, 1, 2$ (modulo 3) and $x \in \hat{C}$, so

$$\mathcal{T}_{(\bar{1},\bar{0})} = \big(C^0 \otimes \iota_0(\hat{C})\big) \oplus d_0(\hat{C}).$$

Consider the bijective linear map

$$\begin{aligned} \Phi\colon \mathcal{T}_{(\bar{1},\bar{0})} &\to C \otimes \hat{C} \\ a \otimes \iota_0(x) &\mapsto -a \otimes x, \\ d_0(x) &\mapsto -\frac{1}{2} 1 \otimes x, \end{aligned}$$

for $a \in C^0$ and $x \in \hat{C}$.

It is clear the $\Phi(\bar{X}) = \overline{\Phi(X)}$ for any $X \in \mathcal{T}_{(\bar{1},\bar{0})}$ since

$$- \tau(a \otimes \iota_0(x)) = -a \otimes \iota_0(\bar{x}) = \bar{a} \otimes \iota_0(\bar{x}), \text{ and}$$
$$- \tau(d_0(x)) = -[l_{\tau(e_1-e_2)}, l_{\tau(\iota_0(x))}] = -[l_{e_2-e_1}, l_{\iota_0(\bar{x})}] = d_0(\bar{x}),$$

for any $a \in C^0$ and $x \in \hat{C}$. Note that the standard involutions of both C and $\hat{C}$ are denoted by the same symbol.

To prove that Φ is an algebra homomorphism, the following instances of $\Phi(X \cdot Y) = \Phi(X)\Phi(Y)$ have to be checked:

(i) $X = d_0(x)$, $Y = d_0(y)$, with $x, y \in C$: Note that one has

$$\begin{aligned}\varphi(d_0(x)) &= \varphi\big([l_{e_1-e_2}, l_{\iota_0(x)}]\big)\\ &= [l_{\varphi(e_1-e_2)}, l_{\varphi(\iota_0(x))}]\\ &= [l_{e_2-e_0}, l_{\iota_1(x)}] = d_1(x),\end{aligned}$$

while $\varphi^2(d_0(y)) = d_2(y)$. Also, the equality $\tau(d_0(x)) = -d_0(\bar{x})$ was checked above. Hence,

$$X \cdot Y = -\tau\big([\varphi(d_0(x)), \varphi^2(d_0(y))]\big) = -\tau\big([d_1(x), d_2(y)]\big).$$

But $[d_1(x), d_2(y)]$ belongs to the subspace $d_0(\hat{C})$ (because of the grading of $\mathfrak{Der} J$ over $\mathbb{Z}_2 \times \mathbb{Z}_2$), so there is an element $z \in \hat{C}$ such that $[d_1(x), d_2(y)] = d_0(z)$. Now, for any $z \in \hat{C}$ and any $i = 0, 1, 2$, one has:

$$\begin{aligned}d_i(z)(e_{i+1}) &= [l_{e_{i+1}-e_{i+2}}, l_{\iota_i(z)}](e_{i+1})\\ &= (e_{i+1} - e_{i+2}) \circ (\iota_i(z) \circ e_{i+1}) - \iota_i(z) \circ ((e_{i+1} - e_{i+2}) \circ e_{i+1})\\ &= \frac{1}{2}(e_{i+1} - e_{i+2}) \circ \iota_i(z) - \iota_i(z) \circ e_{i+1}\\ &= -\frac{1}{2}\iota_i(z).\end{aligned}$$

In the same vein, one obtains:

$$d_i(z)(e_{i+1}) = -d_i(z)(e_{i+2}) = -\frac{1}{2}\iota_i(z), \quad d_i(z)(e_i) = 0, \tag{6.5}$$

for any $z \in \hat{C}$ and $i = 0, 1, 2$. Also,

$$\begin{aligned} d_i(z)(\iota_{i+1}(t)) &= (e_{i+1} - e_{i+2}) \circ (\iota_i(z) \circ \iota_{i+1}(t)) \\ &\quad - \iota_i(z) \circ ((e_{i+1} - e_{i+2}) \circ \iota_{i+1}(t)) \\ &= \frac{1}{2}(e_{i+1} - e_{i+2}) \circ \iota_{i+2}(\overline{zt}) + \frac{1}{2}\iota_i(z) \circ \iota_{i+1}(t) \\ &= \frac{1}{4}\iota_{i+2}(\overline{zt}) + \frac{1}{4}\iota_{i+2}(\overline{zt}) = \frac{1}{2}\iota_{i+2}(\overline{zt}), \end{aligned}$$

and, in the same vein,

$$\begin{aligned} d_i(z)(\iota_{i+1}(t)) &= \frac{1}{2}\iota_{i+2}(\overline{zt}), \\ d_i(z)(\iota_{i+2}(t)) &= -\frac{1}{2}\iota_{i+1}(\overline{tz}), \qquad (6.6) \\ d_i(z)(\iota_i(t)) &= \frac{1}{2}t(z\bar{t})(e_{i+1} - e_{i+2}) \end{aligned}$$

for any $z, t \in \hat{C}$ and $i = 0, 1, 2$.

Thus,

$$\begin{aligned} [d_1(x), d_2(y)](e_1) &= d_1(x)\big(d_2(y)(e_1)\big) - d_2(y)\big(d_1(x)(e_1)\big) \\ &= d_1(x)\big(\frac{1}{2}\iota_2(y)\big) = \frac{1}{4}\iota_0(\overline{xy}), \end{aligned}$$

and therefore,

$$[d_1(x), d_2(y)] = -\frac{1}{2}d_0(\overline{xy})$$

because of (6.5). Hence

$$X \cdot Y = -\tau\big([d_1(x), d_2(y)]\big) = \tau\big(\frac{1}{2}d_0(\overline{xy})\big) = -\frac{1}{2}d_0(xy),$$

and hence

$$\Phi(X \cdot Y) = \frac{1}{4}1 \otimes xy = \big(-\frac{1}{2}1 \otimes x\big)\big(-\frac{1}{2}1 \otimes y\big) = \Phi(X)\Phi(Y).$$

(ii) $X = d_0(x)$, $Y = a \otimes \iota_0(y)$, $x, y \in \hat{C}$, $a \in C^0$. Here

$$X \cdot Y = -\tau\big([\varphi(X), \varphi^2(Y)]\big) = -\tau\big([d_1(x), a \otimes \iota_2(y)]\big)$$

$$= -\tau\big(a \otimes d_1(x)(\iota_2(y))\big) = -\tau\big(a \otimes \frac{1}{2}\iota_0(\overline{xy})\big) \qquad \text{(by (6.6))}$$

$$= -\frac{1}{2} a \otimes \iota_0(xy),$$

so

$$\Phi(X \cdot Y) = \frac{1}{2} a \otimes xy = \big(-\frac{1}{2} 1 \otimes x\big)\big(-a \otimes y\big) = \Phi(X)\Phi(Y).$$

(iii) $X = a \otimes \iota_0(x)$, $Y = d_0(y)$, $a \in C^0$, $x, y \in \hat{C}$. In this case,

$$X \cdot Y = -\tau\big([\varphi(X), \varphi^2(Y)]\big) = -\tau\big([a \otimes \iota_1(x), d_2(y)]\big)$$

$$= \tau\big(a \otimes d_2(y)(\iota_1(x))\big) = \tau\big(-\frac{1}{2} a \otimes \iota_0(\overline{xy})\big) \qquad \text{(by (6.6))}$$

$$= -\frac{1}{2} a \otimes \iota_0(xy),$$

so

$$\Phi(X \cdot Y) = \frac{1}{2} a \otimes xy = \big(-a \otimes x\big)\big(-\frac{1}{2} 1 \otimes y\big) = \Phi(X)\Phi(Y).$$

(iv) $X = a \otimes \iota_0(x)$, $Y = b \otimes \iota_0(y)$, for $a, b \in C^0$ and $x, y \in \hat{C}$. Here

$$X \cdot Y = -\tau\big([\varphi(X), \varphi^2(Y)]\big) = -\tau\big([a \otimes \iota_1(x), b \otimes \iota_2(y)]\big)$$

$$= -\tau\big(([a, b] \otimes \frac{1}{2}\iota_0(\overline{xy})) \, + \, 2t(ab)[l_{\iota_1(x)}, l_{\iota_2(y)}]\big).$$

But $[l_{\iota_1(x)}, l_{\iota_2(y)}] = d_0(z)$ for some $z \in \hat{C}$, and $d_0(z)(e_1) = -\frac{1}{2}\iota_0(z)$ by (6.5), while

$$[l_{\iota_1(x)}, l_{\iota_2(y)}](e_1) = \iota_1(x) \circ (\iota_2(y) \circ e_1) - \iota_2(y) \circ (\iota_1(x) \circ e_1)$$

$$= \frac{1}{2}\iota_1(x) \circ \iota_2(y) = \frac{1}{4}\iota_0(\overline{xy}).$$

Hence $[l_{\iota_1(x)}, l_{\iota_2(y)}] = -\frac{1}{2} d_0(\overline{xy})$, and

$$X \cdot Y = -\tau\big(([a, b] \otimes \frac{1}{2}\iota_0(\overline{xy})) - t(ab) d_0(\overline{xy})\big)$$

$$= -\frac{1}{2}([a, b] \otimes \iota_0(xy)) - t(ab) d_0(xy).$$

Therefore,

$$\begin{aligned}\Phi(X\cdot Y)&=\frac{1}{2}([a,b]\otimes xy)+\frac{1}{2}t(ab)(1\otimes xy)\\&=\frac{1}{2}([a,b]+t(ab)1)\otimes xy.\end{aligned}$$

But $ab+ba=t(ab)1$ for any $a,b\in C^0$, while $ab-ba=[a,b]$. Hence $ab=\frac{1}{2}([a,b]+t(ab)1)$, and

$$\Phi(X\cdot Y)=ab\otimes xy=(-a\otimes x)(-b\otimes y)=\Phi(X)\Phi(Y)$$

also in this case.

□

7. S_4-actions and structurable algebras

Among the irreducible representations of the symmetric group S_4, let us consider the one obtained on the tensor product of the standard representation and the alternating one [**FH91**, §2.3]. This is obtained on a three dimensional vector space $W=kw_0+kw_1+kw_2$ with the action of S_4 given by

$$\begin{cases}\tau_1\colon w_0\mapsto w_0,\ w_1\mapsto -w_1,\ w_2\mapsto -w_2,\\ \tau_2\colon w_0\mapsto -w_0,\ w_1\mapsto w_1,\ w_2\mapsto -w_2,\\ \varphi\colon w_0\mapsto w_1\mapsto w_2\mapsto w_0,\\ \tau\colon w_0\mapsto -w_0,\ w_1\mapsto -w_2,\ w_2\mapsto -w_1.\end{cases}\tag{7.1}$$

(This is the representation that appears on the subspaces spanned by the u_i's and the v_i's in (1.5).)

Then the general Lie algebra $\mathfrak{gl}(W)$ becomes a module for S_4: $\sigma\cdot f=\sigma f\sigma^{-1}$. Thus, S_4 acts by automorphisms on $\mathfrak{gl}(W)$. Identifying $\mathfrak{gl}(W)$ with $\mathrm{Mat}_3(k)$ by means of our basis $\{w_1,w_2,w_0\}$, consider the following basis of $\mathfrak{gl}(W)$:

$$\begin{aligned}&H_0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\quad H_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\quad H_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\\&G_0=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad G_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad G_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\\&D_0=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\quad D_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\quad D_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}.\end{aligned}\tag{7.2}$$

The space W is endowed with a natural nondegenerate symmetric bilinear form $(.|.)\colon W\times W\to k$ given by $(w_i|w_j)=\delta_{ij}$, which is invariant under the action of S_4. Actually, S_4 embeds in the associated special orthogonal group $SO(W)$. The corresponding orthogonal Lie algebra $\mathfrak{so}_3=\{F\in\mathfrak{gl}(W):(F(v)|w)+(v|F(w))=0\,\forall\, v,w\in W\}$ is the span of the D_i's in (7.2). Note that

$$[D_i,D_{i+1}]=D_{i+2}\quad\text{(indices modulo 3)}.$$

As a module for $\mathfrak{so}_3$, $\mathfrak{gl}(W)$ decomposes into the following direct sum of irreducible modules (remember that the characteristic of the ground field k is assumed to be $\neq 2,3$):

$$\mathfrak{gl}(W)=\mathfrak{so}_3\oplus\mathfrak{h}\oplus\mathfrak{z}, \tag{7.3}$$

where $\mathfrak{z}=kI_3$ (I_3 denotes the identity matrix), and $\mathfrak{h}=\{F\in\mathfrak{gl}(W):(F(v)|w)=(v|F(w))\,\forall\, v,w\in W\text{ and }\operatorname{trace}(F)=0\}$.

These three irreducible modules: $\mathfrak{so}_3$, $\mathfrak{h}$, and $\mathfrak{z}$, are invariant under the action by conjugation by the orthogonal group, and hence, in particular, under the action of S_4, but while $\mathfrak{so}_3$ and $\mathfrak{z}$ are irreducible modules under the action of S_4, $\mathfrak{h}$ decomposes as the direct sum of two irreducible modules for S_4:

$$\mathfrak{h}=\operatorname{span}\{G_0,G_1,G_2\}\oplus\operatorname{span}\{H_0-H_1,H_1-H_2\}.$$

A simple computation shows that $\operatorname{span}\{H_0,H_1,H_2\}$ is left elementwise fixed by Klein's 4-group V, and becomes the natural module for $S_3=S_4/V$. On the other hand, $\operatorname{span}\{G_0,G_1,G_2\}$ is the standard module for S_4. Thus, among the five irreducible modules for S_4 (up to isomorphism), only the alternating one is missing in $\mathfrak{gl}(W)$.

Lemma 7.4. *Up to scalars, the following maps are the unique $\mathfrak{so}_3$-invariant linear maps between the $\mathfrak{so}_3$-modules considered:*

- $\mathfrak{so}_3\otimes\mathfrak{so}_3\to\mathfrak{so}_3 : A\otimes B\mapsto[A,B]$,
- $\mathfrak{so}_3\otimes\mathfrak{so}_3\to\mathfrak{h} : A\otimes B\mapsto AB+BA-\frac{2}{3}\operatorname{trace}(AB)I_3$,
- $\mathfrak{so}_3\otimes\mathfrak{so}_3\to\mathfrak{z} : A\otimes B\mapsto\operatorname{trace}(AB)I_3$,
- $\mathfrak{so}_3\otimes\mathfrak{h}\to\mathfrak{so}_3 : A\otimes X\mapsto AX+XA$,
- $\mathfrak{so}_3\otimes\mathfrak{h}\to\mathfrak{h} : A\otimes X\mapsto[A,X]$,
- $\mathfrak{so}_3\otimes\mathfrak{h}\to\mathfrak{z} : A\otimes X\mapsto 0$,
- $\mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{so}_3 : X\otimes Y\mapsto[X,Y]$,
- $\mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{h} : X\otimes Y\mapsto XY+YX-\frac{2}{3}\operatorname{trace}(XY)I_3$,
- $\mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{z} : X\otimes Y\mapsto\operatorname{trace}(XY)I_3$.

Moreover, all these maps are invariant under the action of S_4.

Proof: It is clear that all these maps are invariant under the action of both $\mathfrak{so}_3$ and S_4, because so is the trace form and the associative multiplication in $\mathrm{End}_k(W)$.

Now, to prove the uniqueness it is enough to assume the ground field k to be algebraically closed. In this case, $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2$ and, as a module for $\mathfrak{sl}_2$, $\mathfrak{so}_3$ is isomorphic to $V(2)$, $\mathfrak{h}$ to $V(4)$ and $\mathfrak{z}$ to $V(0)$ (notation as in [**Hum78**, §7]). Note that this makes sense because the characteristic is either 0 or ≥ 5. But for $n, m = 0, 2$ or 4, $V(m) \otimes V(n)$ is generated, as a module for $\mathfrak{sl}_2$, by $R \otimes S$, for a highest weight vector R of $V(m)$ and a lowest weight vector S of $V(n)$. Hence any invariant linear map $V(m) \otimes V(n) \to V(p)$ ($p = 0, 2$, or 4) is determined by the image of $R \otimes S$, which belongs to the weight space of $V(p)$ of weight $2(m-n)$. This is at most one-dimensional, and the result follows. □

Let $\mathfrak{g}$ be an arbitrary Lie algebra over k endowed with an action of the symmetric group S_4 by automorphisms, that is, endowed with a group homomorphism

$$S_4 \to \mathrm{Aut}(\mathfrak{g}).$$

As before, the action of Klein's 4-group induces a grading of $\mathfrak{g}$ over $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the subspace $A = \mathfrak{g}_{(\bar{1},\bar{0})} = \{x \in \mathfrak{g} : \tau_1(x) = x,\ \tau_2(x) = -x\}$ becomes an algebra with involution by means of:

$$\begin{cases} x \cdot y = -\tau\big([\varphi(x), \varphi^2(y)]\big), \\ \bar{x} = -\tau(x). \end{cases}$$

The algebra $(A, \cdot, -)$ will be called the *coordinate algebra* of $\mathfrak{g}$. If this algebra is unital, then it is structurable ([**Oku05**, Theorem 2.6] and [**EO07**, Theorem 2.9]). This is the situation that has already appeared in Theorems 4.4 and 6.4.

Theorem 7.5. *Let $\mathfrak{g}$ be a Lie algebra over k. Then $\mathfrak{g}$ is endowed with an action of S_4 by automorphisms such that the coordinate algebra is unital (or, equivalently, structurable) if and only if there is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{so}_3$, such that, as a module for this subalgebra, $\mathfrak{g}$ is the direct sum of irreducible modules isomorphic either to the adjoint module $\mathfrak{so}_3$, the five dimensional module $\mathfrak{h}$ or the trivial one dimensional module $\mathfrak{z}$.*

Proof: Assume first that $\mathfrak{g}$ contains $\mathfrak{so}_3$ as a subalgebra with the properties stated in the theorem. Then, collecting isomorphic irreducible modules, we may write:

$$\mathfrak{g} = \big(\mathfrak{so}_3 \otimes \mathcal{H}\big) \oplus \big(\mathfrak{h} \otimes \mathcal{S}\big) \oplus \mathfrak{d}, \tag{7.6}$$

for vector subspaces $\mathcal{H}$, $\mathcal{S}$ and $\mathfrak{d}$. The subalgebra $\mathfrak{so}_3$ is then identified to $\mathfrak{so}_3 \otimes 1$ for a distinguished element $1 \in \mathcal{H}$. Here $\mathfrak{d} = \{x \in \mathfrak{g} : [d, x] = 0 \,\forall\, d \in \mathfrak{so}_3\}$ is the sum of the trivial irreducible modules, so $\mathfrak{d}$ is the centralizer of the subalgebra $\mathfrak{so}_3$ and, in particular, it is a subalgebra of $\mathfrak{g}$.

Because of Lemma 7.4, the Lie bracket in $\mathfrak{g}$, which is invariant under the action of the subalgebra $\mathfrak{so}_3$, is given by:

- $\mathfrak{d}$ is a subalgebra of $\mathfrak{g}$,
- $[A \otimes a, B \otimes b] = [A, B] \otimes a \circ b - \big(AB + BA - \frac{2}{3}\operatorname{trace}(AB) I_3\big) \otimes \frac{1}{2}[a, b] + \operatorname{trace}(AB) d_{a,b}$,
- $[A \otimes a, X \otimes x] = -(AX + XA) \otimes \frac{1}{2}[a, x] + [A, X] \otimes a \circ x$,
- $[X \otimes x, Y \otimes y] = [X, Y] \otimes x \circ y - \big(XY + YX - \frac{2}{3}\operatorname{trace}(XY) I_3\big) \otimes \frac{1}{2}[x, y] + \operatorname{trace}(XY) d_{x,y}$,
- $[d, A \otimes a] = A \otimes d(a)$,
- $[d, X \otimes x] = X \otimes d(x)$,

for any $A, B \in \mathfrak{so}_3$, $X, Y \in \mathfrak{h}$, $a, b \in \mathcal{H}$, $x, y \in \mathcal{S}$, and $d \in \mathfrak{d}$, where

- $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$: $(a, b) \mapsto a \circ b$ is a symmetric bilinear map with $1 \circ a = a$ for any $a \in \mathcal{H}$,
- $\mathcal{H} \times \mathcal{H} \to \mathcal{S}$: $(a, b) \mapsto [a, b]$ is a skew symmetric bilinear map with $[1, a] = 0$ for any $a \in \mathcal{H}$,
- $\mathcal{H} \times \mathcal{S} \to \mathcal{H}$: $(a, x) \mapsto [a, x]$ is a bilinear map with $[1, x] = 0$ for any $x \in \mathcal{S}$,
- $\mathcal{H} \times \mathcal{S} \to \mathcal{S}$: $(a, x) \mapsto a \circ x$ is a bilinear map with $1 \circ x = x$ for any $x \in \mathcal{S}$,
- $\mathcal{S} \times \mathcal{S} \to \mathcal{H}$: $(x, y) \mapsto x \circ y$ is a symmetric bilinear map,
- $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$: $(x, y) \mapsto [x, y]$ is a skew symmetric bilinear map,
- $\mathcal{H} \times \mathcal{H} \to \mathfrak{d}$: $(a, b) \mapsto d_{a,b}$ is a skew symmetric bilinear map,
- $\mathcal{S} \times \mathcal{S} \to \mathfrak{d}$: $(x, y) \mapsto d_{x,y}$ is a skew symmetric bilinear map,
- the bilinear maps $\mathfrak{d} \times \mathcal{H} \to \mathcal{H}$: $(d, a) \mapsto d(a)$ and $\mathfrak{d} \times \mathcal{S} \to \mathcal{S}$: $(d, x) \mapsto d(x)$, give two representations of the Lie algebra $\mathfrak{d}$.

Now, define an action of S_4 on $\mathfrak{g}$ by means of the actions by conjugation of S_4 on both $\mathfrak{so}_3$ and $\mathfrak{h}$:

$$\psi\big(A \otimes a + X \otimes x + D\big) = (\psi \cdot A) \otimes a + (\psi \cdot X) \otimes x + D \tag{7.7}$$

for any $\psi \in S_4$, $A \in \mathfrak{so}_3$, $X \in \mathfrak{h}$, $a \in \mathcal{H}$, $x \in \mathcal{S}$ and $D \in \mathfrak{d}$.

The invariance of the maps in Lemma 7.4 under the action of S_4 immediately implies that any $\psi \in S_4$ acts as an automorphism of $\mathfrak{g}$.

Besides, the subspace $\mathfrak{g}_{(\bar{1},\bar{0})} = \{g \in \mathfrak{g} : \tau_1(g) = g = -\tau_2(g)\}$ is precisely the subspace

$$D_0 \otimes \mathcal{H} \oplus G_0 \otimes \mathcal{S}.$$

The involution in the coordinate algebra is given by

$$\overline{D_0 \otimes a + G_0 \otimes x} = -\tau(D_0) \otimes a - \tau(G_0) \otimes x = D_0 \otimes a - G_0 \otimes x,$$

for any $a \in \mathcal{H}$ and $x \in \mathcal{S}$, and the multiplication in the coordinate algebra is given by:

$$\begin{aligned}\big(D_0 \otimes a + G_0 \otimes x\big) &\cdot \big(D_0 \otimes b + G_0 \otimes y\big)\\ &= -\tau\big([\varphi(D_0 \otimes a + G_0 \otimes x), \varphi^2(D_0 \otimes b + G_0 \otimes y)]\big)\\ &= -\tau\big([D_1 \otimes a + G_1 \otimes x, D_2 \otimes b + G_2 \otimes y]\big).\end{aligned}$$

But,

$$\begin{aligned}&[D_1, D_2] = D_0,\ [D_1, G_2] = -G_0,\ [D_2, G_1] = G_0,\ [G_1, G_2] = D_0,\\ &D_1D_2 + D_2D_1 - \frac{2}{3}\,\mathrm{trace}(D_1D_2)I_3 = G_0,\ D_1G_2 + G_2D_1 = -D_0,\\ &D_2G_1 + G_1D_2 = -D_0,\ G_1G_2 + G_2G_1 - \frac{2}{3}\,\mathrm{trace}(G_1G_2)I_3 = G_0,\\ &\mathrm{trace}(D_1D_2) = 0 = \mathrm{trace}(G_1G_2),\end{aligned}$$

so

$$\begin{aligned}\big(D_0 \otimes a + G_0 \otimes x\big) &\cdot \big(D_0 \otimes b + G_0 \otimes y\big)\\ &= -\tau\big([D_1 \otimes a + G_1 \otimes x, D_2 \otimes b + G_2 \otimes y]\big)\\ &= -\tau\Big(D_0 \otimes a \circ b - G_0 \otimes \frac{1}{2}[a,b] + D_0 \otimes \frac{1}{2}[a,y] - G_0 \otimes a \circ y\\ &\quad - D_0 \otimes b \circ x + G_0 \otimes \frac{1}{2}[b,x] + D_0 \otimes x \circ y - G_0 \otimes \frac{1}{2}[x,y]\Big)\\ &= \Big(D_0 \otimes a \circ b + G_0 \otimes \frac{1}{2}[a,b]\Big) + \Big(D_0 \otimes \frac{1}{2}[a,y] + G_0 \otimes \frac{1}{2} a \circ y\Big)\\ &\quad - \Big(D_0 \otimes b \circ x - G_0 \otimes \frac{1}{2}[b,x]\Big) + \Big(D_0 \otimes x \circ y + G_0 \otimes \frac{1}{2}[x,y]\Big).\end{aligned}$$

Define $x \circ b = b \circ x$ and $[x,b] = -[b,x]$ for any $b \in \mathcal{H}$ and $x \in \mathcal{S}$. Now consider the vector space $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$ and define a multiplication on it by means of

$$u \cdot v = u \circ v + \frac{1}{2}[u, v]$$

for any $u, v \in \mathcal{H} \cup \mathcal{S}$, so $u \circ v = \frac{1}{2}(u \cdot v + v \cdot u)$ and $[u, v] = u \cdot v - v \cdot u$. Define too a linear map $-\colon \mathcal{A} \to \mathcal{A}$ such that $\overline{a + x} = a - x$ for any $a \in \mathcal{H}$ and $x \in \mathcal{S}$. Then the linear map $D_0 \otimes a + G_0 \otimes x \mapsto a + x$ gives an isomorphism between the coordinate algebra $\mathfrak{g}_{(\bar{1},\bar{0})}$ and the algebra with involution $(\mathcal{A}, \cdot, -)$. Besides, $1 \in \mathcal{H}$ is the unity element of $\mathcal{A}$.

Conversely, let $\mathfrak{g}$ be a Lie algebra with an action of S_4 by automorphisms such that the coordinate algebra is unital. As in [**EO07**], let $\mathcal{A} = \mathfrak{g}_{(\bar{1},\bar{0})}$ be the coordinate algebra, and for any $x \in \mathcal{A}$ consider the elements:

$$\iota_0(x) = x \in \mathfrak{g}_{(\bar{1},\bar{0})}, \quad \iota_1(x) = \varphi(x) \in \mathfrak{g}_{(\bar{0},\bar{1})}, \quad \iota_2(x) = \varphi^2(x) \in \mathfrak{g}_{(\bar{1},\bar{1})}.$$

(Recall that φ is the cycle (123) in S_4.)

Then [**EO07**, §2], for any $x, y \in \mathcal{A}$ and $i = 0, 1, 2$ (indices modulo 3):

$$[\iota_i(x), \iota_{i+1}(y)] = \iota_{i+2}(\overline{x \cdot y}).$$

Therefore, $\mathfrak{s} = \operatorname{span}\{\iota_0(1), \iota_1(1), \iota_2(1)\}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{so}_3$ (by means of $\iota_i(1) \mapsto D_i$ for any $i = 0, 1, 2$). This is the subalgebra we are looking for.

For any $0 \neq x \in \mathcal{A}$ with $\bar{x} = x$, $\operatorname{span}\{\iota_0(x), \iota_1(x), \iota_2(x)\}$ is a copy of the adjoint module for $\mathfrak{s}$, because

$$\begin{aligned}
[\iota_i(1), \iota_{i+1}(x)] &= \iota_{i+2}(x) = [\iota_i(x), \iota_{i+1}(1)],\\
[\iota_i(1), \iota_i(x)] &= [[\iota_{i+1}(1), \iota_{i+2}(1)], \iota_i(x)]\\
&= [[\iota_{i+1}(1), \iota_i(x)], \iota_{i+2}(1)] + [\iota_{i+1}(1), [\iota_{i+2}(1), \iota_i(x)]]\\
&= -[\iota_{i+2}(x), \iota_{i+2}(1)] + [\iota_{i+1}(1), \iota_{i+1}(x)]\\
&= [\iota_{i+1}(1), \iota_{i+1}(x)] + [\iota_{i+2}(1), \iota_{i+2}(x)],
\end{aligned} \tag{7.8}$$

for any $i = 0, 1, 2$. Adding the resulting equations for $i = 0, 1, 2$ gives $\sum_{i=0}^{2}[\iota_i(1), \iota_i(x)] = 2\big(\sum_{i=0}^{2}[\iota_i(1), \iota_i(x)]\big)$, so $\sum_{i=0}^{2}[\iota_i(1), \iota_i(x)] = 0$ and then (7.8) implies that $[\iota_i(1), \iota_i(x)] = 0$ for any i.

Now take an element $0 \neq x \in \mathcal{A}$ with $\bar{x} = -x$. Let us first check that the $\mathfrak{s}$-submodule generated by $\iota_0(x)$ is $\mathfrak{v} = \operatorname{span}\{\iota_i(x), [\iota_i(1), \iota_i(x)] \colon i = 0, 1, 2\}$. To do so, by symmetry, it is enough to check that this subspace is closed under the action of $\iota_0(1)$, but:

$$
\begin{aligned}
&[\iota_0(1),\iota_1(x)] = \iota_2(\bar{x}) = -\iota_2(x),\\
&[\iota_0(1),\iota_2(x)] = -\iota_1(\bar{x}) = \iota_1(x),
\end{aligned}
$$

$$
\begin{aligned}
[\iota_0(1),[\iota_1(1),\iota_1(x)]] &= [[\iota_0(1),\iota_1(1)],\iota_1(x)] + [\iota_1(1),[\iota_0(1),\iota_1(x)]]\\
&= [\iota_2(1),\iota_1(x)] - [\iota_1(1),\iota_2(x)] = 2\iota_0(x),
\end{aligned}
\tag{7.9}
$$

$$[\iota_0(1),[\iota_2(1),\iota_2(x)]] = 2\iota_0(x) \text{ (same arguments)},$$

and finally, as in [**EO07**, Theorem 2.4], $[\iota_0(1),[\iota_0(1),\iota_0(x)]]=-\iota_0(\delta_0(1,x)(1))$, with $\delta_0(1,x)=-\delta_1(\bar{x},1)-\delta_2(1,x)=2(L_x+R_x)$, where L_x and R_x denote, respectively, the left and right multiplication by x in $\mathcal{A}$. Hence,

$$[\iota_0(1),[\iota_0(1),\iota_0(x)]] = -4\iota_0(x), \tag{7.10}$$

and, therefore, $\mathfrak{v}$ is a submodule. This shows too that $[\iota_0(1),\iota_0(x)]\neq 0$ for any $0\neq x\in\mathcal{A}$ with $\bar{x}=-x$. Moreover,

$$
\begin{aligned}
[\iota_2(1),\iota_2(x)] &= [[\iota_0(1),\iota_1(1)],\iota_2(x)]\\
&= [[\iota_0(1),\iota_2(x)],\iota_1(1)] + [\iota_0(1),[\iota_1(1),\iota_2(x)]]\\
&= -[\iota_1(\bar{x}),\iota_1(1)] + [\iota_0(1),\iota_0(\bar{x})]\\
&= -[\iota_0(1),\iota_0(x)] - [\iota_1(1),\iota_1(x)].
\end{aligned}
$$

Therefore, $\sum_{i=0}^{2}[\iota_i(1),\iota_i(x)] = 0$, and the dimension of $\mathfrak{v}$ is at most 5. But (7.9), (7.10) and their analogues for $i=0,1,2$ show that $[\iota_0(1),\iota_0(x)]$ and $[\iota_1(1),\iota_1(x)]$ are linearly independent elements of $\mathfrak{g}_{(\bar{0},\bar{0})}$. The outcome is that the dimension of $\mathfrak{v}$ is 5. Besides, the assignment $\iota_i(x)\mapsto G_i$, $[\iota_i(1),\iota_i(x)]\mapsto -2(H_{i+1}-H_{i+2})$ (G_i's and H_i's as in (7.2)) shows that $\mathfrak{v}$ is isomorphic to the irreducible module $\mathfrak{h}$. Therefore, $\oplus_{i=0}^{2}\iota_i(\mathcal{A})$ is contained in a sum of irreducible modules for $\mathfrak{s}$ isomorphic either to the adjoint module or to $\mathfrak{h}$.

Now take any element $0\neq d\in\mathfrak{g}_{(\bar{0},\bar{0})}$ and let $\mathcal{U}=\mathcal{U}(\mathfrak{s})$ be the universal enveloping algebra of $\mathfrak{s}$. (Recall that $\mathfrak{s}$ is isomorphic to $\mathfrak{so}_3$.) The $\mathfrak{s}$-module generated by d is $\mathcal{U}d = kd+\sum_{i=0}^{2}\mathcal{U}[d,\iota_i(1)]$. Let $x_i\in\mathcal{A}$ be the element such that $[d,\iota_i(1)]=\iota_i(x_i)$ ($i=0,1,2$). Then $\mathcal{U}d = kd+\sum_{i=0}^{2}\mathcal{U}\iota_i(x_i)$. But the sum $\sum_{i=0}^{2}\mathcal{U}\iota_i(x_i)$ is a finite sum of irreducible modules, each of them isomorphic either to the adjoint module or to $\mathfrak{h}$, and hence, by complete reducibility, to a finite direct sum of irreducible modules of these types. Therefore, to prove that $\mathcal{U}d$ is a sum of irreducible $\mathfrak{s}$-modules which are either trivial, adjoint or isomorphic to $\mathfrak{h}$, it is enough to prove that if M is a module for $\mathfrak{s}$ and N a submodule of M with N either adjoint or isomorphic to $\mathfrak{h}$, and the dimension of M/N is 1, then M contains a one dimensional submodule (which necessarily complements N). But the Casimir element $D_0^2 + D_1^2 + D_2^2 \in \mathcal{U}$ acts as $-2Id$ on the adjoint module, $-6Id$ on $\mathfrak{h}$ and trivially on the trivial module. Hence the one dimensional submodule of M sought for is the kernel of the action of the Casimir element. □

By means of (4.1), actions of the group S_4 on the exceptional Lie algebras were considered. (Note that the simple Lie algebra of type G_2 appears simply as $\mathfrak{Der} C$.) The previous theorem makes easy to embed S_4 in the group of automorphisms of classical Lie algebras.

Examples 7.11. Consider the module W for S_4 in (7.1).

(i) *Orthogonal Lie algebras:* The module W is endowed with a nondegenerate symmetric bilinear form b invariant under the action of S_4: $b(w_i, w_j) = \delta_{ij}$ for any $i, j = 0, 1, 2$. Let (U, b') be any vector space endowed with a nondegenerate symmetric bilinear form. Then the orthogonal Lie algebra of the orthogonal sum $(W \oplus U, b \perp b')$ decomposes as:

$$\mathfrak{so}(W \oplus U, b \perp b') = \mathfrak{so}(W, b) \oplus (W \otimes U) \oplus \mathfrak{so}(U, b'),$$

where $\mathfrak{so}(W, b) = \mathfrak{so}_3$ (respectively $\mathfrak{so}(U, b')$) is identified to the subalgebra of $\mathfrak{so}(W \oplus U, b \perp b')$ which preserves W (resp. U) and annihilates U (resp. W), and for any $w \in W$ and $u \in U$, $w \otimes u$ is identified to the linear map determined by $w' \mapsto b(w, w')u$, $u' \mapsto -b'(u, u')w$, for any $w' \in W$ and $u' \in U$.

As a module for $\mathfrak{so}_3$, W is isomorphic to the adjoint module, so $W \otimes U$ is a direct sum of copies of the adjoint module, while $\mathfrak{so}(U, b')$ is a trivial module. Hence, according to Theorem 7.5, $\mathfrak{so}(W \oplus U, b \perp b')$ is endowed with an action of S_4 by automorphisms.

(ii) *Special Lie algebras:* Let U be now any vector space. Then the special linear Lie algebra $\mathfrak{sl}(W \oplus U)$ decomposes as

$$\mathfrak{sl}(W \oplus U) = \mathfrak{sl}(W) \oplus (W \otimes U^*) \oplus (W^* \otimes U) \oplus \mathfrak{gl}(U)$$

with natural identifications. But as in (7.3), $\mathfrak{sl}(W)$ decomposes as $\mathfrak{sl}(W) = \mathfrak{so}_3 \oplus \mathfrak{h}$ and, as a module for $\mathfrak{so}_3$, W and W^* are both isomorphic to the adjoint module. Then $\mathfrak{sl}(W \oplus U)$ decomposes, as a module for $\mathfrak{so}_3$, as a direct sum of copies of the adjoint module, of $\mathfrak{h}$ (just one copy) and of the trivial module, so $\mathfrak{sl}(W \oplus U)$ (or $\mathfrak{sl}_n$ for $n \geq 3$) is endowed with an action of S_4 by automorphisms.

(iii) *Symplectic Lie algebras:* Let now (U, B') be a vector space endowed with a nondegenerate skew symmetric bilinear form. Also $W \oplus W^*$ is endowed with the natural skew symmetric bilinear form B, where W and W^* are isotropic subspaces and $B(f, w) = f(w)$ for any $f \in W^*$ and $w \in W$. The symplectic Lie algebra of the orthogonal sum $\big((W \oplus W^*) \oplus U, B \perp B'\big)$ decomposes as

$$\begin{aligned}\mathfrak{sp}\big((W \oplus W^*) \oplus U, B \perp B'\big)\\ = \mathfrak{sp}(W \oplus W^*, B) \oplus \big((W \oplus W^*) \otimes U\big) \oplus \mathfrak{sp}(U, B').\end{aligned}$$

But $\mathfrak{gl}(W)$ is naturally embedded in $\mathfrak{sp}(W \oplus W^*, B)$ as the subalgebra that leaves both W and W^* invariant. Hence $\mathfrak{so}_3$, which is contained in $\mathfrak{gl}(W)$, embeds in $\mathfrak{sp}(W \oplus W^*, B)$ which, as a module for $\mathfrak{so}_3$ is the direct sum of $\mathfrak{so}_3$, three copies of $\mathfrak{h}$ and three copies of $\mathfrak{z}$. Again this shows that $\mathfrak{sp}\big((W \oplus W^*) \oplus U, B \perp B'\big)$ is endowed with an action of S_4 by automorphisms.

Remark 7.12. The Lie algebras over a field of characteristic 0 containing a three dimensional simple Lie algebra $\mathfrak{s}$ such that, as modules for $\mathfrak{s}$, are a direct sum of copies of the adjoint, the unique five dimensional irreducible module and the trivial module have been thoroughly studied in [**Sel88**, Chapter 7]. □

Remark 7.13. The Lie algebras $\mathfrak{g}$ whose Lie algebras of derivations contain a subalgebra isomorphic to $\mathfrak{so}_3$ and such that, as modules for this subalgebra, they are a direct sum of irreducible modules isomorphic either to the adjoint module $\mathfrak{so}_3$, the five dimensional module $\mathfrak{h}$ or the trivial one dimensional module $\mathfrak{z}$, can be shown to admit a group of automorphisms isomorphic to S_4 exactly as in the proof of Theorem 7.5. In particular, if J is any Jordan algebra (not necessarily unital) and $\mathfrak{d}$ is a Lie subalgebra of $\mathfrak{Der}(J)$ containing the inner derivations, then the Lie algebra $\mathfrak{g} = \big(\mathfrak{so}_3 \otimes J\big) \oplus \mathfrak{d}$, where $\mathfrak{d}$ is a subalgebra and the bracket is determined by (see [**Tit62**]) $[A \otimes x, B \otimes y] = [A, B] \otimes xy + \frac{1}{2}\,\mathrm{trace}(AB)[L_x, L_y]$ and $[d, (A \otimes x)] = A \otimes d(x)$ for any $A, B \in \mathfrak{so}_3$, $x, y \in J$, and $d \in \mathfrak{d}$, is a Lie algebra satisfying the conditions above. Note that in case $-1 \in k^2$, then $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2$, and the construction above becomes the well-known Tits-Kantor-Koecher construction $TKK(J)$ (see [**EO07**, Example 3.2]). In particular, if J is the Jordan superalgebra of type D_t or F, then this construction will give the Lie superalgebra of type $D(2, 1; t)$ or $F(4)$ respectively. □

Two more comments are in order here. In a previous paper [**EO07**], the authors have shown how to define an action of S_4 on the Lie algebra $\mathcal{K}(\mathcal{A}, -)$ attached to a structurable algebra $(\mathcal{A}, -)$ by means of Kantor's construction [**All79**] in case -1 is a square on the ground field. The previous theorem provides a natural interpretation: The Lie algebra $\mathcal{K}(\mathcal{A}, -)$ contains a subalgebra isomorphic to $\mathfrak{sl}_2$ such that, as a module for $\mathfrak{sl}_2$, $\mathcal{K}(\mathcal{A}, -)$ is a direct sum of copies of $\mathfrak{sl}_2$, of its five dimensional irreducible module in $\mathfrak{gl}(\mathfrak{sl}_2)$ and the trivial module. However, if -1 is a square, then $\mathfrak{sl}_2$ is isomorphic to $\mathfrak{so}_3$ and, after identifying $\mathfrak{sl}_2 \simeq \mathfrak{so}_3$, the five dimensional irreducible module is the module $\mathfrak{h}$ considered so far.

Also, the Lie algebras which contain a subalgebra isomorphic to $\mathfrak{sl}_2$ and which, as a module for $\mathfrak{sl}_2$, are direct sums of copies of irreducible modules of three types: the adjoint module $\mathfrak{sl}_2$, the five dimensional irreducible module in $\mathfrak{gl}(\mathfrak{sl}_2)$ and the trivial module, are essentially the BC_1-graded Lie algebras of type B_1 (see [**BSm03**]) and the references therein). These Lie algebras present a decomposition as in (7.6):

$$\mathfrak{g} = \big(\mathfrak{sl}_2 \otimes \mathcal{H}\big) \oplus \big(\mathfrak{h} \otimes \mathcal{S}\big) \oplus \mathfrak{d}.$$

Take the standard basis $\{e, f, h\}$ of $\mathfrak{sl}_2$ with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. The action of $\operatorname{ad} h$ gives a 5-grading: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} : [h, x] = ix\}$ $(i = -2, -1, 0, 1, 2)$. There is just one extra condition in the definition of the BC_1-graded Lie algebras of type B_1: $\mathfrak{g}_0 = [\mathfrak{g}_{-2}, \mathfrak{g}_2] + [\mathfrak{g}_{-1}, \mathfrak{g}_1]$. With the notations as in the proof of Theorem 7.5, this is equivalent to the condition $\mathfrak{d} = d_{\mathcal{H},\mathcal{H}} + d_{\mathcal{S},\mathcal{S}}$.

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