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SHARP GROWTH ESTIMATES FOR DYADIC b-INPUT $T(b)$ THEOREMS

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Abstract

The following deals with the $T(b)$ theorems of David, Journé, and Semmes [**7**] considered in a dyadic setting. We find sharp growth estimates for a global and a local dyadic $T(b)$ Theorem. We use multiscale analysis and Haar wavelets in the local case.

1. Background

The $T(1)$ theorem of David and Journé [**6**] gives a necessary and sufficient condition for a singular integral operator to be bounded on the space $L^2(\mathbb{R}^n)$. Both the properties of the operator T and cancellation properties of its associated kernel K are considered. The hypotheses of the $T(1)$ theorem are sometimes difficult to verify directly, but in certain instances this problem can be somewhat alleviated by replacing the constant function 1 by a function b whose mean is bounded away from zero. We consider growth conditions on b and $T(b)$ which are necessary and sufficient for T to be bounded sharply, and find the dependency of the bounds on T by the bounds on b.

There have been several reformulations of the $T(b)$ problem. In 1990, Christ [**3**] gave an L^∞ criterion for L^2-boundedness of a singular integral operator and posed questions about the application of the theorems to analytic capacity. In their 2002 paper, Nazarov, Treil, and Volberg [**12**] proved an equivalent condition to that of Christ, only they considered a nondoubling measure, making their result valid for a nonhomogeous space. Furthermore, they allowed the image of the operator to be in the space of bounded mean oscillation (BMO), rather than in L^∞. In 2003, Tolsa [**14**] used the non-doubling $T(b)$ theorem in [**12**] in his answer to the Painlevé problem and his proof of the semiadditivity of analytic capacity of a compact set $E \subset \mathbb{C}$. In 2002, Auscher, Hofmann, Muscalu,

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Tao, and Thiele [**2**] proved several dyadic $T(b)$ theorems in the context of Carleson measures and trees. Much of the following work is done in this context using a dyadic setting which will be described rigorously in a later section.

In general, the dyadic $T(b)$ theorems deal with conditions on a singular integral operator T, its dual T^*, and on a function b (in the global case) or functions b_I for each dyadic interval I (in the local case) that will insure L^2 boundedness of T. These theorems are generalizations of the $T(1)$ theorems in which a function b is substituted for the constant function 1 (see [**6**] and [**7**]).

We normalize the bounds in the theorems so that the bound on the mean of b is a constant not dependent on a small constant γ, $0 < \gamma \ll 1$, but the norm of b does depend on γ. The conclusion of each $T(b)$ theorem states that a dyadic singular integral operator is bounded on L^2, i.e. $\|Tf\|_2 \lesssim \|f\|_2$, where the implicit constant depends on γ. We prove each $T(b)$ theorem as in the paper by Auscher, Hofmann, Muscalu, Tao, and Thiele *and* track the power of γ in the implicit constant. Then we show that the power on γ is sharp by providing an example.

Historically, the $T(b)$ theorems have been symmetric, in that the hypotheses confirm control of the size of both $T(b)$ and $T^*(b)$. The function b is sufficiently controlled (say, in L^2, L^∞, or BMO) as well. The symmetric dyadic case, labeled the two-sided $T(b)$ theorem in [**2**], is quite complicated, and as a natural first step in finding the sharp power on γ in the symmetric dyadic case, we look at a simpler case in which we control the norm of $T(b)$ and $T^*(1)$.

Our search for the sharp power on γ was initially motivated by a desire to apply $T(b)$-style results to problems in analytic capacity [**15**]. We quickly discovered that by breaking the symmetric, two-sided case into a one-sided case, we could more easily see where the powers of γ were lost in the proof. Additionally, we could formulate and prove new $T(b)$ theorems in which the BMO-norm is weighted by the function b [**11**].

Though the notation will be developed in the following section, we state the two main results here and direct the reader to Sections 4 and 5 for proofs and examples.

The Dyadic b-input Global $T(b)$ Theorem. *Let b be a function on $[0,1)$ such that*

$$|[b]_I| \geq 1$$

for all $I \in \Delta$ (i.e. b is 1-pseudo-accretive for all $I \in \Delta$) and such that for some $0 < \gamma \ll 1$,

$$\|b\|_{\mathrm{BMO}} \leq \gamma^{-1}.$$

Let T be a dyadic singular integral operator such that the following hold:

$$|\langle T\psi_I, \psi_I\rangle| \leq 1$$
$$\|Tb\|_{\mathrm{BMO}} \leq G_1\gamma^{-1}$$
$$\|T^*(1)\|_{\mathrm{BMO}} \leq G_2.$$

Then T is bounded on L^2, and

$$\|Tf\|_2 \leq C(1+\gamma^{-1}(1+G_1+G_2)+G_2)\|f\|_2.$$

The Dyadic b-input Local $T(b)$ Theorem. *Let T be a dyadic singular integral operator obeying*

$$\|T^*(1)\|_{\mathrm{BMO}} \leq G$$
$$|\langle T\psi_I, \psi_I\rangle| \leq 1$$

for a fixed G and for all $I \in \Delta$. Suppose that for every $I \in \Delta$, there exists a function $b_I \in \mathbf{S}(I)$ with mean 1 on I such that

$$\|b_I\|_{L^2(I)} \leq \frac{|I|^{\frac{1}{2}}}{\gamma}$$

and

$$\|Tb_I\|_{L^2(I)} \leq \frac{|I|^{\frac{1}{2}}}{\gamma}$$

for some $0 < \gamma \ll 1$. Then T is bounded on L^2 and moreover,

$$\|Tf\|_2 \leq C\|f\|_2\left(1+\frac{C(2+G)}{\gamma^2}+G\right).$$

As mentioned earlier, this work is part of a program to understand the dependency of the bounds on T by the bounds on b in $T(b)$ theorems. The power on γ changes significantly as we alter the hypotheses of the theorem to include b-weighted BMO (a b-output theorem), or control over $T(b)$ and $T^*(b)$ for a function b not equal to the constant function 1. The latter case is closest to the symmetric $T(b)$ theorems found, for example, in [**3**], and as yet the sharp power on the constant in this case is open. However, we do know that the sharp power is between -1 and -13 in the global case, and between -2 and -13 in the local case [**11**]. Furthermore, the dependency of the bounds on T by the bounds on b in a continuous (non-dyadic) case is still an open question.

2. Definitions and Notation

We consider the real line $\mathbb{R}$ decomposed into dyadic intervals, I.

Definition 2.1. $I \subset \mathbb{R}$ is a *dyadic interval* if it is of the form $[j2^k, (j+1)2^k)$, for some $j, k \in \mathbb{Z}$.

From this point on, we consider only dyadic intervals.

Given an interval $I \subset \mathbb{R}$, we use the notation $|I|$ to denote the Lebesgue measure of I. There is a unique $k \in \mathbb{Z}$ (from the definition of I above) such that $|I| = 2^k$. That the left side of the interval is closed and the right side is open is merely a convention which we adopt. Because we are often integrating over intervals, this convention does not change our calculations. However, adopting this convention is necessary for two reasons: first, we would like to have intervals partition $\mathbb{R}$, and second, we would like dyadic intervals to nest nicely.

Proposition 2.1 (Nesting Property of Dyadic Intervals)**.** *Given two dyadic intervals I and J, one of the following situations occurs:*

- $I = J$
- $I \cap J = \emptyset$
- $I \subset J$
- $J \subset I$.

Notice that given any collection of dyadic intervals, the subset of intervals which are maximal with respect to inclusion are disjoint. This property will be used heavily in the following.

We refer to dyadic intervals using generational terms. Given a dyadic interval I, we refer to the left and right halves of I, denoted I_l and I_r respectively, as the *children* of I. Each dyadic interval has exactly two children, four grandchildren, eight great-grandchildren, and so on. It also has a unique *parent*, of which it is either a *left or right child*.

As in [**2**], we restrict ourselves to a finite set of dyadic intervals on the half-line. We fix a large M, and let

$$\Delta_M = \{I = [j2^k, (j+1)2^k) : j, k \in \mathbb{Z}, -M \le k \le M, \text{ and } I \subseteq [0, 2^M)\}.$$

Our estimates will be independent of M, and so we freely take $\Delta_M = \Delta$. As such, we can use a standard translation and limiting argument to get bounds over the non-truncated dyadic line.

We will also use the language of *trees* in the lemmas and proofs.

Definition 2.2. A *dyadic tree* (henceforth abbreviated *tree*) is a collection of dyadic intervals $\mathcal{T} \subseteq \Delta$ with a top interval (called the *top* of the tree), denoted $I_{\mathcal{T}}$, which is the unique dyadic interval in $\mathcal{T}$ such that for all $J \in \mathcal{T}$, we have that $J \subseteq I_{\mathcal{T}}$.

A tree $\mathcal{T}$ is said to be *complete* if $J \in \mathcal{T}$ for all $J \subseteq I_{\mathcal{T}}$. We let $\text{Tree}(I)$ denote the complete tree with top I.

In our analysis, we use Haar wavelets,

$$\psi_I = \frac{1}{|I|^{\frac{1}{2}}}(\chi_{I_l} - \chi_{I_r}),$$

where $I \in \Delta$, and where χ_J is the characteristic function on the interval J. We also use an L^2-normalized characteristic function $\widetilde{\chi_I} = \frac{1}{|I|^{\frac{1}{2}}}\chi_I$. The set $\{\psi_I\}_{I\in\Delta} \cup \chi_{[0,1)}$ is an orthonormal basis for $L^2([0,1))$.

Given a function f defined on an interval I, we let

$$[f]_I = \frac{1}{|I|}\int_I f\,dx$$

be the mean value of f on I. As such, $[\psi_I]_I = 0$.

We study operators of a particular type, following the notation from [**4**]. By *singular integral operator*, we mean an operator which is defined as an integral against a kernel which is in some way singular. This definition may be formal, as

$$Tf(x) = \int K(x,y)f(y)\,dy \tag{1}$$

may not be finite for all values of x. The kernel K is a function from $(\mathbb{R}\times\mathbb{R})\setminus\{x=y\}$ to $\mathbb{R}$ which is integrable off the diagonal. In initial work on $T(1)$ and $T(b)$ theorems, in particular [**6**] and [**7**], the authors find the conditions under which such operators are bounded on $L^2(\mathbb{R})$. As mentioned earlier, we wish to find the sharp growth estimates for these theorems.

For reasons of simplicity, we limit ourselves to one-dimensional analysis. To sharpen the formal definition (1), we use the following from [**4**]:

Definition 2.3. A kernel K on $(\mathbb{R}\times\mathbb{R})\setminus\{x=y\}$ is said to satisfy *standard estimates* if there exist $\delta > 0$ and $C < \infty$ such that for all distinct $x, y \in \mathbb{R}$ and all z such that $|x-z| < \frac{|x-y|}{2}$:

(1) $|K(x,y)| \leq \frac{C}{|x-y|}$

(2) $|K(x,y) - K(z,y)| \leq C\frac{|x-z|^\delta}{|x-y|^{1+\delta}}$

(3) $|K(y,x) - K(y,z)| \leq C\frac{|x-z|^\delta}{|x-y|^{1+\delta}}$.

We will refer to a function satisfying the above estimates as a *standard kernel*.

This definition changes slightly in the dyadic setting. We define a dyadic metric on $\mathbb{R}$ in the following manner: given $x, y \in \mathbb{R}$, let $|x - y|_{\text{dyadic}}$ be the length of the smallest dyadic interval containing both x and y. We adapt the definition of a standard kernel to this metric, normalizing so that the constant $C = 1$.

Definition 2.4. A kernel K on $(\mathbb{R}\times\mathbb{R})\setminus\{x = y\}$ is said to satisfy *dyadic standard estimates* if

(1) For all $(x, y) \in (\mathbb{R} \times \mathbb{R}) \setminus \{x = y\}$,

$$|K(x, y)| \leq \frac{1}{|x - y|_{\text{dyadic}}}.$$

(2) For all $x, x' \in I$ and $y \in J$ for sibling dyadic intervals I and J,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| = 0.$$

The second condition implies that given a Whitney decomposition of the square $[0, 2^M) \times [0, 2^M)$, K is constant on squares of the decomposition which don't intersect the diagonal.

A dyadic singular integral operator, T, is an operator which is defined as integration against such a kernel. Therefore, given $f \in C_0^\infty$, T a dyadic singular integral operator, and provided that x is not in the support of f, we can let

$$Tf(x) = \int K(x, y)f(y)\, dy$$

without sacrificing rigor.

As in [**6**], the adjoint operator T^* is defined by $\langle T^*f, g\rangle = \langle f, Tg\rangle$ and is associated to a related standard kernel given by $L(x, y) = K(y, x)$. As we focus on the real Hilbert space, we need not consider the complex conjugate in our definitions.

In the continuous case in [**6**], defining the action of T on the constant function 1 is problematic and must be done carefully using distributions. In the truncated dyadic case defined earlier, we look only at a finite number of scales contained in one large dyadic interval, so these problems cease to exist and we may, without losing rigor, refer to $T(1)$ without confusion. (Similarly, $T^*(1)$ is defined rigorously and intuitively on our space.)

We find sharp growth bounds for standard dyadic singular integral operators. In the local case, we use the L^1- or L^2 norms to measure growth, but in the global case, we turn to the space of bounded mean oscillation (BMO) and its associated norm. For our purposes, we will deal only with functions defined on $\mathbb{R}$, and will use the common notation BMO = BMO($\mathbb{R}$). Furthermore, we will look only at *dyadic* BMO, a norm for which is defined below for locally integrable functions. For more information on dyadic BMO, see [**10**].

Definition 2.5. A locally integrable function f will be said to belong to dyadic BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_{I\in\Delta} \frac{1}{|I|^{\frac{1}{2}}}\left(\int_I |f-[f]_I|^2\,dx\right)^{\frac{1}{2}} = \sup_{I\in\Delta} \frac{1}{|I|^{\frac{1}{2}}}\left(\sum_{J\subseteq I} |\langle f,\psi_J\rangle|^2\right)^{\frac{1}{2}} < \infty.$$

The space of locally integrable functions of bounded mean oscillation is a set of equivalence classes of functions where $f \sim g$ if and only if $f-g=c$ for some constant c. Notice that for any constant function $f \equiv C$, we have $\|f\|_{\mathrm{BMO}} = 0$.

3. Lemmas

The following lemma demonstrates that we can separate any dyadic singular integral operator T into three parts. The proof can be found in [**2**].

Lemma 3.1 (The Splitting Lemma)**.** *If T is a dyadic singular integral operator, then for all f in the span of* $\{\psi_I : I\in\Delta\}\cup\widetilde{\chi}_{[0,1)}$,

$$\begin{aligned} T(f) &= \sum_{I\in\Delta}\langle T\psi_I,\psi_I\rangle\langle f,\psi_I\rangle\psi_I \\ &\quad+ \sum_{I\in\Delta}\frac{1}{|I|^{\frac{1}{2}}}\langle T(1),\psi_I\rangle\langle f,\widetilde{\chi}_I\rangle\psi_I \\ &\quad+ \sum_{I\in\Delta}\frac{1}{|I|^{\frac{1}{2}}}\langle T^*(1),\psi_I\rangle\langle f,\psi_I\rangle\widetilde{\chi}_I. \end{aligned}$$

The first sum is referred to as the diagonal part, the second as the $T(1)$ paraproduct, and the third as the $T^*(1)$ paraproduct.

When the above lemma is used, then paired with the following paraproduct estimates, also from [**2**], we can find growth bounds on dyadic singular operators. These paraproduct estimates are a corollary of the Carleson Embedding Theorem (see [**13**]).

Lemma 3.2 ($L^2 \times \mathrm{BMO} \to L^2$ Paraproduct Estimates)**.** [**2**, Corollary 5.2] *For all f, g in the span of $\{\psi_I : I \in \Delta\} \cup \widetilde{\chi}_{[0,1)}$,*

$$\left\| \sum_{I \in \Delta} \frac{1}{|I|^{\frac{1}{2}}} \langle f, \psi_I \rangle \langle g, \widetilde{\chi}_I \rangle \psi_I \right\|_2 \leq C \|f\|_2 \|g\|_\infty, \tag{2}$$

and

$$\left\| \sum_{I \in \Delta} \frac{1}{|I|^{\frac{1}{2}}} \langle g, \psi_I \rangle \langle f, \widetilde{\chi}_I \rangle \psi_I \right\|_2 \leq C \|f\|_2 \|g\|_{\mathrm{BMO}} \tag{3}$$

$$\left\| \sum_{I \in \Delta} \frac{1}{|I|^{\frac{1}{2}}} \langle g, \psi_I \rangle \langle f, \psi_I \rangle \widetilde{\chi}_I \right\|_2 \leq C \|f\|_2 \|g\|_{\mathrm{BMO}}. \tag{4}$$

Using the above lemmas, we track the constants in the following dyadic $T(1)$ theorems. Proofs of the theorems without constant tracking can be found in [**2**], though the constants are fairly simple at this stage.

Theorem 3.3 (Dyadic Global $T(1)$ Theorem)**.** *Let T be a dyadic singular integral operator such that*

$$\|T(1)\|_{\mathrm{BMO}} \leq G_1,$$
$$\|T^*(1)\|_{\mathrm{BMO}} \leq G_2,$$

and T has the weak boundedness property

$$|\langle T\psi_I, \psi_I \rangle| \leq 1,$$

for all $I \in \Delta$. Then T is bounded on L^2 and in particular,

$$\|Tf\|_2 \leq C(1 + G_1 + G_2)\|f\|_2.$$

Next, we look at the local version of the dyadic $T(1)$ theorem. Its hypotheses are local analogues of those of the global theorem. The major difference is that we switch from the BMO norm to the L^1 norm. This is a reasonable switch, however, since the BMO norm is already *local* in that it looks at a function over one interval at a time. It is also important to note that while we measure $\|T(1)\|_{L^1(I)}$, we could as easily have measured using the $L^2(I)$ norm and would have reached the same conclusion. The local theorem follows as a corollary to the global theorem.

Theorem 3.4 (Dyadic Local $T(1)$ Theorem)**.** *Let T be a dyadic singular integral operator such that*

$$\|T\chi_I\|_{L^1(I)} \le G_1|I|$$
$$\|T^*\chi_I\|_{L^1(I)} \le G_2|I|$$

for all $I \in \Delta$. Then T is bounded on L^2. In particular,

$$\|Tf\|_2 \le C(2(1+\min(G_1,G_2)) + G_1 + G_2)\|f\|_2.$$

As mentioned earlier, the dyadic b-input $T(b)$ theorems have similar hypotheses to the $T(1)$ theorems above. In order to prove the $T(b)$ theorems, we must use a stopping-time argument which relies on the following two lemmas. The first tells us that to bound the maximal size of a function on a tree $\mathcal{T}$, it suffices to do so outside a set of intervals $\mathbb{J}$ where

$$\sum_{I\in\mathbb{J}} |I| \le (1-\eta) I_{\mathcal{T}}$$

for some $\eta > 0$. The second lemma makes a claim of the same flavor about functions of large mean. Specifically, if a function b has large mean on the top of a tree, then $|[b]_I| < \frac{1}{4}$ on a non-trivial set of intervals.

Lemma 3.5. *Suppose $\mathbb{I} \subseteq \Delta$ is a collection of dyadic intervals and $a\colon \mathbb{I} \to \mathbb{R}^+$ is a function. Suppose also that we have constants $A > 0$ and $0 < \eta < 1$ such that for every tree $\mathcal{T}$ we have*

$$\frac{1}{|I_{\mathcal{T}} \setminus \bigcup_{\mathcal{T}'\in\mathbb{T}_{\mathcal{T}}} I_{\mathcal{T}'}|} \sum_{I\in\mathcal{T}\setminus\bigcup_{\mathcal{T}'\in\mathbb{T}_{\mathcal{T}}}\mathcal{T}'} a(I) \le A$$

for some collection $\mathbb{T}_{\mathcal{T}}$ of trees in $\mathcal{T}$ whose tops cover at most $(1-\eta)$ of $I_{\mathcal{T}}$, i.e.

$$\sum_{\mathcal{T}'\in\mathbb{T}_{\mathcal{T}}} |I_{\mathcal{T}'}| \le (1-\eta)|I_{\mathcal{T}}|.$$

Then we have

$$\mu = \sup_{\mathcal{T}\in\mathbb{I}} \frac{1}{|I_{\mathcal{T}}|} \sum_{I\in\mathcal{T}} a(I) \le \frac{A}{\eta}.$$

Note that while we allow $\mathbb{T}_{\mathcal{T}}$ to depend on $\mathcal{T}$, we will use the shorthand notation $\mathbb{T}_{\mathcal{T}} = \mathbb{T}$.

In the application, we will take $\eta = \gamma^2$.

Proof: Let $\mathcal{T}$ be any tree in $\mathbb{I}$. From the hypothesis, we have the following:

$$\begin{aligned}\sum_{I\in\mathcal{T}} a(I) &= \sum_{I\in\mathcal{T}\setminus\cup_{\mathcal{T}'\in\mathbb{T}}\mathcal{T}'} a(I) + \sum_{\mathcal{T}'\in\mathbb{T}}\sum_{I\in\mathcal{T}'} a(I)\\ &\le A|I_{\mathcal{T}}| + \sum_{\mathcal{T}'\in\mathbb{T}} \mu|I_{\mathcal{T}'}|\\ &\le A|I_{\mathcal{T}}| + \mu(1-\eta)|I_{\mathcal{T}}|\\ &= |I_{\mathcal{T}}|(A+(1-\eta)\mu).\end{aligned}$$

Therefore

$$\frac{1}{|I_{\mathcal{T}}|}\sum_{I\in\mathcal{T}} a(I) \le A + (1-\eta)\mu.$$

We take the supremum of the left side and the desired result follows. $\square$

Lemma 3.6. *Let $\mathcal{T}_0 \subseteq \Delta$ be a convex tree and let b be a function such that*

$$\left\|\sum_{I\in\mathcal{T}_0}\langle b,\psi_I\rangle\psi_I\right\|_2 \le \frac{C_0|I_{\mathcal{T}_0}|^{\frac{1}{2}}}{\gamma}$$

for some $C_0 > 0$, and $0 < \gamma \ll 1$ and such that $|[b]_{I_{\mathcal{T}_0}}| > 1$. Then there exists a family $\mathbb{T}$ of disjoint convex subtrees of $\mathcal{T}_0$ such that $|[b]_I| > \frac{1}{4}$ for all $I \in \mathcal{T}_0\setminus\bigcup_{\mathcal{T}\in\mathbb{T}}\mathcal{T}$ and the tops of the trees in $\mathbb{T}$ cover at most $(1-\frac{\gamma^2}{2C_0^2})$ of $I_{\mathcal{T}_0}$. Furthermore $|[b]_{I_{\mathcal{T}}}| \le \frac{1}{4}$ for all $\mathcal{T}\in\mathbb{T}$.

Proof: Let $\mathbf{I}$ denote the collection of dyadic intervals I contained in $\mathcal{T}_0$ for which $|[b]_I| \le \frac{1}{4}$ and which are maximal with respect to inclusion. By maximality these intervals are disjoint and by construction they obey

$$\left|\int_I b\right| \le \frac{1}{4}|I|.$$

We must show that $\mathbf{I}$ covers at most $(1-\frac{\gamma^2}{2C_0^2})$ of $I_{\mathcal{T}_0}$. That is, we need to show that

$$\sum_{I\in\mathbf{I}}|I| \le \left(1-\frac{\gamma^2}{2C_0^2}\right)|I_{\mathcal{T}_0}|.$$

Suppose, for sake of contradiction, that

$$\sum_{I\in\mathbf{I}} |I| > \left(1 - \frac{\gamma^2}{2C_0^2}\right) |I_{\mathcal{T}_0}|.$$

Let $E = \bigcup_{I\in\mathbf{I}} I$, where the union is disjoint. Then

$$\left|\int_E b\right| \le \sum_{I\in\mathbf{I}} \left|\int_I b\right| \le \frac{1}{4}\sum_{I\in\mathbf{I}} |I| \le \frac{1}{4}|I_{\mathcal{T}_0}|,$$

where the last inequality is true because the intervals are disjoint and contained in $I_{\mathcal{T}_0}$. By assumption, $|[b]_{I_{\mathcal{T}_0}}| > 1$, so

$$\begin{aligned}
|I_{\mathcal{T}_0}| &< \left|\int_{I_{\mathcal{T}_0}} b\right| \\
&= \left|\int_E b + \int_{I_{\mathcal{T}_0}\setminus E} b\right| \\
&\le \left|\int_E b\right| + \left|\int_{I_{\mathcal{T}_0}\setminus E} b\right| \\
&< \frac{1}{4}|I_{\mathcal{T}_0}| + \|b\|_2 |I_{\mathcal{T}_0} \setminus E|^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\frac{3}{4}|I_{\mathcal{T}_0}| < \|b\|_2 |I_{\mathcal{T}_0} \setminus E|^{\frac{1}{2}} < \frac{C_0 |I_{\mathcal{T}_0}|^{\frac{1}{2}}}{\gamma} |I_{\mathcal{T}_0} \setminus E|^{\frac{1}{2}},$$

so

$$\frac{9}{16}\frac{\gamma^2}{C_0^2}|I_{\mathcal{T}_0}| < |I_{\mathcal{T}_0} \setminus E|.$$

However, we assumed that

$$|E| > \left(1 - \frac{\gamma^2}{2C_0^2}\right) |I_{\mathcal{T}_0}|,$$

which is contradicted by the above conclusion. If E covers at least $(1 - \frac{\gamma^2}{2C_0^2})$ of $I_{\mathcal{T}_0}$ then $I_{\mathcal{T}_0} \setminus E$ covers less than $\frac{\gamma^2}{2C_0^2}$ of $I_{\mathcal{T}_0}$. $\square$

The constant C_0 in the lemma above does not depend on γ. We adopt the convention that when it appears, the constant C may depend on C_0. As we are making claims about the *sharp* power of γ, we insist that all constants other than γ do not depend on γ.

4. Dyadic b-input Global $T(b)$ Theorem

Theorem 4.1 (Dyadic b-input Global $T(b)$ Theorem)**.** *Let b be a function on $[0,1)$ such that*

$$|[b]_I| \geq 1$$

for all $I \in \Delta$ (i.e. b is 1-pseudo-accretive for all $I \in \Delta$) and such that for some $0 < \gamma \ll 1$,

$$\|b\|_{\mathrm{BMO}} \leq \gamma^{-1}.$$

Let T be a dyadic singular integral operator such that the following hold:

$$|\langle T\psi_I, \psi_I\rangle| \leq 1$$

$$\|Tb\|_{\mathrm{BMO}} \leq G_1\gamma^{-1}$$

$$\|T^*(1)\|_{\mathrm{BMO}} \leq G_2.$$

Then T is bounded on L^2, and

$$\|Tf\|_2 \leq C(1+\gamma^{-1}(1+G_1+G_2)+G_2)\|f\|_2.$$

Proof: From Lemma 3.1, the splitting lemma, we know that

$$\begin{aligned} Tb = &\sum_{I\in\Delta} \langle T\psi_I, \psi_I\rangle\langle b, \psi_I\rangle\psi_I \\ &+ \sum_{I\in\Delta} \frac{1}{|I|^{\frac{1}{2}}}\langle T(1), \psi_I\rangle\langle b, \widetilde{\chi}_I\rangle\psi_I \\ &+ \sum_{I\in\Delta} \frac{1}{|I|^{\frac{1}{2}}}\langle T^*(1), \psi_I\rangle\langle b, \psi_I\rangle\widetilde{\chi}_I. \end{aligned}$$

We wish to solve this for $T(1)$, and then find an upper bound on $\|T(1)\|_{\mathrm{BMO}}$ so that we can apply the global $T(1)$ theorem. Fix $J \subseteq [0,1)$. We pair Tb with ψ_J for this J. We note that $\langle \psi_I, \psi_J\rangle = \delta_{IJ}$ and

$$\langle \widetilde{\chi}_I, \psi_J\rangle = \begin{cases} \langle \widetilde{\chi}_I, \psi_J\rangle & \text{for all } I \subsetneq J \\ 0 & \text{for all } J \subseteq I \text{ and } J \cap I = \emptyset. \end{cases}$$

We calculate that

$$\begin{aligned} \langle Tb, \psi_J\rangle = \langle T\psi_J, \psi_J\rangle\langle b, \psi_J\rangle &+ \frac{1}{|J|^{\frac{1}{2}}}\langle T(1), \psi_J\rangle\langle b, \widetilde{\chi}_J\rangle \\ &+ \sum_{I\subsetneq J} \frac{1}{|I|^{\frac{1}{2}}}\langle T^*(1), \psi_I\rangle\langle b, \psi_I\rangle\langle \widetilde{\chi}_I, \psi_J\rangle. \end{aligned}$$

Therefore

$$\langle T(1), \psi_J\rangle = \frac{1}{[b]_J}\langle Tb, \psi_J\rangle - \frac{1}{[b]_J}\langle T\psi_J, \psi_J\rangle\langle b, \psi_J\rangle - \frac{1}{[b]_J}\sum_{I\subset J}\frac{1}{|I|^{\frac{1}{2}}}\langle T^*(1), \psi_I\rangle\langle b, \psi_I\rangle\langle \widetilde{\chi}_I, \psi_J\rangle.$$

By hypothesis, $|[b]_J|^{-1} \leq 1$. Using this and the triangle inequality, we see that

$$\begin{aligned}\|T(1)\|_{\mathrm{BMO}} &\leq \|Tb\|_{\mathrm{BMO}} + |\langle T\psi_J, \psi_J\rangle|\|b\|_{\mathrm{BMO}} + \left\|\sum_{I\subset J}\frac{1}{|I|^{\frac{1}{2}}}\langle T^*(1), \psi_I\rangle\langle b, \psi_I\rangle\widetilde{\chi}_I\right\|_{\mathrm{BMO}}\\ &\leq \frac{G_1}{\gamma} + \frac{1}{\gamma} + \frac{G_2}{\gamma},\end{aligned}$$

where the first two estimates in the last line are made using the hypotheses of the theorem and the last is made using a paraproduct estimate. See, for example, [**2**].

We apply the $T(1)$ theorem to get the result. $\square$

The following function and operator give a class of examples which proves that the power of γ in the Dyadic b-input Global $T(b)$ Theorem is best possible.

Example 4.1. Let $b \equiv 1$. Then $\|b\|_{\mathrm{BMO}} = 0$ and $|[b]_I| = 1$ for all $I \in \Delta$. Pick any Carleson sequence a_I and define

$$Tf = \sum_{I\in\Delta}\sqrt{a_I}\frac{1}{|I|^{\frac{1}{2}}}\langle f, \widetilde{\chi_I}\rangle\psi_I.$$

Recall that we define the Carleson measure of a sequence as follows:

Definition. Given a real sequence of positive numbers indexed by dyadic intervals, $(a_I)_{I\in\Delta}$, we say

$$\|\sqrt{a_I}\|_{\mathrm{BMO}} = \|\sqrt{a_I}\|_{\mathcal{C}} = \sup_{I\in\Delta}\frac{1}{|I|^{\frac{1}{2}}}\left(\sum_{J\subset I}a_J\right)^{\frac{1}{2}}.$$

Scale the sequence $(\sqrt{a_I})_{I\in\Delta}$ so that

$$\|\sqrt{a_I}\|_{\mathrm{BMO}} = \gamma^{-1}.$$

For example, we could let $a_{[0,1]} = \gamma^{-2}$ and $a_I = 0$ for all other I. Using this scaling, we have that

$$Tb = \sum_{I\in\Delta} \sqrt{a_I}\psi_I$$

so $\|Tb\|_{\mathrm{BMO}} = \gamma^{-1}$. For a general $f \in L^2([0,1))$,

$$Tf = \sum_{I\in\Delta} \sqrt{a_I}[f]_I\psi_I,$$

so by a paraproduct estimate,

$$\|Tf\|_2 \le C\|\sqrt{a_I}\|_{\mathrm{BMO}}\|f\|_2 \le C\|f\|_2\gamma^{-1}.$$

Let J be the interval on which $(\sqrt{a_I})_{I\in\Delta}$ attains the Carleson measure γ^{-1}. Let $f = \widetilde{\chi}_J \in L^2$ with $\|f\|_{L^2(J)} = 1$. Let $g = \frac{1}{|J|^{\frac{1}{2}}}\sum_{K\subset J}\sqrt{a_K}\psi_K \in L^2$. Then

$$\begin{aligned}
\langle Tf, g\rangle &= \left\langle \sum_{I\in\Delta} \sqrt{a_I}[f]_I\psi_I, \frac{1}{|J|^{\frac{1}{2}}}\sum_{K\subset J}\sqrt{a_K}\psi_K \right\rangle \\
&= \sum_{I\in\Delta} \frac{1}{|J|^{\frac{1}{2}}}\sum_{K\subset J}\sqrt{a_I}[f]_I\sqrt{a_K}\langle\psi_I,\psi_K\rangle \\
&= \sum_{I\subset J} a_I\frac{1}{|J|^{\frac{1}{2}}}[f]_I \\
&= \frac{1}{|J|}\sum_{I\subset J} a_I \\
&= \gamma^{-1} \le \sup\{\|Tf\|_2 : \|f\|_{L^2(J)} = 1\} = \|T\|.
\end{aligned}$$

Given the above example, we see that the power on γ is sharp in the b-input global $T(b)$ Theorem.

5. Dyadic b-input Local $T(b)$ Theorem

We now turn to the local version of the global $T(b)$ theorem. We shift from a globally defined function b to a set of functions $\{b_I\}_{I\in\Delta}$, with each function b_I supported on its corresponding dyadic interval I. Furthermore, we no longer look at the BMO norm of the function b_I, but instead bound its L^2 norm. We note that given a global b, we can find a local b_I by restricting b to the interval I, but it is not generally true that this yields the sharp power on γ.

The proof of the local theorem is intrinsically more difficult than that of the global theorem. We require, as indicated below, that the absolute value of the mean of b_I on I be at least 1, but we do not control the mean of b_I on subintervals $J \subsetneq I$. We must therefore use a stopping-time argument to prove the theorem. We will also use the nesting property of dyadic intervals.

We form the following definition for the sole purpose of making the notation easier later on.

Definition 5.1. Given a dyadic interval $I \in \Delta$, we define $\mathbf{S}(I)$ to be the span of $\{\psi_J : J \subseteq I\} \cup \widetilde{\chi_I}$.

Theorem 5.1 (Dyadic b-input Local $T(b)$ Theorem)**.** *Let T be a dyadic singular integral operator obeying*

$$\|T^*(1)\|_{\mathrm{BMO}} \leq G$$

$$|\langle T\psi_I, \psi_I\rangle| \leq 1$$

for a fixed G and for all $I \in \Delta$. Suppose that for every $I \in \Delta$, there exists a function $b_I \in \mathbf{S}(I)$ with mean 1 on I such that

$$\|b_I\|_{L^2(I)} \leq \frac{|I|^{\frac{1}{2}}}{\gamma}$$

and

$$\|Tb_I\|_{L^2(I)} \leq \frac{|I|^{\frac{1}{2}}}{\gamma}$$

for some $0 < \gamma \ll 1$. Then T is bounded on L^2 and moreover,

$$\|Tf\|_2 \leq C\|f\|_2 \left(1 + \frac{C(2+G)}{\gamma^2} + G\right).$$

Note. We will see in the proof that the theorem also follows if we substitute

$$\inf_c \|Tb_I - c\chi_I\|_2 \leq \frac{|I|^{\frac{1}{2}}}{\gamma}$$

for the hypothesis on $\|Tb_I\|_2$.

Proof: By Lemma 3.5, it suffices to show that given any interval $J \in \Delta$ we can show that

$$\sum_{I \in \mathrm{Tree}(J) \setminus \bigcup_{\mathcal{T}' \in \mathbb{T}} \mathcal{T}'} |\langle T(1), \psi_I\rangle|^2 \leq C|J|\gamma^{-2}$$

for a collection $\mathbb{T}$ of trees in $\mathrm{Tree}(J)$ whose tops are disjoint and satisfy

$$\sum_{\mathcal{T}' \in \mathbb{T}} |I_{\mathcal{T}'}| < (1 - \gamma^2)|J|.$$

If such a collection $\mathbb{T}$ exists, then

$$\|T(1)\|^2_{\mathrm{BMO}} \le \frac{C}{\gamma^4}$$

and we may apply the $T(1)$ Theorem to get the result.

Fix $J \in \Delta$. By hypotheses of the theorem there exists $b_J \in \mathbf{S}(J)$ with $|[b_J]_J| = 1$ and such that b_J and Tb_J satisfy the L^2 estimates in the hypotheses of the theorem. Furthermore, by Lemma 3.6, there exists a collection of trees $\mathbb{T} \subset \mathrm{Tree}(J)$ whose tops are disjoint and cover at most $(1 - C\gamma^2)$ of J for some constant C independent of γ. For all $T' \in \mathbb{T}$, $|[b_J]_{I_{T'}}| \le \frac{1}{4}$.

Let $F = \mathrm{Tree}(J) \setminus \bigcup_{T' \in \mathbb{T}} T'$. By construction of $\mathbb{T}$ in Lemma 3.6, we know that $|[b_J]_I| > \frac{1}{4}$ for all $I \in F$. As in the global b-input $T(b)$ theorem, we look at the pairing

$$\langle T(1), \psi_I\rangle = \frac{1}{[b_J]_I}\langle Tb_J, \psi_I\rangle + \frac{1}{[b_J]_I}\langle T\psi_I, \psi_I\rangle\langle b_J, \psi_I\rangle + \frac{1}{[b_J]_I}\sum_{K\subset I}\frac{1}{|K|^{\frac{1}{2}}}\langle T^*(1), \psi_K\rangle\langle b_J, \psi_K\rangle\langle \widetilde{\chi}_K, \psi_I\rangle. \tag{5}$$

We observe that we can write equation (5) only for those intervals on which the mean of b_J is large in absolute value. That is, we must restrict equation (5) to those I which are in F to be sure we are never dividing by zero. Using the triangle inquality on (5), we see that for $I \in F$,

$$|\langle T(1), \psi_I\rangle| \le C(|\langle Tb_J, \psi_I\rangle| + |\langle T\psi_I, \psi_I\rangle||\langle b_J, \psi_I\rangle|) + C\left|\sum_{K\subset I}\frac{1}{|K|^{\frac{1}{2}}}\langle T^*(1), \psi_K\rangle\langle b_J, \psi_K\rangle\langle \widetilde{\chi}_K, \psi_I\rangle\right|,$$

and therefore

$$\left(\sum_{I\in F}|\langle T(1), \psi_I\rangle|^2\right)^{\frac{1}{2}} \le C\left(\left(\sum_{I\in F}|\langle Tb_J, \psi_I\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{I\in F}|\langle b_J, \psi_I\rangle|^2\right)^{\frac{1}{2}}\right) + C\left(\sum_{I\in F}\left|\sum_{K\subset I}\frac{1}{|K|^{\frac{1}{2}}}\langle T^*(1), \psi_K\rangle\langle b_J, \psi_K\rangle\langle \widetilde{\chi}_K, \psi_I\rangle\right|^2\right)^{\frac{1}{2}}.$$

Using the hypotheses on the first two summands and a paraproduct estimate from Lemma 3.2 on the third, we get that

$$\left(\sum_{I\in F}|\langle T(1),\psi_I\rangle|^2\right)^{\frac{1}{2}} \le C\left(\frac{|J|^{\frac{1}{2}}}{\gamma}+\frac{|J|^{\frac{1}{2}}}{\gamma}+\frac{G|J|^{\frac{1}{2}}}{\gamma}\right)=\frac{C(2+G)|J|^{\frac{1}{2}}}{\gamma}.$$

We note here that as we are pairing Tb_J with the mean zero function ψ_I,

$$\langle Tb_J - c\chi_J,\psi_I\rangle = \langle Tb_J,\psi_I\rangle$$

and either hypothesis will yield the desired result.

Therefore

$$\sum_{I\in S}|\langle T(1),\psi_I\rangle|^2 \le \frac{C(2+G)|J|}{\gamma^2},$$

so

$$\|T(1)\|^2_{\mathrm{BMO}} \le \frac{C(2+G)}{\gamma^4}.$$

By the $T(1)$ Theorem,

$$\|Tf\|_2 \le \|f\|_2\left(1+\frac{C(2+G)}{\gamma^2}+G\right). \qquad \square$$

The following is an example which shows that the power on γ in the above theorem is sharp.

Example 5.1. We can relax the hypotheses slightly and allow $1 \le |[b_I]_I| \le 2$, as such a change can be countered later by normalization. As usual, for simplicity we dilate the intervals in $\Delta_M = \Delta$ so that our largest interval is $[0,1)$. We also assume $\gamma = 2^{-N}$ for large $N \in \mathbb{Z}^+$. For any dyadic interval $K \subset [0,1]$ we note that $|K| = 2^{-n}$ for some $n \in \mathbb{Z}^+$.

For any dyadic subinterval $K \subseteq [0,1)$, let K_0 be the rightmost descendent of K with $|K_0| = \gamma^2|K|$. Let $J = [0,1)$. Then $J_0 = [1-\gamma^2, 1)$.

Now define the function b_K as follows:

$$b_K(t) = \begin{cases}\gamma & \text{if } t \in K\setminus K_0\\ \gamma^{-2} & \text{if } t\in K_0.\end{cases}$$

Then

$$\begin{aligned}|[b_K]_K| &= \frac{1}{|K|}\int_K b_K\\ &= \frac{1}{|K|}\left(\gamma|K|(1-\gamma^2)+\gamma^2|K|\gamma^{-2}\right) = \gamma-\gamma^2+1 \in (1,2),\end{aligned}$$

and

$$\|b_K\|^2_{L^2(K)} = \gamma^2|K|(1-\gamma^2)+\gamma^2|K|\gamma^{-4} = \frac{C|K|}{\gamma^2}.$$

Now we select a Carleson sequence $(a_I)_{I\in\Delta}$ and then we define

$$Tf = \sum_{I\in\Delta} \sqrt{a_I}\frac{1}{|I|^{\frac{1}{2}}}\langle f, \widetilde{\chi}_I\rangle\psi_I.$$

Let

$$\tilde{\mathbb{I}} = \{I_l : I \not\subseteq K_0, \forall\, K \subseteq [0,1)\}.$$

$I \in \tilde{\mathbb{I}}$ if and only if I is the left child of an interval which is not contained in K_O for any K. Let $\mathbb{I} \subset \tilde{\mathbb{I}}$ be those intervals in $\tilde{\mathbb{I}}$ which are maximal with respect to inclusion. By this construction, the intervals in $\mathbb{I}$ are disjoint and no interval in $\mathbb{I}$ is contained in J_0. Therefore

$$\sum_{I\in\mathbb{I}} |I| \leq (1-\gamma^2)|J| = 1-\gamma^2.$$

Let $K \subset J$. Let

$$\tilde{\mathbb{I}}_K = \{I_l : I \not\subseteq L_0, \forall\, L \subseteq K\}$$

and let $\mathbb{I}_K$ be the maximal such intervals. Then we know, as above, that $\mathbb{I}_K$ covers at most $1-\gamma^2$ of K.

Let

$$a_I = \begin{cases} |I|\gamma^{-2} & I \in \tilde{\mathbb{I}} \\ 0 & \text{otherwise.} \end{cases}$$

Then for any dyadic interval $L \subset [0,1)$,

$$\begin{aligned} \sum_{I\subseteq L} a_I = \sum_{I\in\tilde{\mathbb{I}}_L} |I|\gamma^{-2} = \gamma^{-2} &\sum_{I\in\tilde{\mathbb{I}}_L} |I| \\ &\leq \gamma^{-2}|L| \sum_{n\in\mathbb{Z}^+} \frac{1}{2}(1-\gamma^2)^n \\ &= C|L|\gamma^{-4} \end{aligned}$$

by geometric series. This implies that

$$\|\sqrt{a_I}\|_{\mathrm{BMO}} = C\gamma^{-2}.$$

For any $K \subset [0,1)$

$$Tb_K = \sum_{I\in\Delta} \sqrt{a_I}\frac{1}{|I|^{\frac{1}{2}}}\langle b_K, \widetilde{\chi}_I\rangle\psi_I = \sum_{I\in\tilde{\mathbb{I}}} \gamma^{-1}\langle b_K, \widetilde{\chi}_I\rangle\psi_I.$$

Therefore

$$\|Tb_K\|_2 = \gamma^{-1}\left(\sum_{I\in\tilde{\mathbb{I}}} |\langle b_K, \widetilde{\chi}_I\rangle|^2\right)^{\frac{1}{2}}.$$

This sum can be split up into three parts, where we sum over intervals I such that $I \cap K = \emptyset, K \subseteq I$, or $I \subset K$. We calculate $|\langle b_K, \widetilde{\chi}_I\rangle|^2 = \frac{1}{|I|}\left|\int_I b_K\right|^2$ for each of these cases.

If $I \cap K = \emptyset$, then $\int_I b_K = 0$ as b_K is supported on K.

If $K \subseteq I$, then

$$\frac{1}{|I|}\left|\int_I b_K\right|^2 = \frac{|K|^2}{|I|}|[b_K]_K|^2 \leq 4\frac{|K|^2}{|I|}.$$

If $I \subsetneq K$, then

$$\frac{1}{|I|}\left|\int_I b_K\right|^2 = \frac{1}{|I|}\left|\int_I \gamma\right|^2 = \gamma^2|I|$$

because $I \in \tilde{\mathbb{I}}$ and so by the nesting properties of dyadic intervals, $I \not\subset K_0$.

Therefore,

$$\begin{aligned}
\|Tb_K\|_{L^2(K)} &= \frac{1}{\gamma}\left(\sum_{I\in\tilde{\mathbb{I}}} |\langle b_K, \widetilde{\chi}_I\rangle|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\gamma}\left(\sum_{I\in\tilde{\mathbb{I}}, I\subsetneq K} \gamma^2|I| + \sum_{I\in\tilde{\mathbb{I}}, K\subset I} 4\frac{|K|^2}{|I|}\right)^{\frac{1}{2}} \\
&= C|K|^{\frac{1}{2}},
\end{aligned}$$

where the first sum is bounded by $C|K|$ using the same calculation used to bound the measure of $(a_I)_{I\in\Delta}$, and the second sum is bounded by $C|K|$ using geometric series and that $|K| < |I|$.

This shows that all of the hypotheses of the theorem are satisfied. Now we must find $\|T(1)\|_{\mathrm{BMO}}$ so that we can apply the $T(1)$ theorem.

We know that

$$T(1) = \sum_{I\in\mathbb{I}} \frac{1}{\gamma}\frac{1}{|I|^{\frac{1}{2}}}\langle 1, \widetilde{\chi}_I\rangle\psi_I = \sum_{I\in\mathbb{I}} \frac{1}{\gamma}|I|^{\frac{1}{2}}\psi_I.$$

Therefore

$$\begin{aligned}
\|T(1)\|_{\mathrm{BMO}}^2 &= \sup_K \frac{1}{|K|}\sum_{I\in\mathbb{I}\cap K} \frac{1}{\gamma^2}|I| \\
&= \sup_K \frac{1}{|K|}\sum_{I\in\mathbb{I}\cap K} a_I \\
&= \|\sqrt{a_I}\|_{\mathrm{BMO}}^2 = C\gamma^{-4}
\end{aligned}$$

by previous calculation. We can conclude that

$$\|T(1)\|_{\mathrm{BMO}} = \frac{C}{\gamma^2}.$$

By application of the $T(1)$ theorem, we conclude that

$$\|Tf\|_2 \leq C\|f\|_2\gamma^{-2}.$$

Let $g = \psi_{[0,\frac{1}{2})}$ and let $f = \frac{2}{\sqrt{2}\gamma}\chi_{[0,\frac{1}{2})}$. Then

$$\begin{aligned}\langle Tf, g\rangle &= \sum_{I\in\mathbb{I}} \frac{1}{\gamma}\langle f, \widetilde{\chi_I}\rangle\langle\psi_I, g\rangle \\ &= \frac{1}{\gamma}\langle f, \widetilde{\chi}_{[0,\frac{1}{2})}\rangle \\ &= \frac{1}{\gamma^2} \leq C\|T\|.\end{aligned}$$

Therefore

$$\|Tf\|_2 = C\gamma^{-2}\|f\|_2,$$

and we see that this example shows that the power of -2 is sharp.

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