Collect. Math. **59**, 2 (2008), 215–237

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Hodge spaces of real toric varieties

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Received September 17, 2007. Revised December 12, 2007.

ABSTRACT

We define the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$ of a rational fan Σ . If Σ is the normal fan of a reflexive polytope Δ then we use polyhedral duality to compute the $\mathbb{Z}_2$ Hodge Spaces of Σ . In particular, if the cones of dimension at most e in the face fan Σ^* of Δ are smooth then we compute $H_{pq}(\Sigma)$ for $p < e - 1$. If Σ^* is a smooth fan then we completely determine the spaces $H_{pq}(\Sigma)$ and we show X_Σ is maximal, meaning that the sum of the $\mathbb{Z}_2$ Betti numbers of $X_\Sigma(\mathbb{R})$ is equal to the sum of the $\mathbb{Z}_2$ Betti numbers of $X_\Sigma(\mathbb{C})$.

1. Introduction

In this paper, we define and study the $\mathbb{Z}_2$ Hodge spaces of a rational fan Σ . The $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$ are indexed by pairs of integers p, q with $0 \leq q \leq p \leq \dim \Sigma$ and are defined to be the homology groups $H_p(\wedge^q \mathcal{E})$, where $\wedge^q \mathcal{E}$ is the q th exterior power of the cosheaf $\mathcal{E}$ on Σ . Geometrically, for $\sigma \in \Sigma$ the stalk E_σ of the cosheaf $\mathcal{E}$ is $(\mathbb{Z}_2)^r$, a compact subset of the real orbit $O_\sigma(\mathbb{R}) \cong (\mathbb{R}^*)^r$ of the real toric variety $X_\Sigma(\mathbb{R})$. The terminology $\mathbb{Z}_2$ Hodge spaces is inspired by work of Brion [7] who considered similar spaces associated to a fan Σ . When Σ is a smooth fan, we have $H_{pq}(\Sigma) = 0$ for $p \neq q$. However, for $p > q$ the spaces $H_{pq}(\Sigma)$ are not generally well understood. While these spaces are interesting from a purely combinatorial viewpoint, they are also important in understanding the topology of toric varieties. The $\mathbb{Z}_2$ Hodge spaces of Σ are related to the topology of both the real and complex points of the toric variety X_Σ in that $H_{pq}(\Sigma) = \overline{E}_{p,q}^1 = E_{p,q}^2$, where $(\overline{E}^r, \overline{d}^r)$ and (E^r, d^r) are two spectral sequences with

$$\overline{E}_{p,q}^1 \implies H_p(X_\Sigma(\mathbb{R}); \mathbb{Z}_2) \quad \text{and} \quad E_{p,q}^2 \implies H_{p+q}(X_\Sigma(\mathbb{C}); \mathbb{Z}_2) .$$

Keywords: Real toric variety, Hodge space, reflexive polytope, Smith-Thom inequality.

MSC2000: Primary 14M25; Secondary 55T99, 52B12.

When the sum of the $\mathbb{Z}_2$ Betti numbers for $X_\Sigma(\mathbb{R})$ is equal to the sum of the $\mathbb{Z}_2$ Betti numbers for $X_\Sigma(\mathbb{C})$, we say X_Σ is *maximal*. As discussed in Section 3.2, if we have $\overline{E}^\infty = \overline{E}^1$ then $E^\infty = E^2$ and X_Σ is maximal. In [5], Bihan, Franz, McCrory, and van Hamel conjectured that all toric varieties are maximal, and in [13] we found a counterexample to this conjecture.

In this paper, we mainly restrict our attention to fans which are the normal fans of reflexive polytopes. The set of reflexive polytopes has a duality operation $*$ which is very beneficial for our purposes. If Δ is a reflexive polytope then we write Σ for the normal fan of Δ and Σ^* for the normal fan of the dual (or polar) polytope Δ^* . It can occur that X_Σ is highly singular while X_{Σ^*} is nonsingular. For example, if Δ is the d dimensional cross polytope then Δ^* is the d dimensional cube. In this case, $X_{\Sigma^*} = \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ and the singular locus of X_Σ has codimension 2. The main question we seek to answer is the following. What happens to the $\mathbb{Z}_2$ Hodge spaces of Σ (and the topology of X_Σ) when Σ^* is a smooth fan? We answer this question by using the polyhedral duality mentioned above and using the language of cosheaves on a fan.

Section 4 is devoted to establishing the following theorem.

Theorem 1.0.1

Let Δ be a reflexive polytope and Σ^ the face fan of Δ . If the cones in Σ^* of dimension e are $\mathbb{Z}_2$ regular then*

- (1) $H_{pq}(\Sigma) = 0$ for $q < p < e - 1$
- (2) $H_{qq}(\Sigma) = \mathbb{Z}_2$ for $q < e - 1$

where Σ is the normal fan of Δ .

In Section 5 we assume Σ^* consists of $\mathbb{Z}_2$ regular cones. We show that the spectral sequence $(\overline{E}^r, \overline{d}^r)$ for X_Σ collapses at $\overline{E}^1$, and hence X_Σ is maximal.

Some notation: $N \cong \mathbb{Z}^d$ and $M = \text{Hom}(N, \mathbb{Z})$ are dual lattices with dual pairing denoted $\langle \cdot, \cdot \rangle$. All tensor products are taken over $\mathbb{Z}$. We write $\Sigma \subset N$ when $\Sigma \subset N \otimes \mathbb{R}$ is the normal fan of a d dimensional lattice polytope $\Delta \subset M \otimes \mathbb{R}$ (written $\Delta \subset M$). Except where noted, all rational fans are assumed to be polytopal. Many statements hold for the more general complete fans, but the class of polytopal fans is enough for our purposes. For $\sigma \in \Sigma$ we will abuse notation by writing $\sigma \cap N$ for the lattice of points in the linear span of σ . The cones in Σ (resp. faces of σ) of dimension k are denoted $\Sigma(k)$ (resp. $\sigma(k)$).

All homology and cohomology groups will be with $\mathbb{Z}_2$ coefficients, unless otherwise stated.

I would like to thank Matthias Franz and Tom Braden for helpful discussions at the June 2006 Toric Topology conference in Osaka, my advisor Clint McCrory for our ongoing discussions, and Bernd Sturmfels for introducing me to the beauty of real toric varieties.

2. The real points of a toric variety

2.1 The natural cell structure on $X_\Sigma(\mathbb{R})$

A real toric variety is a normal real variety that contains a real algebraic torus $(\mathbb{R}^*)^d$ as a dense subset. When $X_\Sigma = X_\Sigma(\mathbb{C})$ is a projective toric variety embedded in $\mathbb{CP}^r$, the real variety $X_\Sigma(\mathbb{R})$ is the intersection of X_Σ with the real projective space $\mathbb{RP}^r$. A discussion of real toric varieties can be found in [20]. Suppose X_Σ is projective with moment map $\mu = \mu_{\mathbb{C}}$ (see [10, §4.2]). Since $\mu_{\mathbb{C}}$ is invariant under complex conjugation, we have a map $\mu_{\mathbb{R}}$ for the real toric variety $X_\Sigma(\mathbb{R})$. We write the complex torus as $T = (\mathbb{R}_{>0})^d \times (N \otimes \mathbb{R})/N$ and the real torus as $T(\mathbb{R}) = (\mathbb{R}_{>0})^d \times \frac{1}{2}N/N$. The maps $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ are shown below

$$\begin{array}{ccc}
 (\mathbb{R}_{>0})^d \times (N \otimes \mathbb{R})/N & \subset & X_\Sigma(\mathbb{C}) \\
 \parallel & \cup & \cup \\
 (\mathbb{R}_{>0})^d \times \frac{1}{2}N/N & \subset & X_\Sigma(\mathbb{R}) \\
 \cong \downarrow & & \downarrow \mu_{\mathbb{R}} \\
 \text{int } \Delta & \subset & \Delta
 \end{array}
 \quad .$$

Suppose $p \in \text{int } f$, $f < \Delta$, and $\sigma \in \Sigma$ the cone dual to f . Then, we have the following

$$\begin{aligned}
 \mu_{\mathbb{C}}^{-1}\{p\} &= \frac{(N \otimes \mathbb{R})/\sigma}{N/\sigma \cap N} \cong (S^1)^{\dim f} \\
 \cup & \quad \cup \quad \cup \\
 \mu_{\mathbb{R}}^{-1}\{p\} &= \frac{\frac{1}{2}N/N}{\sigma \cap \frac{1}{2}N/\sigma \cap N} \cong (S^0)^{\dim f} .
 \end{aligned}
 \tag{1}$$

In the sequel, we will use $N/2N$ rather than $\frac{1}{2}N/N$ and $N(\sigma) := \frac{N/2N}{\sigma \cap N/\sigma \cap 2N}$ instead of $\frac{\frac{1}{2}N/N}{\sigma \cap \frac{1}{2}N/\sigma \cap N}$ for convenience. The group $\sigma \cap N/\sigma \cap 2N$ will be denoted N_σ . The real moment map $\mu_{\mathbb{R}}$ induces a polyhedral cell structure on $X_\Sigma(\mathbb{R})$. Explicitly, we realize $X_\Sigma(\mathbb{R})$ as the quotient space

$$\begin{aligned}
 X_\Sigma(\mathbb{R}) &\cong \frac{\Delta \times N/2N}{\sim} \\
 (x, t) &\sim (x', t') \iff x = x' \quad \text{and} \quad t - t' \in N_\gamma
 \end{aligned}
 \tag{2}$$

where γ is dual to $g < \Delta$ and $x \in \text{int } g$. Thus, the cells in (2) are of the form (g, t) with $t \in N(\gamma)$. A discussion of this cell structure can be found in [12, §11.5 B].

Next, we discuss the first step in an algorithm for identifying the faces of the 2^d copies of Δ . This algorithm basically follows from [12, Theorem 5.4 in Chapter 11]. The entire algorithm is presented in [14] and involves at step j gluing faces of codimension j . When we identify two faces, we will glue the closed faces. The reason for this is as

follows. Let (g, t) and (g, t') be two open faces identified in the relation $\sim$ of (2). Then, $t - t' \in N_\gamma$ where γ is dual to g . Suppose $f < g$ in Δ and β is the cone dual to f . We have $\gamma < \beta$ and $N_\gamma \subset N_\beta$ yielding $t - t' \in N_\beta$. Thus, (f, t) and (f, t') are identified in the relation $\sim$.

Step 1: Let $f_1, f_2, \dots, f_p$ be the facets of Δ and r_i the ray in Σ dual to f_i . If R_i is the image of r_i in $N/2N$ then (f_i, t) is identified with (f_i, t') if and only if $t - t' = R_i$.

Lemma 2.1.1

After Step 1 we are left with 2^{d-s} connected components, where

$$s = \text{rank}(\text{span}\{R_1, R_2, \dots, R_p\})$$

with R_i the image of r_i in $N/2N$ and $\{r_1, r_2, \dots, r_p\}$ the rays of Σ .

To prove Lemma 2.1.1 we fix $t \in N/2N$. The d -cell (Δ, t) will be glued to the d -cells $(\Delta, t+R_1), (\Delta, t+R_2), \dots, (\Delta, t+R_p)$ along the facets $f_1, f_2, \dots, f_p$ respectively. Next, note that the cell $(\Delta, t+R_i+R_j)$ is in the same component as (Δ, t) since the cell $(\Delta, t+R_i)$ is glued to $(\Delta, (t+R_i)+R_j)$ along the facet f_j . Similarly, we can see that $(\Delta, t+\sum_{i \in I} R_i)$ is in the same connected component as (Δ, t) where $\sum_{i \in I} R_i \in \text{span}\{R_1, R_2, \dots, R_p\}$. As the rank of the $\mathbb{Z}_2$ vector space is s , there are exactly 2^s d -cells in this connected component yielding $2^d/2^s = 2^{d-s}$ connected components in total after Step 1. $\square$

Note that in Steps 2 and higher, only faces of codimension greater than 1 are identified. Hence, Lemma 2.1.1 gives

$$\text{rank} H_d(X_\Sigma(\mathbb{R})) = 2^{d-s}.$$

2.2 T-homeomorphic versus algebraically isomorphic

Two real toric varieties may be homeomorphic even though they are not isomorphic as algebraic varieties. Since our main interest lies in the topological type of real toric varieties, we introduce the notion of T-homeomorphic real toric varieties. We will first develop the notion of T-homeomorphism for the real variety $X_\Sigma(\mathbb{R})$ in the projective case assuming Δ is full dimensional. We then discuss this notion for the real part of an orbit closure corresponding to a face f of Δ . Suppose $\Sigma_1 \subset N_1$ and $\Sigma_2 \subset N_2$ are two fans with $N_1 \cong N_2 \cong \mathbb{Z}^d$. Let's assume A is a $d \times d$ integer valued matrix with $\det A$ an odd integer. We consider A as a map between the d dimensional lattices N_1 and N_2 . Then we have $A(2N_1) \subset 2N_2$ and hence A determines a map

$$A' : N_1/2N_1 \longrightarrow N_2/2N_2.$$

Since the determinant of A is odd, the preimage $A^{-1}\{2N_2\}$ is contained in $2N_1$. Thus A' is an injective map between $\mathbb{Z}_2$ vector spaces of the same rank yielding A' is an isomorphism of vector spaces.

DEFINITION 2.2.1 Suppose the $d \times d$ matrix A determines a rational isomorphism of fans. Then A induces a *T-homeomorphism* $\tilde{A} : X_{\Sigma_1}(\mathbb{R}) \xrightarrow{\cong} X_{\Sigma_2}(\mathbb{R})$ provided $\det A = 1 \pmod{2}$. We say $X_{\Sigma_1}(\mathbb{R})$ and $X_{\Sigma_2}(\mathbb{R})$ are *T-homeomorphic*.

Note that for $\sigma_1 \in \Sigma_1$ the restriction $A'|_{N_{\sigma_1}}$ is an isomorphism between N_{σ_1} and $N_{\sigma_2} = A'(N_{\sigma_1})$ where $\sigma_2 \in \Sigma_2$. Note that the dual matrix A^* gives a combinatorial equivalence between the polytopes Δ_2 and Δ_1 which define X_{Σ_2} and X_{Σ_1} , respectively. Hence, the T-homeomorphism $\tilde{A}$ is a homeomorphism which respects the natural cell structure on $X_{\Sigma_1}(\mathbb{R})$

$$\frac{\Delta_1 \times N_1/2N_1}{\sim_1} \xrightarrow[\cong]{\tilde{A}} \frac{\Delta_2 \times N_2/2N_2}{\sim_2}.$$

Suppose $f_1 < \Delta_1$ with $\sigma_1 \in \Sigma_1$ dual to f . Using equation (1) in Section 2.1, we see $V(\sigma_1)(\mathbb{R})$ inherits a cell structure from the cell structure (2) on $X_{\Sigma_1}(\mathbb{R})$. The notation $V(\sigma_1)$ refers to the closure of the torus orbit associated to the cone σ_1 . Specifically, we have

$$\begin{aligned} V(\sigma_1)(\mathbb{R}) &\cong \frac{f \times N(\sigma_1)}{\sim_f} \\ (x, t) \sim^f (x', t') &\iff x = x' \quad \text{and} \quad t - t' \in N_\gamma/N_\sigma \end{aligned} \tag{3}$$

where γ is dual to $g < f$ and $x \in \text{int}g$. Next, the isomorphism $A'|_{N_{\sigma_1}} : N_{\sigma_1} \xrightarrow{\cong} N_{\sigma_2}$ mentioned above determines an isomorphism between the quotients $N(\sigma_1) \xrightarrow{\cong} N(\sigma_2)$. We obtain a homeomorphism induced by the T-homeomorphism $\tilde{A}$

$$\frac{f_1 \times N(\sigma_1)}{\sim_{f_1}^1} \xrightarrow{\cong} \frac{f_2 \times N(\sigma_2)}{\sim_{f_2}^2},$$

where $f_2 < \Delta_2$ is the face for which $A^*(f_2) = f_1$. When this occurs we say $V(\sigma_1)(\mathbb{R})$ and $V(\sigma_2)(\mathbb{R})$ are T-homeomorphic.

3. The $\mathbb{Z}_2$ Hodge spaces and the spectral sequence $\overline{E}^r$

3.1 The cosheaf $\mathcal{E}$ and the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$

In this section, we define the notion of a cosheaf on a fan Σ and cosheaf homology. In later sections, we will also use sheaves on Σ and sheaf cohomology. The definition and properties of sheaves on fans are similar to those of cosheaves and are developed in [7, §1.1]. Sheaves on fans are also studied in [6, 1]. The main difference between the sheaves we consider and those in [7] is that our sheaves are sheaves of $\mathbb{Z}_2$ vector spaces.

A *cosheaf* $\mathcal{F}$ of $\mathbb{Z}_2$ vector spaces on a fan Σ is a collection of vector spaces $(F_\sigma)_{\sigma \in \Sigma}$ over $\mathbb{Z}_2$ together with face extension maps $\rho_{\sigma, \tau} : F_\sigma \longrightarrow F_\tau$ for $\sigma < \tau$ satisfying the following two conditions.

- If $\sigma < \tau < \beta$ then $\rho_{\tau, \beta} \circ \rho_{\sigma, \tau} = \rho_{\sigma, \beta}$
- $\rho_{\sigma, \sigma}$ is the identity map

Note that throughout our work, we will use script letters for the cosheaf (or sheaf) and Roman letters for their stalks.

We define $H_p(\mathcal{F})$ to be the p th homology group of the complex $(C_*(\mathcal{F}), \partial_*)$, where

$$C_p(\mathcal{F}) := \bigoplus_{\sigma \in \Sigma(d-p)} F_\sigma$$

and $\partial_p : C_p(\mathcal{F}) \rightarrow C_{p-1}(\mathcal{F})$ is the direct sum of the maps

$$\sum_{\sigma \in \tau(d-p)} \rho_{\sigma, \tau} : F_\sigma \rightarrow \bigoplus_{\sigma \in \tau(d-p)} F_\tau. \quad (4)$$

Note that if we were working with cosheaves of k vector spaces with $\text{char } k \neq 2$ then we would need to introduce signs

in (4) to guarantee $\partial^2 = 0$.

Next, we define the cosheaf $\mathcal{N}$ on Σ by

$$N_\sigma := \sigma \cap N / \sigma \cap 2N \quad \text{for } \sigma \in \Sigma$$

with the extension map $\rho_{\sigma, \tau}$ given by inclusion

$$\rho_{\sigma, \tau} : N_\sigma \xrightarrow{\subset} N_\tau \quad \text{for } \sigma < \tau.$$

If $\dim \sigma = q$ then N_σ is a rank q vector space over $\mathbb{Z}_2$. We encountered the cosheaf $\mathcal{N}$ in Section 2.1 when identifying the 2^d copies of Δ .

The cosheaf $\mathcal{E}$ is defined so that

$$E_\sigma := N(\sigma) = \frac{N/2N}{N_\sigma}.$$

If $\sigma < \tau$ then the extension map $\varpi_{\sigma, \tau} : N(\sigma) \rightarrow N(\tau)$ is induced from the identity on $N/2N$ which takes N_σ to N_τ . Thus, the cosheaf $\mathcal{E}$ is the cokernel of the inclusion $\mathcal{N} \hookrightarrow N/2N$, where $N/2N$ is the constant cosheaf assigning $N/2N$ to each cone in Σ .

DEFINITION 3.1.1 The $\mathbb{Z}_2$ Hodge spaces of Σ , denoted $H_{pq}(\Sigma)$, are defined to be the homology groups

$$H_{pq}(\Sigma) := H_p(\wedge^q \mathcal{E}),$$

where $\wedge^q \mathcal{E}$ is the q th exterior power of the cosheaf $\mathcal{E}$.

Remark 3.1.2 Since $\wedge^q E_\sigma$ is zero for $q > \text{codim } \sigma$, we have $C_p(\wedge^q \mathcal{E}) = 0$ for $p < q$. Hence the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$ are indexed by integers p, q with $0 \leq q \leq p \leq d$.

3.2 The spectral sequence $\overline{E}^r$

We recall that the cells in $X_\Sigma(\mathbb{R})$ are of the form (f, t) with $f < \Delta$ and $t \in N(\sigma)$, $\sigma \in \Sigma$ is dual to f . It follows that

$$C_j(X_\Sigma(\mathbb{R})) = \bigoplus_{\sigma \in \Sigma(d-j)} H_0(N(\sigma)), \quad (5)$$

where $H_0(N(\sigma))$ is the group algebra of the finite group $N(\sigma)$. Note that elements in $H_0(N(\sigma))$ are formal combinations of elements in $N(\sigma)$ with coefficients in $\mathbb{Z}_2$. The

boundary map ∂ for the cellular chain complex $C_*(X_\Sigma(\mathbb{R}))$ is induced from the face extension maps $\varpi_{\sigma,\tau} : N(\sigma) \longrightarrow N(\tau)$ from Section 3.1, and ∂_j is the direct sum of the maps

$$\sum_{\sigma \in \tau(d-j)} (\varpi_{\sigma,\tau})_* : H_0(N(\sigma)) \longrightarrow \bigoplus_{\sigma \in \tau(d-j)} H_0(N(\tau)).$$

In [5] Bihan et al. want to understand the relationship between the topology of $X_\Sigma(\mathbb{R})$ and that of $X_\Sigma(\mathbb{C})$. The authors show that the chain complex $C_*(X_\Sigma(\mathbb{R}))$ can be filtered in such a way that the associated spectral sequence $(\bar{E}^r, \bar{d}^r)$ converges to $H_*(X(\mathbb{R}))$ and is known to collapse at $\bar{E}^1$ when X_Σ is complete and has isolated singularities or when the dimension of X_Σ is at most 3 [5]. Our notation is slightly different than in [5] and hence we briefly review their construction.

Using (5) we may specify a filtration on $C_*(X_\Sigma(\mathbb{R}))$ by giving a filtration of $H_0(N(\sigma))$, the $\mathbb{Z}_2$ group algebra of $N(\sigma)$. We use the *augmentation* homomorphism ϵ_σ

$$\begin{aligned} \epsilon_\sigma : H_0(N(\sigma)) &\longrightarrow \mathbb{Z}_2 \\ \sum_{g \in N(\sigma)} n_g g &\longmapsto \sum n_g. \end{aligned}$$

We define I_σ , an ideal in $H_0(N(\sigma))$, via $I_\sigma := \ker \epsilon_\sigma$. This gives a filtration of $H_0(N(\sigma))$

$$0 = I_\sigma^{j+1} \subset I_\sigma^j \subset \dots \subset I_\sigma^2 \subset I_\sigma \subset I_\sigma^0 = H_0(N(\sigma))$$

where $j = \text{rank } N(\sigma) = \text{codim } \sigma$. We reindex by setting $J_\sigma^p = I_\sigma^{d-p}$ so that

$$0 = J_\sigma^{d-j-1} \subset J_\sigma^{d-j} \subset \dots \subset J_\sigma^{d-2} \subset J_\sigma^{d-1} \subset J_\sigma^d = H_0(N(\sigma)) \quad (6)$$

is an increasing filtration of $H_0(N(\sigma))$. The filtrations of $H_0(N(\sigma))$ of the form (6) for $\sigma \in \Sigma$ determine an increasing filtration F of $C_*(X_\Sigma(\mathbb{R}))$ and $\partial_j(F_q C_j(X_\Sigma(\mathbb{R}))) \subset F_q C_{j-1}(X_\Sigma(\mathbb{R}))$. We let $(\tilde{E}^r, \tilde{d}^r)$ be the associated spectral sequence as in [16]. Using [5, Proposition 7.1] we arrive at the following equality

$$\tilde{E}_{p,q}^1 \cong H_{p+q, d-p}(\Sigma),$$

where $H_{p+q, d-p}(\Sigma)$ is the $\mathbb{Z}_2$ Hodge space of Σ defined in Section 3.1. Next, since $\tilde{E}_{p,q}^1 = 0$ for $q < d - 2p$, we obtain a natural homomorphism

$$\tilde{E}_{d-(p+q), 2(p+q)-d}^1 \longrightarrow H_{p+q}(X_\Sigma(\mathbb{R})). \quad (7)$$

At this point, we introduce reindexing which we will use in the sequel. We rotate the $\tilde{E}^1$ term clockwise by 90° and then shear so that the diagonal lines $p + q = c$ become vertical lines $p = c$. We write $(\bar{E}^r, \bar{d}^r)$ for the spectral sequence obtained after reindexing.

The total grading of $(\bar{E}^r, \bar{d}^r)$ is p , and furthermore we have the identity $\bar{E}_{p,q}^1 = H_{pq}(\Sigma)$. Figure 1 shows the terms $\tilde{E}_{p,q}^1$ and $\bar{E}_{p,q}^1$ for a 4 dimensional fan Σ .

For the spectral sequence $(\bar{E}^r, \bar{d}^r)$ the boundary map $\bar{d}^r$ satisfies $\bar{d}_{p,q}^r : \bar{E}_{p,q}^r \longrightarrow \bar{E}_{p-1, q+r}^r$. We show the first five boundary maps in Figure 2.

Figure 1: The terms $\tilde{E}_{p,q}^1$ (left) and $\bar{E}_{p,q}^1$ (right) Figure 2: The differentials $\bar{d}^0, \bar{d}^1, \bar{d}^2, \bar{d}^3$, and $\bar{d}^4$

Next, we point out that after reindexing the natural homomorphism (7) becomes a map which we denote $f_*^{\mathbb{R}}$ and will use in future sections where

$$f_q^{\mathbb{R}} : H_{qq}(\Sigma) = \bar{E}_{q,q}^1 \longrightarrow H_q(X_{\Sigma}(\mathbb{R})). \quad (8)$$

From [5, Proposition 7.1] we have

$$\bar{E}_{p,q}^1 \cong E_{p,q}^2 \quad \text{for } 0 \leq p, q \leq d \quad (9)$$

where (E^r, d^r) is the $\mathbb{Z}_2$ Leray spectral sequence for the moment map $\mu_{\mathbb{C}}$ for $X_{\Sigma}(\mathbb{C})$. We refer the reader to [16, §XI.7] or [17, §5] for construction of the Leray spectral sequence and to Section 2.1 for properties of $\mu_{\mathbb{C}}$.

We now have a way to compare the topology of the real and complex points of a toric variety. The Smith-Thom inequality [8, §1.1] states the sum of the $\mathbb{Z}_2$ Betti numbers for $X_{\Sigma}(\mathbb{R})$ is less than or equal to the sum of the $\mathbb{Z}_2$ Betti numbers for $X_{\Sigma}(\mathbb{C})$. We say X_{Σ} is *maximal* if equality is obtained. Using the the Smith-Thom inequality and (9) we have the following diagram

$$\begin{array}{ccc}
 \sum \text{rank}(\bar{E}_{p,q}^1) & = & \sum \text{rank}(E_{p,q}^2) \\
 \vee & & \vee \\
 \sum b_i(X(\mathbb{R})) & \leq & \sum b_j(X(\mathbb{C})) .
 \end{array} \quad (10)$$

If the spectral sequence $\overline{E}^r$ collapses at $\overline{E}^1$ then the left vertical inequality in (10) is equality. This forces both the right vertical inequality and the lower horizontal inequality to both be equalities. In this case we obtain X_Σ is maximal.

3.3 The diagonal entries $H_{qq}(\Sigma)$

In this section we work with the $\mathbb{Z}_2$ torus invariant Chow groups of X_Σ . The integral torus invariant Chow groups are discussed in [10, §5.1] or [11]. When considering the $\mathbb{Z}_2$ torus invariant Chow groups, we define $A_q^T(X_\Sigma) := Z_q^T(X_\Sigma)/R_{q+1}^T(X_\Sigma)$, where $Z_q^T(X_\Sigma)$ is the $\mathbb{Z}_2$ vector space generated by cycles $[V(\tau)]$ for $\tau \in \Sigma$ of codimension q . The relations are generated by torus invariant divisors of the following form. For $\sigma \in \Sigma$ of codimension $q+1$, each $u \in \sigma^\perp \cap M$ gives a rational function χ^u on $V(\sigma)$. The subspace $R_{q+1}^T(X_\Sigma)$ is generated by $\text{div} \chi^u$ except the coefficients are taken mod 2. That is, the coefficient of $[V(\tau)]$ in the relation $\text{div} \chi^u$ is

$$\langle u, n_{\sigma, \tau} \rangle \pmod{2} \quad (11)$$

where $n_{\sigma, \tau}$ is a lattice generator of $\tau \cap N / \sigma \cap N$. Equivalently, we may take $u \in M(\sigma) := \sigma^\perp \cap M / \sigma^\perp \cap 2M$ and $n_{\sigma, \tau}$ the nonzero element of N_τ / N_σ to obtain the coefficient (11). By tensoring the integer complexes with $\mathbb{Z}_2$ we have $H_{qq}(\Sigma) \cong A_q^T(X_\Sigma)$, for $0 \leq q \leq d$. A discussion of this canonical isomorphism may also be found in [14].

Lemma 3.3.1

Assume for each $\tau \in \Sigma(d-k-1)$ we have $V(\tau)(\mathbb{R})$ is T -homeomorphic to $\mathbb{RP}^{k+1}$. If $q \leq k$ then $A_q^T(X_\Sigma)$ is generated by the orbit closure of any q dimensional torus orbit in X_Σ .

Proof. We first look at the relations coming from a single cone σ of codimension $q+1$. Since σ is a face of a cone in $\Sigma(d-k-1)$, $V(\sigma)(\mathbb{R})$ is T -homeomorphic to $\mathbb{RP}^{q+1}$. Thus, $M(\sigma) \cong \text{Hom}(N(\sigma), \mathbb{Z}_2)$ is generated by elements $a_1, a_2, \dots, a_{q+1}$ and σ is contained in the $q+2$ cones $\omega_1, \omega_2, \dots, \omega_{q+2}$ of codimension q . Let $\check{a}_i \in N(\sigma)$ be dual to $a_i \in M(\sigma)$ for $1 \leq i \leq q$. Then, since $V(\sigma)(\mathbb{R})$ is T -homeomorphic to $\mathbb{RP}^{q+1}$, we have n_{σ, ω_i} , the nonzero element of $N_{\omega_i} / N_\sigma \subset N(\sigma)$, is of the form

$$n_{\sigma, \omega_i} = \begin{cases} \check{a}_i & \text{if } 1 \leq i \leq q+1 \\ \check{a}_1 + \check{a}_2 + \dots + \check{a}_{q+1} & \text{if } i = q+2. \end{cases}$$

Hence, the map

$$M(\sigma) \longrightarrow Z_q^T(X_\Sigma) \quad (12)$$

is given by basis elements as follows

$$\begin{aligned} a_1 &\longmapsto [V(\omega_1)] + [V(\omega_{q+2})] \\ a_2 &\longmapsto [V(\omega_2)] + [V(\omega_{q+2})] \\ &\dots \\ a_{q+1} &\longmapsto [V(\omega_{q+1})] + [V(\omega_{q+2})]. \end{aligned}$$

Extending linearly to all of $M(\sigma)$, we see that in the image of (12) we obtain the sum of any even number of $[V(\omega_i)]$'s. For any r with $1 \leq r \leq q+2$, the cycle $[V(\omega_r)]$ is

not in the image of (12). Hence for any r , the cycle $[V(\omega_r)]$ is a representative of the generator of $\text{coker}(M(\sigma) \longrightarrow \bigoplus_{i=1}^{q+2} [V(\omega_i)])$. Next, we consider the map

$$\bigoplus_{\sigma \in \Sigma(d-q-1)} M(\sigma) \longrightarrow \bigoplus_{\omega \in \Sigma(d-q)} [V(\omega)], \quad (13)$$

where the map on each $M(\sigma)$ is described above.

Claim 3.3.2 If ω and ω' are codimension q cones in Σ then $[V(\omega)] + [V(\omega')]$ is in the image of (13).

To see the claim, note that if $\omega \cap \omega'$ is a codimension $q+1$ cone then $[V(\omega)] + [V(\omega')]$ is in the image of (13), as discussed above. Else, find a sequence of codimension q and codimension $q+1$ cones

$$\begin{array}{ccccccc} \omega = \omega_0 & & \omega_1 & & \omega_2 & \cdots & \omega_{n-1} & & \omega_n = \omega' \\ & \searrow & \nearrow & \searrow & \nearrow & & \searrow & \nearrow & \\ & \sigma_{0,1} & & \sigma_{1,2} & & \cdots & & \sigma_{n-1,n} & \end{array}$$

where $\sigma_{i-1,i} \subset \omega_{i-1}$ and $\sigma_{i-1,i} \subset \omega_i$ for $1 \leq i \leq n$. There exists $u_{i-1,i} \in M(\sigma_{i-1,i})$ with $\text{div}(\chi^{u_{i-1,i}}) = [V(\omega_{i-1})] + [V(\omega_i)]$. Adding, we obtain

$$\sum_{i=1}^n u_{i-1,i} \in \bigoplus_{\sigma \in \Sigma(d-q-1)} M(\sigma)$$

and

$$\sum_{i=1}^n u_{i-1,i} \longmapsto [V(\omega)] + [V(\omega')]$$

which proves the claim. Thus, the sum of any even number of $[V(\omega)]$'s is in the image of (13) and the cycle $[V(\omega')]$ is not. Hence, $A_q^T(X_\Sigma)$ is generated by $[V(\omega')]$ where ω' is any cone of codimension q . $\square$

Next, we recall the natural map

$$f_q^{\mathbb{R}} : A_q^T(X_\Sigma) \cong H_{qq}(\Sigma) \longrightarrow H_q(X_\Sigma(\mathbb{R}))$$

given in (8) which arises from the edge homomorphisms of the spectral sequence $(\tilde{E}^r, \tilde{d}^r)$. Thinking cellularly, the real points of an algebraic cycle $[V(\beta)]$ in $Z_q^T(X_\Sigma)$ is represented by the sum of all 2^q cells in the orbit $O_\beta(\mathbb{R})$. Hence, if g is a q dimensional face of Δ with β the cone dual to g then $f_q^{\mathbb{R}}([V(\beta)])$ is the homology class of C_β where $C_\beta := \sum_{t \in N(\beta)} (g, t)$ is the cellular chain in $C_q(X_\Sigma)$ obtained by adding all 2^q copies of g in $X_\Sigma(\mathbb{R})$.

We recall that a cone $\sigma \subset N$ is $\mathbb{Z}_2$ regular provided the image in $N/2N$ of the rays of σ forms a basis for the $\mathbb{Z}_2$ vector space N_σ . Bihan et al. [5] show that if Σ consists of $\mathbb{Z}_2$ regular cones then the map $f_q^{\mathbb{R}}$ is an isomorphism for $0 \leq q \leq d$. The counterexample in [13] shows that in general $f_q^{\mathbb{R}}$ is neither injective nor surjective.

3.4 The right-most column $H_{dq}(\Sigma)$

Let $\{r_1, r_2, \dots, r_k\}$ be the rays of Σ and R_i the image of r_i in $N/2N \cong (\mathbb{Z}_2)^d$.

Proposition 3.4.1

Suppose the rank of the $\mathbb{Z}_2$ vector space $V := \text{span}(R_1, R_2, \dots, R_k)$ is $s \leq d$. Then,

$$\text{rank} H_{dq}(\Sigma) = \binom{d-s}{q-s} \quad \text{and} \quad \bar{d}_{d,q}^r = 0 \quad \text{for} \quad r \geq 1.$$

Proof. We choose a basis for V consisting of a subset of the R_i , and reorder if needed so that $\{R_1, R_2, \dots, R_s\}$ is a basis for V . Note that the elements of the form

$$R_i \wedge * \wedge * \wedge \dots \wedge *$$

generate

$$\ker(\wedge^{qN/2N} \longrightarrow \wedge^q N(r_i)),$$

where the $*$ -entries are any elements of $N/2N$. Hence,

$$\ker(\wedge^{qN/2N} \longrightarrow \bigoplus_{i=1}^s \wedge^q N(r_i))$$

is generated by elements of the following form

$$R_1 \wedge R_2 \wedge \dots \wedge R_s \wedge * \wedge * \wedge \dots \wedge *.$$

This is a subspace of $\wedge^{qN/2N}$ of rank $\binom{d-s}{q-s}$, where $\binom{n}{k} = 0$ if $k < 0$. Hence, the kernel of the boundary map

$$\bar{d}_{d,q}^0 : \wedge^{qN/2N} \longrightarrow \bigoplus_{r \in \Sigma(1)} \wedge^q N(r)$$

has rank at most $\binom{d-s}{q-s}$. This shows

$$b_d(X_\Sigma(\mathbb{R})) \leq \sum_{q=s}^d \binom{d-s}{q-s} = 2^{d-s} \quad (14)$$

and equality holds if and only if for each q , $\ker \bar{d}_{d,q}^0$ has rank $\binom{d-s}{q-s}$ and all higher differentials $\bar{d}_{d,q}^r$, $r \geq 1$ with source the rightmost column are zero. From Lemma 2.1.1 we have $b_d(X_\Sigma(\mathbb{R})) = 2^{d-s}$ yielding $\text{rank} H_{dq}(\Sigma) = \binom{d-s}{q-s}$ and the higher boundaries $\bar{d}_{d,q}^r$, $r \geq 1$ are zero. $\square$

4. $\mathbb{Z}_2$ Hodge spaces for reflexive polytopes

4.1 A correspondence between sheaves and cosheaves

A *reflexive* polytope is a lattice polytope Δ with $0 \in \text{int}\Delta$ and such that the polar polytope Δ^* is also a lattice polytope. A discussion of reflexive polytopes can be found in [2, §4.1]. Throughout this section we will use the following notation

$\Delta \subset M$	a reflexive polytope
$\Delta^* \subset N$	the polar polytope of Δ
$\Sigma \subset N$	the normal fan of Δ (= the face fan of Δ^*)
$\Sigma^* \subset M$	the normal fan of Δ^* (= the face fan of Δ) .

If $\tau \in \Sigma$ and $\dim \tau = j > 0$ then $\tau = \text{poshull} f^*$ where $f^* < \Delta^*$ is a face of dimension $j - 1$. We define τ^* to be the cone in Σ^* of dimension $d - j + 1$ which is dual to f^* . The correspondence

$$\tau \in \Sigma \longleftrightarrow \tau^* \in \Sigma^*$$

gives a one-to-one inclusion reversing correspondence between the cones in Σ of positive dimension and the positive dimensional cones in Σ^* , where $\dim \tau^* = d - \dim \tau + 1$.

Let $\mathcal{H}$ be a sheaf on the fan Σ^* . We use the correspondence above to create a cosheaf $\widehat{\mathcal{H}}$ on Σ by defining for $\tau \in \Sigma$

$$\widehat{H}_\tau := \begin{cases} H_{\tau^*} & \text{if } \dim \tau > 0 \\ 0 & \text{if } \dim \tau = 0 \end{cases} \quad (15)$$

with extension map for $\sigma < \tau$

$$\widehat{\rho}_{\sigma, \tau} := \begin{cases} \rho_{\tau^*, \sigma^*} & \text{if } \dim \sigma > 0 \\ 0 & \text{if } \dim \sigma = 0 \end{cases} \quad (16)$$

where ρ_{τ^*, σ^*} is a restriction map for the sheaf $\mathcal{H}$. Using (15) we have equality of chain groups for the sheaf $\mathcal{H}$ on Σ^* and the cosheaf $\widehat{\mathcal{H}}$ on Σ , as depicted below.

$$\begin{array}{ccccccccccc} C_d(\widehat{\mathcal{H}}) & \longrightarrow & C_{d-1}(\widehat{\mathcal{H}}) & \longrightarrow & C_{d-2}(\widehat{\mathcal{H}}) & \longrightarrow & \cdots & \longrightarrow & C_2(\widehat{\mathcal{H}}) & \longrightarrow & C_1(\widehat{\mathcal{H}}) & \longrightarrow & C_0(\widehat{\mathcal{H}}) \\ \parallel & & \parallel & & \parallel & & & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & C^0(\mathcal{H}) & \longrightarrow & C^1(\mathcal{H}) & \longrightarrow & \cdots & \longrightarrow & C^{d-3}(\mathcal{H}) & \longrightarrow & C^{d-2}(\mathcal{H}) & \longrightarrow & C^{d-1}(\mathcal{H}) \end{array}$$

From (16) the horizontal sheaf and cosheaf boundary maps are equal. Hence we may equate sheaf cohomology groups for $\mathcal{H}$ on Σ^* with cosheaf homology groups for $\widehat{\mathcal{H}}$ on Σ

$$H_p(\widehat{\mathcal{H}}) \cong H^{d-p-1}(\mathcal{H}), \quad p > 0.$$

One more construction which we will use in the sequel is that of the cosheaf $\mathcal{A}^\circ$. If $\mathcal{A}$ is a cosheaf on Σ then $\mathcal{A}^\circ$ is defined by

$$A_\sigma^\circ := \begin{cases} A_\sigma & \text{if } \dim \sigma > 0 \\ 0 & \text{if } \dim \sigma = 0 \end{cases}$$

with the extension map for $\sigma < \tau$ defined by

$$\rho_{\sigma,\tau}^\circ := \begin{cases} \rho_{\sigma,\tau} & \text{if } \dim \sigma > 0 \\ 0 & \text{if } \dim \sigma = 0. \end{cases}$$

Note that by definition

$$C_p(\mathcal{A}) = C_p(\mathcal{A}^\circ) \quad \text{for } p \leq d-1,$$

and hence we have

$$H_p(\mathcal{A}) = H_p(\mathcal{A}^\circ) \quad \text{for } p \leq d-2.$$

4.2 The cosheaves $\widehat{\mathcal{F}}$, $\widehat{\mathcal{G}}$, and $\mathcal{C}$ on Σ

The sheaf $\mathcal{F}$ on Σ^* is defined as follows. For $\sigma^* \in \Sigma^*$ the stalk is

$$F_{\sigma^*} := \frac{(\sigma^*)^\perp \cap N}{(\sigma^*)^\perp \cap 2N}$$

and the face restriction map

$$\rho_{\tau^*,\sigma^*} : F_{\sigma^*} \xrightarrow{\subseteq} F_{\tau^*}$$

is given by inclusion for $\tau^* < \sigma^*$ in Σ^* . The sheaf $\mathcal{G}$ on Σ^* is then defined to be the cokernel of the inclusion $\mathcal{F} \hookrightarrow N/2N$ so that

$$0 \longrightarrow \mathcal{F} \longrightarrow N/2N \longrightarrow \mathcal{G} \longrightarrow 0 \quad (17)$$

is a short exact sequence of sheaves on Σ^* .

Claim 4.2.1 For $\tau \in \Sigma$, $\tau^* \in \Sigma^*$ as in the previous section with $\dim \tau = l > 0$, we have the following containment of $\mathbb{Z}_2$ vector spaces

$$\frac{(\tau^*)^\perp \cap N}{(\tau^*)^\perp \cap 2N} \subset \frac{\tau \cap N}{\tau \cap 2N}. \quad (18)$$

Note that the $\mathbb{Z}_2$ vector space on the left has rank $l-1$ while the one on the right has rank l . To prove the claim, let $f < \Delta$ and $f^* < \Delta^*$ be such that

$$\begin{aligned} \tau &= \text{poshull } f^* & \tau^* &= \text{poshull } f \\ &= \text{poshull } \{q_1, q_2, \dots, q_s\} & &= \text{poshull } \{p_1, p_2, \dots, p_k\}. \end{aligned}$$

Then we have $\langle p_i, q_j \rangle = -1 \quad \forall i, j$. This gives $(\tau^*)^\perp = \text{span}\{q_i - q_j \mid 1 \leq i, j \leq s\}$ and the claim follows. Next, we note that the inclusion (18) is compatible with the face extension maps for cones in Σ . Thus, we obtain an injective homomorphism of cosheaves $\widehat{\mathcal{F}} \hookrightarrow \mathcal{N}$. We define the cosheaf $\mathcal{C}$ to be the cokernel of this homomorphism so that

$$0 \longrightarrow \widehat{\mathcal{F}} \longrightarrow \mathcal{N} \longrightarrow \mathcal{C} \longrightarrow 0 \quad (19)$$

is an exact sequence of cosheaves on Σ . Note that if $\dim \sigma = 0$ then the stalks $\widehat{F}_\sigma$, N_σ , and C_σ are all zero. Moreover, for each $\sigma \in \Sigma$ of positive dimension the stalk C_σ is a rank one $\mathbb{Z}_2$ vector space.

Lemma 4.2.2

For $\sigma < \tau$ in Σ with $\dim \sigma > 0$ the map $C_\sigma \longrightarrow C_\tau$ is an isomorphism.

Proof. We prove the lemma by contradiction. Assume we have $\tau_1 < \tau_2$ in Σ and $C_{\tau_1} \longrightarrow C_{\tau_2}$ is the zero map. We have the follow diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{(\tau_1^*)^\perp \cap N}{(\tau_1^*)^\perp \cap 2N} & \longrightarrow & \frac{\tau_1 \cap N}{\tau_1 \cap 2N} & \longrightarrow & C_{\tau_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \phi \\ 0 & \longrightarrow & \frac{(\tau_2^*)^\perp \cap N}{(\tau_2^*)^\perp \cap 2N} & \longrightarrow & \frac{\tau_2 \cap N}{\tau_2 \cap 2N} & \longrightarrow & C_{\tau_2} \longrightarrow 0 \end{array}$$

where the rows are short exact sequences and vertical maps are cosheaf extension maps. We have assumed $\phi = 0$ and hence

$$\frac{\tau_1 \cap N}{\tau_1 \cap 2N} \subset \frac{(\tau_2^*)^\perp \cap N}{(\tau_2^*)^\perp \cap 2N}. \quad (20)$$

Suppose

$$\tau_2 = \text{poshull}\{q_1, \dots, q_l\}, \tau_1 = \text{poshull}\{q_1, \dots, q_{l-r}\},$$

and

$$\tau_2^* = \text{poshull}\{p_1, \dots, p_k\}.$$

Due to the inclusion (20) we have $\langle q_1, p_1 \rangle = 0 \pmod{2}$. This contradicts the fact the Δ is reflexive which guarantees $\langle q_1, p_1 \rangle = -1$. $\square$

Remark 4.2.3 If we were considering that analogous sheaves of $\mathbb{Z}$ modules or of $\mathbb{Q}$ vector spaces then the map $C_\sigma \longrightarrow C_\tau$ from Lemma 4.2.2 would be injective without the assumption of reflexivity.

Next, note that

$$0 \longrightarrow \widehat{\mathcal{F}} \longrightarrow \widehat{N/2N} \longrightarrow \widehat{\mathcal{G}} \longrightarrow 0 \quad (21)$$

is a short exact sequence of cosheaves of Σ where $\widehat{N/2N} = N/2N^\circ$ as cosheaves on Σ . We can combine (21) with (19) into the following commutative diagram

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & 0 & & 0 & & \mathcal{K} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \widehat{\mathcal{F}} & \longrightarrow & \widehat{N/_{2N}} & \longrightarrow & \widehat{\mathcal{G}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \Phi \\
0 & \longrightarrow & \mathcal{N} & \longrightarrow & N/_{2N}^\circ & \longrightarrow & \mathcal{E}^\circ \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathcal{C} & & 0 & & 0 \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}$$

where the rows and columns are exact and $\mathcal{K} := \ker \Phi$.

Next, we use the Snake Lemma on the commutative diagram to obtain $\mathcal{K} \cong \mathcal{C}$ as cosheaves yielding

$$0 \longrightarrow \mathcal{C} \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathcal{E}^\circ \longrightarrow 0 \quad (22)$$

is a short exact sequence of cosheaves on Σ . As $(\wedge^q \mathcal{E})^\circ = \wedge^q \mathcal{E}^\circ$ we have $H_p(\wedge^q \mathcal{E}) = H_p(\wedge^q \mathcal{E}^\circ)$ for $p \leq d-2$.

4.3 Vanishing of the homology groups $H_p(\wedge^q \widehat{\mathcal{G}})$

We remind the reader that a cone $\sigma \in \Sigma$ is $\mathbb{Z}_2$ regular provided the image in $N/_{2N}$ of the rays of σ forms a basis for N_σ .

Lemma 4.3.1

Assume the cones in Σ^* of dimension at most e are $\mathbb{Z}_2$ regular. Then

$$H_p(\wedge^q \widehat{\mathcal{G}}) = 0 \quad \text{for } 1 \leq p < e-1.$$

Proof. First, we will assume that $e = d$. Following [7, §1.2], as the cones in Σ^* are $\mathbb{Z}_2$ regular the sheaf $\mathcal{G}$ on Σ^* can be written as follows.

$$\mathcal{G} = \bigoplus_{r_i \in \Sigma^*(1)} \mathcal{G}(r_i)$$

is a direct sum of the sheaves $\mathcal{G}(r_i)$ with

$$G(r_i)_{\tau^*} = \begin{cases} \mathbb{Z}_2 & \text{if } r_i \in \tau^*(1) \\ 0 & \text{else.} \end{cases}$$

Moreover, we have

$$\wedge^q \mathcal{G} = \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} \mathcal{G}(r_1, r_2, \dots, r_k)$$

is the direct sum of sheaves $\mathcal{G}(r_1, r_2, \dots, r_k)$ with

$$G(r_1, r_2, \dots, r_k)_{\tau^*} = \begin{cases} \mathbb{Z}_2 & \text{if } r_1, r_2, \dots, r_k \in \tau^*(1) \\ 0 & \text{else.} \end{cases}$$

The proof of the proposition in [7, Section 1.2] yields $H^p(\mathcal{G}(r_1, r_2, \dots, r_k)) = 0$ for $p > 0$ because Σ^* is a complete fan. Hence we have

$$\begin{aligned} H^p(\wedge^q \mathcal{G}) &= H^p\left(\bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} \mathcal{G}(r_1, r_2, \dots, r_k)\right) \\ &\cong \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} H^p(\mathcal{G}(r_1, r_2, \dots, r_k)) \\ &= 0 \quad \text{for } p > 0. \end{aligned}$$

Hence, as $\wedge^q \widehat{\mathcal{G}} = \widehat{\wedge^q \mathcal{G}}$ we have $H_p(\wedge^q \widehat{\mathcal{G}}) = H^{d-p-1}(\wedge^q \mathcal{G}) = 0$ for p such that $p > 0$ and $d - p - 1 > 0$, which proves Lemma 4.3.1 when $e = d$.

Next, assume $e < d$. We still have $H^p(\mathcal{G}(r_1, r_2, \dots, r_k)) = 0$ for $p > 0$ however the sheaf $\mathcal{G}$ cannot be written as $\bigoplus_{r_i \in \Sigma^*(1)} \mathcal{G}(r_i)$. Let $\Sigma_{\leq e}^* := \bigcup_{i \leq e} \Sigma^*(i)$ be the subfan of Σ^* consisting of the cones of dimension at most e , and let $\mathcal{G}'$ be the restriction of $\mathcal{G}$ to $\Sigma_{\leq e}^*$. Then as $\Sigma_{\leq e}^*$ consists of $\mathbb{Z}_2$ regular cones, we have the following equality

$$\mathcal{G}' = \bigoplus_{r_i \in \Sigma^*(1)} \mathcal{G}(r_i)'$$

where $\mathcal{G}(r_i)'$ is the restriction of $\mathcal{G}(r_i)$ to $\Sigma_{\leq e}^*$. Moreover, we have

$$\wedge^q \mathcal{G}' = \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} \mathcal{G}(r_1, r_2, \dots, r_k)'.$$

Hence we obtain

$$\begin{aligned} H^p(\wedge^q \mathcal{G}) &\cong H^p(\wedge^q \mathcal{G}') \\ &\cong \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} H^p(\mathcal{G}(r_1, r_2, \dots, r_k)') \\ &\cong \bigoplus_{r_1, r_2, \dots, r_k \text{ distinct}} H^p(\mathcal{G}(r_1, r_2, \dots, r_k)) \\ &= 0, \end{aligned}$$

for $p > d - e$. In the above equalities we used that $H^p(\mathcal{A}) = H^p(\mathcal{A}')$ for $p > d - e$, where $\mathcal{A}$ is any sheaf on Σ^* and $\mathcal{A}'$ is the restriction to $\Sigma_{\leq e}^*$. Thus, $H_p(\wedge^q \widehat{\mathcal{G}}) = 0$ for p such that $p > 0$ and $d - p - 1 > d - e$. $\square$

4.4 Vanishing of the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$

In this section, we use the results from the previous two sections to determine the vanishing of certain $\mathbb{Z}_2$ Hodge spaces. See Section 6 for examples.

Theorem 4.4.1

Assume the cones in Σ^* of dimension at most e are $\mathbb{Z}_2$ regular. Then

$$H_p(\wedge^q \mathcal{E}) = 0 \quad \text{for } q < p < e - 1.$$

Proof. We use the short exact sequence $0 \rightarrow \mathcal{C} \rightarrow \widehat{\mathcal{G}} \rightarrow \mathcal{E}^\circ \rightarrow 0$ of cosheaves of Σ and the associated Koszul degree q exact sequence

$$0 \rightarrow S^q \mathcal{C} \rightarrow \wedge^1 \widehat{\mathcal{G}} \otimes S^{q-1} \mathcal{C} \rightarrow \dots \rightarrow \wedge^{q-1} \widehat{\mathcal{G}} \otimes S^1 \mathcal{C} \rightarrow \wedge^q \widehat{\mathcal{G}} \rightarrow \wedge^q \mathcal{E}^\circ \rightarrow 0, \quad (23)$$

where $S^{q-k} \mathcal{C}$ is the $(q-k)$ th symmetric power of the cosheaf $\mathcal{C}$. A similar exact sequence of sheaves of $\mathbb{Q}$ vector spaces can be found in [7, §1.2] and a treatment of this sequence is in [14, §3.3]. We break (23) into short exact sequences below

$$0 \rightarrow S^q \mathcal{C} \rightarrow \wedge^1 \widehat{\mathcal{G}} \otimes S^{q-1} \mathcal{C} \rightarrow \mathcal{W}_1 \rightarrow 0 \quad (24)$$

$$0 \rightarrow \mathcal{W}_1 \rightarrow \wedge^2 \widehat{\mathcal{G}} \otimes S^{q-2} \mathcal{C} \rightarrow \mathcal{W}_2 \rightarrow 0 \quad (25)$$

$$\dots \quad (26)$$

$$0 \rightarrow \mathcal{W}_{q-1} \rightarrow \wedge^q \widehat{\mathcal{G}} \rightarrow \wedge^q \mathcal{E}^\circ \rightarrow 0. \quad (27)$$

These induce long exact sequences on homology groups. From Lemma 4.2.2 we see that

$$\begin{aligned} \mathcal{C} &\cong (\mathbb{Z}_2)^\circ \\ S^{q-k} \mathcal{C} &\cong (\mathbb{Z}_2)^\circ \end{aligned}$$

where $\mathbb{Z}_2$ is the constant cosheaf on Σ . Thus we have

$$H_p(\wedge^k \widehat{\mathcal{G}} \otimes S^{q-k} \mathcal{C}) \cong H_p(\wedge^k \widehat{\mathcal{G}}) \otimes S^{q-k} \mathcal{C} = 0 \quad \text{for } 1 \leq p < e - 1,$$

where we have used Lemma 4.3.1. We begin with the long exact sequence induced from (24)

$$\dots \rightarrow H_p(\wedge^1 \widehat{\mathcal{G}} \otimes S^{q-1} \mathcal{C}) \rightarrow H_p(\mathcal{W}_1) \rightarrow H_{p-1}(S^q \mathcal{C}) \rightarrow \dots$$

We have

$$\begin{aligned} H_p(\wedge^1 \widehat{\mathcal{G}} \otimes S^{q-1} \mathcal{C}) &= 0 \quad \text{for } 1 \leq p < e - 1 \\ H_{p-1}(S^q \mathcal{C}) &= 0 \quad \text{for } 1 \leq p - 1 < e - 1 \end{aligned}$$

and hence $H_p(\mathcal{W}_1) = 0$ for $1 < p < e - 1$. Next we use the long exact sequence induced from (25)

$$\dots \rightarrow H_p(\wedge^2 \widehat{\mathcal{G}} \otimes S^{q-2} \mathcal{C}) \rightarrow H_p(\mathcal{W}_2) \rightarrow H_{p-1}(\mathcal{W}_1) \rightarrow \dots$$

We have

$$\begin{aligned} H_p(\wedge^2 \widehat{\mathcal{G}} \otimes S^{q-2} \mathcal{C}) &= 0 & \text{for } 1 \leq p < e-1 \\ H_{p-1}(\mathcal{W}_1) &= 0 & \text{for } 1 < p-1 < e-1 \end{aligned}$$

and hence $H_p(\mathcal{W}_2) = 0$ for $2 < p < e-1$. We continue this process to obtain $H_p(\wedge^q \mathcal{E}^\circ) = 0$ for $q < p < e-1$. Moreover, as $e-1 \leq d-1$ we have $H_p(\wedge^q \mathcal{E}) = H_p(\wedge^q \mathcal{E}^\circ)$ for $q < p < e-1$. $\square$

When $\Sigma_{\leq e}^*$ consists of $\mathbb{Z}_2$ regular cones, the proposition below gives additional information about the $\mathbb{Z}_2$ Hodge spaces of Σ .

Proposition 4.4.2

If the cones in $\Sigma_{\leq e}^$ are $\mathbb{Z}_2$ regular then for $q < e-1$ we have $H_{qq}(\Sigma)$ has rank 1 and is generated by the orbit closure of any q dimensional torus orbit.*

Let $\tau^* \in \Sigma^*(e)$, $\tau^* = \text{poshull } f$ where $f = \text{conv}\{p_1, p_2, \dots, p_e\}$. The proposition follows from Lemma 3.3.1 and the following claim.

Claim 4.4.3 The real variety $V(\tau)(\mathbb{R})$ in $X_\Sigma(\mathbb{R})$ is T-homeomorphic to $\mathbb{RP}^{e-1}$.

Proof. Since τ^* is $\mathbb{Z}_2$ regular, we can find a matrix A of the form

$$A := \begin{bmatrix} | & | & & | & | & & | \\ p_1 & p_2 & \cdots & p_e & t_1 & \cdots & t_{d-e} \\ | & | & & | & | & & | \end{bmatrix}$$

with $t_1, t_2, \dots, t_{d-e} \in M$, $\det A = 1 \pmod{2}$, and A giving a combinatorial equivalence between a polytope Ψ and Δ . Moreover, if x_i is the i th standard basis vector for $\mathbb{R}^d$ then we have $Ax_i = p_i$ for $1 \leq i \leq e$ and hence $A(\text{conv}\{x_1, x_2, \dots, x_e\}) = f$. The matrix A^* determines a rational isomorphism between Σ and Γ , the normal fan of Ψ . As $\det A^* = 1 \pmod{2}$, we obtain a T-homeomorphism between $X_\Sigma(\mathbb{R})$ and $X_\Gamma(\mathbb{R})$. Using Section 2.2 we have $V(\tau)(\mathbb{R})$ is T-homeomorphic to $\mathbb{RP}^{e-1}$. $\square$

5. Collapsing of the spectral sequence $\overline{E}^r$

In this section, we show that the spectral sequence $\overline{E}^r$ for X_Σ collapses at $\overline{E}^1$ when the cones in Σ^* are $\mathbb{Z}_2$ regular. Using our work in Section 4.4, we have that the ranks of the entries in the $\overline{E}^1$ term are as follows

$$\begin{array}{cccccccc}
& & & & & & & 1 \\
& & & & & & * & * \\
& & & & & 1 & * & * \\
& & & 1 & 0 & * & * \\
q & & 1 & 0 & 0 & * & * \\
& & 1 & 0 & 0 & 0 & * & * \\
& & 1 & 0 & 0 & 0 & 0 & * & * \\
& 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & p
\end{array} \tag{28}$$

where the $*$ entries are possibly nonzero and occur in the columns $p > d - 2$.

Remark 5.0.4 We can completely determine the $*$ entries. From Section 3.4, if s is the rank of the image of the rays of Σ in $N/2N$ then the rank of $\overline{E}_{d,q}^1$ is the binomial coefficient $\binom{d-s}{q-s}$. It follows from [7, §1.4] that the Euler characteristic of the q th row is $(-1)^q h_q$ where $h = (h_0, h_1, h_2, \dots, h_d)$ is the h -vector of the polytope Δ . Thus, we may determine the ranks of the vector spaces $\overline{E}_{d-1,q}^1$, giving us knowledge of the ranks of all the entries in $\overline{E}_{p,q}^1$. Examples can be found in Section 6.

By looking at the $\overline{E}^1$ term for X_Σ , we see that the only possible nonzero higher differentials have target $\overline{E}_{d-2,d-2}^1$. To show that the spectral sequence $\overline{E}^r$ for X_Σ collapses at $\overline{E}^1$, we need only show the following lemma.

Lemma 5.0.5

The map

$$f_{d-2}^{\mathbb{R}} : H_{d-2,d-2}(\Sigma) \longrightarrow H_{d-2}(X_\Sigma(\mathbb{R}))$$

from (8) is nonzero.

Proof. Let g be a $d-2$ dimensional face of Δ with $\beta = \text{poshull}\{q_1, q_2, \dots, q_k\}$ the cone dual to g . As mentioned in Section 3.3, $f_{d-2}^{\mathbb{R}}$ sends $[V(\beta)]$ to the homology class of $C_\beta := \sum_{t \in N(\beta)} (g, t)$. Suppose

$$C_\beta = \partial C, \quad C = (g_1, t_1) + (g_2, t_2) + \dots + (g_r, t_r) \in C_{d-1}(X_\Sigma)$$

where $g_i < \Delta$, $\sigma_i \in \Sigma$ is dual to g_i , and $t_i \in N(\sigma_i)$. We include $N \subset \widetilde{N}$ where $\widetilde{N} \cong N \oplus \mathbb{Z}$ and $\Delta^* \times [-1, 1] \subset \widetilde{N}$. Note that $\Delta^* \times [-1, 1]$ is reflexive and the normal fan of $\Delta^* \times [-1, 1]$ consists of $\mathbb{Z}_2$ regular cones. Let Ξ be the face fan of $\Delta^* \times [-1, 1]$. Then, Ξ is the normal fan of $B_\Delta := (\Delta^* \times [-1, 1])^*$ the bipyramid with base Δ . We define the cone $\tilde{\beta} \in \Xi$ to be the positive hull of the rays $\{(q_1, 1), (q_1, -1), (q_2, 1), (q_2, -1), \dots, (q_k, 1), (q_k, -1)\}$ in $\tilde{N}$. Note that $\tilde{\beta}$ is

dual to g considered as a face in B_Δ . Moreover, we have

$$\begin{aligned}
 N(\tilde{\beta}) &= \frac{\tilde{N}/_{2\tilde{N}}}{\tilde{\beta} \cap \tilde{N} / \tilde{\beta} \cap 2\tilde{N}} \\
 &\cong \frac{N/_{2N} \oplus \langle e_{d+1} \rangle}{\beta \cap N / \beta \cap 2N \oplus \langle e_{d+1} \rangle} \\
 &\cong \frac{N/_{2N}}{\beta \cap N / \beta \cap 2N} \\
 &= N(\beta),
 \end{aligned}$$

where $\langle e_{d+1} \rangle$ is the $\mathbb{Z}_2$ vector space generated by the $(d+1)$ st standard basis vector. This gives that inclusion of $[V(\tilde{\beta})]$ in $H_{d-2}(X_\Xi(\mathbb{R}))$ is also represented cellularly by C_β . Similarly, for each $\sigma_i \in \Sigma$ appearing in the chain C , $\tilde{\sigma}_i \in \Xi$, with rays of the form $\{(r, 1), (r, -1) \mid r \text{ is a ray of } \sigma\}$ satisfies $N(\tilde{\sigma}_i) \cong N(\sigma_i)$. Hence, C can be viewed as a chain in $C_{d-1}(X_\Xi(\mathbb{R}))$. We have

$$\tilde{\partial}C = C_\beta \tag{29}$$

where $\tilde{\partial}$ is the cellular boundary map for $X_\Xi(\mathbb{R})$. Equation (29) holds because $\tilde{\sigma}_i < \gamma$ in Ξ implies γ must be of the form $\gamma = \tilde{\sigma}$ with $\sigma \in \Sigma$. As $[V(\tilde{\beta})]$ generates $H_{d-2,d-2}(\Xi)$, the map $H_{d-2,d-2}(\Xi) \longrightarrow H_{d-2}(X_\Xi(\mathbb{R}))$ must be zero. However, the ranks of the entries $\overline{E}^1$ for X_Ξ have the form (28) and $\overline{E}_{d-2,d-2}^1$ is in the 4th column from the right. There cannot be higher boundaries with target $\overline{E}_{d-2,d-2}^1$ which contradicts the fact that $H_{d-2,d-2}(\Xi) \longrightarrow H_{d-2}(X_\Xi(\mathbb{R}))$ is the zero map. $\square$

This shows that the spectral sequence $\overline{E}^r$ for X_Σ collapses at $\overline{E}^1$ and hence we have the following.

Theorem 5.0.6

If Σ^ consists of $\mathbb{Z}_2$ regular cones then X_Σ is maximal.*

The following definition can be found in [3] where Batyrev studies Fano polyhedra, a very special class of reflexive polytopes.

DEFINITION 5.0.7 Let $\text{vert}\Delta = \{v_1, v_2, \dots, v_n\} \subset \mathbb{Z}^d$. The d dimensional polytope Δ is a *Fano polyhedron* provided

- (1) $0 \in \text{int}\Delta$
- (2) Each face of Δ is a simplex
- (3) If $v_{i_1}, v_{i_2}, \dots, v_{i_d}$ are the vertices of a $(d-1)$ dimensional face of Δ then

$$\det[v_{i_1} v_{i_2} \cdots v_{i_d}] = \pm 1.$$

We have proved the maximality of toric varieties associated to the Fano polyhedra, which are also called smooth Fano polyhedra by some authors (e.g. [18]).

If Δ is a Fano polyhedron then Δ^* defines one of the so called smooth toric Fano manifolds. The toric Fano manifolds $X_{\Sigma^*}(\mathbb{C})$ are important in algebraic geometry since their anticanonical sheaf is very ample. A classification of the Fano polyhedra is known for dimension at most 4. There are 5 Fano polyhedra of dimension 2. Batyrev classified the 18 Fano polyhedra of dimension 3 in [4] and the 123 Fano polyhedra of dimension 4 in [3].

6. Examples

In this section, we illustrate our results with a few examples. We use `torhom` [9] to compute the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$ and the f -vector of the polytope Δ .

EXAMPLE 6.0.8 The five dimensional cross polytope.

Let Δ be the five dimensional cross polytope. Then, Δ is the convex hull of the ten vertices

$$\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5\}.$$

The polar polytope Δ^* is the five dimensional cube, which defines the nonsingular toric variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The f -vector for Δ is $(10, 40, 80, 80, 32)$. We compute the ranks of the $\mathbb{Z}_2$ Hodge spaces for Σ

$$\begin{array}{rcccccc} & & & & & 1 & \\ & & & & & 31 & 4 \\ & & & & 1 & 45 & 6 \\ q & & 1 & 0 & 25 & 4 & \\ & 1 & 0 & 0 & 5 & 1 & \\ & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & p & & . \end{array}$$

We compute the $\mathbb{Z}_2$ Betti numbers for the real points $X_{\Sigma}(\mathbb{R})$ by adding along the columns

$$[1 \ 1 \ 1 \ 1 \ 106 \ 16].$$

We compute the $\mathbb{Z}_2$ Betti numbers for the complex points $X_{\Sigma}(\mathbb{C})$ by adding along the diagonals

$$[1 \ 0 \ 1 \ 0 \ 1 \ 5 \ 27 \ 49 \ 37 \ 4 \ 1].$$

EXAMPLE 6.0.9 A seven dimensional example.

We define Δ to be the convex hull of the following nine vertices

$$\{-e_1, -e_2, -e_3, -e_4, -e_5, -e_6, -e_7, e_1 + e_2 + e_3 + e_4, e_5 + e_6 + e_7\}.$$

The f -vector for Δ is $(9, 36, 84, 125, 120, 70, 20)$. The polar polytope Δ^* is the product $P_3 \times P_4$, where P_i defines i dimensional projective space. Moreover, Δ^*

defines the nonsingular toric variety $\mathbb{P}^3 \times \mathbb{P}^4$. Below are the ranks of the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$

$$\begin{array}{ccccccccc}
 & & & & & & & & 1 \\
 & & & & & & & & 15 & 2 \\
 & & & & & & 1 & 31 & 1 \\
 & & & & 1 & 0 & 34 & 0 \\
 q & & & 1 & 0 & 0 & 21 & 0 \\
 & & 1 & 0 & 0 & 0 & 7 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & p & & & & .
 \end{array}$$

We compute the $\mathbb{Z}_2$ Betti numbers for the real points $X_\Sigma(\mathbb{R})$ by adding along the columns

$$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 109 \ 4]$$

and for the complex points $X_\Sigma(\mathbb{C})$ by adding along the diagonals

$$[1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 8 \ 21 \ 35 \ 31 \ 16 \ 2 \ 1].$$

EXAMPLE 6.0.10 A six dimensional example.

Let Δ be the convex hull of the twelve vertices below

$$\{-e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, e_6, e_1 - e_6, -e_1 - e_2 - e_3 - e_4 - e_5 - e_6\}.$$

The f -vector for Δ is $(12, 62, 174, 267, 207, 64)$. The polar polytope Δ^* has 64 vertices. Using **polymake** we determine that Δ^* defines a toric variety with isolated singularities. That is, $\Sigma_{\leq 5}^*$ consists of $\mathbb{Z}_2$ regular cones. The results of Section 4.4 give the $\mathbb{Z}_2$ Hodge spaces $H_{pq}(\Sigma)$ for $p < 4$, as shown by the **torhom** computation below

$$\begin{array}{ccccccccc}
 & & & & & & & & 1 \\
 & & & & & & & & 58 & 0 \\
 & & & & & & 15 & 113 & 0 \\
 & & & 1 & 38 & 96 & 0 \\
 q & & & 1 & 0 & 34 & 45 & 0 \\
 & & 1 & 0 & 0 & 10 & 10 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & p & & & & .
 \end{array}$$

We do not have the theory to guarantee collapsing of the spectral sequence $\overline{E}^r$ at $\overline{E}^1$. However, using **torhom** we compute the $\mathbb{Z}_2$ Betti numbers for the real points $X_\Sigma(\mathbb{R})$

$$[1 \ 1 \ 1 \ 1 \ 97 \ 322 \ 1].$$

We conclude that the spectral sequence $\overline{E}^r$ collapses at $\overline{E}^1$. Again, we use Equation (10) to obtain the collapsing of the spectral sequence E^r at E^2 . We can

therefore compute the $\mathbb{Z}_2$ Betti numbers for $X_\Sigma(\mathbb{C})$ by adding along the diagonals

$$[1 \ 0 \ 1 \ 0 \ 1 \ 10 \ 45 \ 83 \ 111 \ 113 \ 58 \ 0 \ 1].$$

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