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ACM sets of points in multiprojective space

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Abstract

If $\mathbb{X}$ is a finite set of points in a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$, then $\mathbb{X}$ may or may not be arithmetically Cohen-Macaulay (ACM). For sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ there are several classifications of the ACM sets of points. In this paper we investigate the natural generalizations of these classifications to an arbitrary multiprojective space. We show that each classification for ACM points in $\mathbb{P}^1 \times \mathbb{P}^1$ fails to extend to the general case. We also give some new necessary and sufficient conditions for a set of points to be ACM.

1. Introduction

Let $\mathbb{X}$ be a finite set of points in a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, and let $R/I_{\mathbb{X}}$ denote the associated $\mathbb{N}^r$-graded coordinate ring. When $r = 1$, then $R/I_{\mathbb{X}}$ is always Cohen-Macaulay. On the other hand, if $r \geq 2$, then $R/I_{\mathbb{X}}$ may or may not be Cohen-Macaulay. More precisely, we know that $\dim R/I_{\mathbb{X}} = r$, the number of projective spaces. However, the depth of $R/I_{\mathbb{X}}$ may take on any value in the set $\{1, \ldots, r\}$. When $R/I_{\mathbb{X}}$ is Cohen-Macaulay, that is, depth $R/I_{\mathbb{X}} = \dim R/I_{\mathbb{X}} = r$, then $\mathbb{X}$ is called an **arithmetically Cohen-Macaulay** (ACM) set of points.

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Because a set of points in a multiprojective space may or may not be ACM, a natural problem arises:

Problem 1. *Find a classification of ACM sets of points in* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ *for* $r \geq 2$.

Little is known about this problem except in the case that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. In this situation there are several classifications. Giuffrida, Maggioni, and Ragusa [7], who helped to initiate the study of points in multiprojective spaces (see, for example [8, 9, 12, 13, 14, 18, 20, 21, 22] for more on these points), provided the first classification. They showed that ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ can be classified via their Hilbert functions. The two authors [11, 21] independently gave geometric classifications of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. More recently, L. Marino [16] used the notion of a separator to provide a new classification of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In this paper we will consider the natural generalizations of the above classifications to an arbitrary multiprojective space. As we shall show, these natural generalizations no longer classify ACM sets of points, thus suggesting a solution to Problem 1 is quite subtle. We give a partial answer to Problem 1 by giving some necessary and sufficient conditions for a set of points to be ACM in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

Before proceeding, we should point out that Problem 1 is a refinement of the following question: if $X_1, \ldots, X_s$ are linear subspaces of $\mathbb{P}^n$, then when is $X = \bigcup_{i=1}^s X_i$ ACM? To see this, note that if we consider only the graded structure of $R/I_{\mathbb{X}}$, then the defining ideal of each point is also the defining ideal of a linear subspace in a projective space. This paper, therefore, can be seen as one attack on this more general question. Alternatively, this paper can viewed as part of the program to understand when a multigraded ring is Cohen-Macaulay (for example, see [4]).

We now expand upon the results of this paper. We start in Section 2 by recalling the relevant results and definitions about sets of points in a multiprojective space. In Section 3 we study the Hilbert function of an ACM set of points. As mentioned above, ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ can be classified via their Hilbert function $H_{\mathbb{X}}$; precisely, $\mathbb{X}$ is ACM in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta H_{\mathbb{X}}$, a generalized first difference function, is the Hilbert function of a bigraded artinian quotient of $k[x_1, y_1]$. One direction of this characterization extends to any multiprojective space, as first proved by the second author [21]:

Theorem 1.1 (Theorem 3.2)

Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ *be a finite set of points with Hilbert function* $H_{\mathbb{X}}$. *If* $\mathbb{X}$ *is ACM, then* $\Delta H_{\mathbb{X}}$ *is the Hilbert function of an* $\mathbb{N}^r$*-graded artinian quotient of* $k[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{r,1}, \ldots, x_{r,n_r}]$ *with* $\deg x_{i,j} = e_i$.

It was not known whether the converse held (in fact, this question was raised in the second author's thesis [19, Question 1.3.9]). We give the first example (see Example 3.3) of a set of points in $\mathbb{P}^2 \times \mathbb{P}^2$ for which the converse fails and it can be extended to any multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with at least two n_i's greater than or equal to 2. This example together with Example 3.4 demonstrates that we cannot expect a classification of ACM sets of points based only upon the Hilbert function. However, if all but one of the n_i's equal one, we expect the converse of Theorem 3.2 to hold. We give partial evidence for this claim in Theorem 3.7 where we show that the converse holds for sets of points $\mathbb{X}$ in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (r times) with $\operatorname{depth}(R/I_{\mathbb{X}}) = r - 1$. In fact, Theorem 3.7, combined with Theorem 1.1, will allow us to give a new proof of Giuffrida, Maggioni, and Ragusa's result.

In Section 4 we examine how the geometry of a set of points influences its ACMness. If $\mathbb{X}$ is a set of points in $\mathbb{P}^n \times \mathbb{P}^m$, we say that $\mathbb{X}$ satisfies **property** $(\star)$ if whenever $P_1 \times Q_1$ and $P_2 \times Q_2$ are in $\mathbb{X}$ with $P_1 \neq P_2$ and $Q_1 \neq Q_2$, then either $P_1 \times Q_2$ or $P_2 \times Q_1$ (or both) are in $\mathbb{X}$. The two authors independently showed (see [11, 21]) that $\mathbb{X}$ is ACM in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\mathbb{X}$ satisfies property $(\star)$. We extend one direction of this classification:

Theorem 1.2 (Theorem 4.5)

Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ be a finite set of points. If $\mathbb{X}$ satisfies property $(\star)$, then $\mathbb{X}$ is ACM.

The converse, however, is false, as shown in Example 4.9 where we give an example of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^2$ which fails to satisfy property $(\star)$. At the end of Section 4, we show how to use Theorem 1.2 to easily construct ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^n$.

In Section 5 we study the connection between the separators of a point and the ACMness of a set of points. If $P \in \mathbb{X}$, then the multihomogeneous form $F \in R$ is a **separator for** P if $F(P) \neq 0$ and $F(Q) = 0$ for all $Q \in \mathbb{X} \setminus \{P\}$. The **degree of a point** $P \in \mathbb{X}$ is the set

$$\deg_{\mathbb{X}}(P) \ = \ \min\{\deg F \mid F \text{ is a separator for } P \in \mathbb{X}\}.$$

(We are using the partial order on $\mathbb{N}^r$ defined by $(i_1, \ldots, i_r) \succeq (j_1, \ldots, j_r)$ whenever $i_t \geq j_t$ for $t = 1, \ldots, r$.) Note that if $r \geq 1$, then we may have $|\deg_{\mathbb{X}}(P)| > 1$. Separators for points in $\mathbb{P}^1 \times \mathbb{P}^1$ were first introduced by Marino [14], who extended the original definition for points in $\mathbb{P}^n$ due to Orecchia [17]. Marino has recently shown [16] that a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM if and only if $|\deg_{\mathbb{X}}(P)| = 1$ for all $P \in \mathbb{X}$. We show that one direction of Marino's result holds in an arbitrary multiprojective space:

Theorem 1.3 (Theorem 5.7)

Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a finite set of points. If $\mathbb{X}$ is ACM, then $|\deg_{\mathbb{X}}(P)| = 1$ for every point $P \in \mathbb{X}$.

The converse of Theorem 1.3 fails to hold; Example 5.10 gives an example of a set of points $\mathbb{X} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ where every point $P \in \mathbb{X}$ has $|\deg_{\mathbb{X}}(P)| = 1$, but $\mathbb{X}$ fails to be ACM.

Finally, we note that examples of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, especially the counterexamples to the converses of Theorems 1.1, 1.2, and 1.3, play a prominent role in this paper. Instrumental in finding these examples was the computer program `CoCoA` [5]. To encourage further experimentation, our `CoCoA` scripts and examples can be found on the second author's webpage.[1]

[1] `http://flash.lakeheadu.ca/~avantuyl/research/ACMexamples_Guardo_VanTuyl.html`

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2. Preliminaries

We begin by recalling some relevant results about points in a multiprojective space. A more thorough introduction to points in a multiprojective space can be found in [20, 21]. In this paper k denotes an algebraically closed field of characteristic zero.

We shall write $(i_1,\dots,i_r)\in\mathbb{N}^r$ as $\underline{i}$. We induce a partial order on the set $\mathbb{N}^r$ by setting $(a_1,\dots,a_r)\succeq(b_1,\dots,b_r)$ if $a_i\geq b_i$ for $i=1,\dots,r$. The coordinate ring of the **multiprojective space** $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ is the $\mathbb{N}^r$-graded ring

$$R=k[x_{1,0},\dots,x_{1,n_1},x_{2,0},\dots,x_{2,n_2},\dots,x_{r,0},\dots,x_{r,n_r}]$$

where $\deg x_{i,j}=e_i$, the ith standard basis vector of $\mathbb{N}^r$. We use

$$R=k[x_0,\dots,x_n,y_0,\dots,y_m]$$

if considering the multiprojective space $\mathbb{P}^n\times\mathbb{P}^m$. If

$$P=[a_{1,0}:\cdots:a_{1,n_1}]\times\cdots\times[a_{r,0}:\cdots:a_{r,n_r}]\in\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$$

is a **point** in this space, then the ideal I_P of R associated to P is a prime ideal of the form

$$I_P=(L_{1,1},\dots,L_{1,n_1},\dots,L_{r,1},\dots,L_{r,n_r})$$

where $\deg L_{i,j}=e_i$ for $j=1,\dots,n_i$. When $\mathbb{X}=\{P_1,\dots,P_s\}$ is a set of s distinct points in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$, then $I_{\mathbb{X}}=I_{P_1}\cap\cdots\cap I_{P_s}$, where I_{P_i} is the ideal associated to the point P_i, is the ideal generated by all the multihomogeneous forms that vanish at all the points of $\mathbb{X}$. The ideal $I_{\mathbb{X}}$ is a **multihomogeneous** (or simply, **homogeneous**) ideal of R.

Theorem 2.1

Let $\mathbb{X}\subseteq\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ be a finite set of points. Then

$$\dim R/I_{\mathbb{X}}=r \quad\textit{and}\quad 1\leq\operatorname{depth}R/I_{\mathbb{X}}\leq r.$$

In fact, for any $l\in\{1,\dots,r\}$, there exists a set of points $\mathbb{X}_l$ with $\operatorname{depth}R/I_{\mathbb{X}_l}=l$.

Proof. Because each prime ideal I_{P_i} with $P_i\in\mathbb{X}$ has height $\sum_{i=1}^r n_i$, it follows that $\dim R/I_{\mathbb{X}}=r$. For the statement about the depth, see [21, Proposition 2.6]. □

DEFINITION 2.2 A set of points $\mathbb{X}\subseteq\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ is **arithmetically Cohen-Macaulay** (ACM) if $\dim R/I_{\mathbb{X}}=\operatorname{depth}R/I_{\mathbb{X}}=r$, that is, if $R/I_{\mathbb{X}}$ is Cohen-Macaulay (CM).

Remark 2.3 When $r=1$, it is clear from Theorem 2.1 that $\mathbb{X}$ is always ACM. However, when $r\geq 2$, it is possible that $\operatorname{depth}R/I_{\mathbb{X}}<\dim R/I_{\mathbb{X}}$, and consequently, $\mathbb{X}$ will not be ACM. The sets $\mathbb{X}_l$ in Theorem 2.1 can be constructed with the property that $|\mathbb{X}_l|=2$.

We will periodically require the following result which guarantees the existence of a regular sequence of specific degrees.

Theorem 2.4

Suppose that $\mathbb{X}$ *is set of points in* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ *with* $\operatorname{depth} R/I_{\mathbb{X}} = l \geq 1$. *Then there exist elements* $\overline{L}_1, \ldots, \overline{L}_l$ *in* $R/I_{\mathbb{X}}$ *such that* $L_1, \ldots, L_l$ *give rise to a regular sequence in* $R/I_{\mathbb{X}}$ *and* $\deg L_i = e_i$ *for* $i = 1, \ldots, l$.

Proof. One can adapt the proof of [21, Proposition 3.2] to get the desired conclusion. □

Remark 2.5 By Theorem 2.4, when $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, then there exists $L_1, \ldots, L_r$ with $\deg L_i = e_i$ such that the L_i's give rise to a regular sequence. After a change of coordinates, we may assume that $L_i = x_{i,0}$ for $i = 1, \ldots, r$. Note that $x_{i,0}$ does not vanish at any point of $\mathbb{X}$. Furthermore, since each $x_{i,0}$ is homogeneous, any permutation of $\{x_{1,0}, \ldots, x_{r,0}\}$ is also a regular sequence on $R/I_{\mathbb{X}}$.

The **coordinate ring of** $\mathbb{X}$, that is, $R/I_{\mathbb{X}}$, inherits the $\mathbb{N}^r$-grading of R. We can then apply the following definition:

Definition 2.6 Let I be a multihomogeneous ideal of R. The **Hilbert function of** $S = R/I$ is the numerical function $H_S : \mathbb{N}^r \to \mathbb{N}$ defined by

$$H_S(\underline{i}) := \dim_k S_{\underline{i}} = \dim_k R_{\underline{i}} - \dim_k (I)_{\underline{i}} \quad \text{for all } \underline{i} \in \mathbb{N}^r.$$

When $S = R/I_{\mathbb{X}}$ is the coordinate ring of a set of points $\mathbb{X}$, then we usually say H_S is the **Hilbert function of** $\mathbb{X}$, and write $H_{\mathbb{X}}$.

Remark 2.7 An interesting open problem is to classify what functions can be the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. When $r = 1$, then the set of valid Hilbert functions for sets of points in $\mathbb{P}^n$ was first classified by Geramita, Maroscia, and Roberts [6]. However, very few results are known if $r \geq 2$. See [7, 20, 21] for more on this problem, and some necessary conditions.

In the study of the Hilbert functions of points in $\mathbb{P}^n$, one can use the first difference Hilbert function to ascertain certain geometric and algebraic information about the set of points. As we shall see in the next section, a generalized first difference Hilbert function is a tool that provides information about the ACMness of sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. The definition that we shall require is:

Definition 2.8 If $H : \mathbb{N}^r \to \mathbb{N}$ is a numerical function, then the **first difference function of** H, denoted ΔH, is defined to be

$$\Delta H(\underline{i}) := \sum_{\underline{0} \,\preceq\, \underline{l}=(l_1,\ldots,l_r) \preceq (1,\ldots,1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_r - l_r)$$

where $H(\underline{j}) = 0$ if $\underline{j} \not\succeq \underline{0}$ and $|\underline{l}| = l_1 + \cdots + l_r$.

Note that when $r = 1$, we recover the well known first difference function $\Delta H(i) = H(i) - H(i-1)$ where $H(i) = 0$ if $i < 0$.

3. ACM sets of points and their Hilbert functions

In this section we revisit a classification of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ due to Giuffrida, Maggioni, and Ragusa [7]:

Theorem 3.1

Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of points with Hilbert function $H_{\mathbb{X}}$. Then $\mathbb{X}$ is ACM if and only if $\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded artinian quotient of $k[x_1, y_1]$.

As shown by the second author [21], one direction of this classification extends quite naturally to an arbitrary multiprojective space, thus giving us a necessary condition for a set of points to be ACM. However, what was not known was whether or not the converse statement held; we show via an example that the converse fails, thus showing that ACM sets of points cannot be classified by Hilbert functions. We also give a new proof for Theorem 3.1. We begin by recalling the partial generalization of Theorem 3.1.

Theorem 3.2 ([21, Corollary 3.5])

Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a finite set of points with Hilbert function $H_{\mathbb{X}}$. If $\mathbb{X}$ is ACM, then $\Delta H_{\mathbb{X}}$ is the Hilbert function of an $\mathbb{N}^r$-graded artinian quotient of $k[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{r,1}, \ldots, x_{r,n_r}]$.

Proof. We sketch out the idea of the proof. Because $\mathbb{X}$ is ACM, by Theorem 2.4 there exists a regular sequence $\overline{L}_1, \ldots, \overline{L}_r \in R/I_{\mathbb{X}}$ with $\deg L_i = e_i$. Let $J_0 = I_{\mathbb{X}}$ and $J_i = (J_{i-1}, L_i)$ for $i = 1, \ldots, r$. Then, for each i we have a short exact sequence

$$0 \to R/J_{i-1}(-e_i) \xrightarrow{\times \overline{L}_i} R/J_{i-1} \longrightarrow R/J_i \to 0.$$

Using the r short exact sequences, one can show that $\Delta H_{\mathbb{X}}$ is the Hilbert function of R/J_r. Furthermore, because $R/I_{\mathbb{X}}$ is ACM and $J_r = I_{\mathbb{X}} + (L_1, \ldots, L_r)$, we have that R/J_r is artinian. By Remark 2.5, if we make a change of coordinates so that $L_i = x_{i,0}$, then

$$R/J_r \cong (R/(x_{1,0}, \ldots, x_{r,0}))/((I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r,0})/(x_{1,0}, \ldots, x_{r,0}))$$

and $R/(x_{1,0}, \ldots, x_{r,0}) \cong k[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{r,1}, \ldots, x_{r,n_r}]$. □

The following two examples show that the converse to Theorem 3.2 is not true in general; these examples are the first known counterexamples to the converse statement.

EXAMPLE 3.3 Let $P_1, \ldots, P_6$ be six points in general position in $\mathbb{P}^2$. By general position we mean that no more than two points lie on line, and no more than five points lie on a conic. Set $Q_{i,j} := P_i \times P_j \in \mathbb{P}^2 \times \mathbb{P}^2$, and let $\mathbb{X}$ be the following set of 27 points:

$$\begin{aligned} \mathbb{X} = \{ & Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}, Q_{1,5}, Q_{1,6}, Q_{2,1}, Q_{2,3}, Q_{2,4}, Q_{2,6}, Q_{3,1}, Q_{3,2}, Q_{3,5}, Q_{3,6}, \\ & Q_{4,1}, Q_{4,2}, Q_{4,5}, Q_{4,6}, Q_{5,1}, Q_{5,3}, Q_{5,6}, Q_{6,1}, Q_{6,2}, Q_{6,3}, Q_{6,4}, Q_{6,5}, Q_{6,6}\}. \end{aligned}$$

Then $\mathbb{X}$ is not ACM since $R/I_{\mathbb{X}}$ has projective dimension 5 (and not 4 for $R/I_{\mathbb{X}}$ to be Cohen-Macaulay) because the minimal graded resolution is

$$0 \to R \to R^{13} \to R^{38} \to R^{42} \to R^{17} \to R \to R/I_{\mathbb{X}} \to 0$$

where we have suppressed the bigraded shifts. For this set of points, $H_{\mathbb{X}}$ and $\Delta H_{\mathbb{X}}$ are

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 3 & 6 & 6 & \cdots \\ 3 & 9 & 18 & 18 & \cdots \\ 6 & 18 & 27 & 27 & \cdots \\ 6 & 18 & 27 & 27 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad \Delta H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & 0 & \cdots \\ 2 & 4 & 6 & 0 & \cdots \\ 3 & 6 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(The (i, j)th entry of the above matrix corresponds to the value of the Hilbert function at (i, j), where we start our indexing at $(0, 0)$.) However, $\Delta H_{\mathbb{X}}$ equals $H_{S/I}$, the Hilbert function of S/I where $S = k[x_1, x_2, y_1, y_2]$ and

$$I = (x_1, x_2)^3 + (y_1, y_2)^3 + (x_1, x_2)^2(y_1, y_2)^2.$$

Note that S/I is artinian since $H_{S/I}(\underline{i}) = 0$ for all but a finite number of $\underline{i} \in \mathbb{N}^2$. This shows that the converse of Theorem 3.2 cannot hold because $\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded artinian quotient, but $\mathbb{X}$ is not ACM.

As the following example illustrates, we cannot expect any general classification of ACM sets of points to be based solely upon the Hilbert function.

EXAMPLE 3.4 Let $P_{ij} = [1 : i : j] \in \mathbb{P}^2$, and let $Q_{ijkl} = P_{ij} \times P_{kl} \in \mathbb{P}^2 \times \mathbb{P}^2$. Consider the following 27 points in $\mathbb{P}^2 \times \mathbb{P}^2$:

$$\begin{aligned} \mathbb{Y} = \{ & Q_{1121}, Q_{1122}, Q_{1131}, Q_{1221}, Q_{1222}, Q_{1231}, Q_{1321}, Q_{1322}, Q_{1331}, \\ & Q_{2111}, Q_{2112}, Q_{2113}, Q_{2121}, Q_{2122}, Q_{2131}, Q_{2211}, Q_{2212}, Q_{2213}, \\ & Q_{2221}, Q_{2222}, Q_{2231}, Q_{3111}, Q_{3112}, Q_{3113}, Q_{3121}, Q_{3122}, Q_{3131}\}. \end{aligned}$$

Using `CoCoA` to compute the resolution of $R/I_{\mathbb{Y}}$ we get

$$0 \to R^{12} \to R^{38} \to R^{42} \to R^{17} \to R \to R/I_{\mathbb{Y}} \to 0,$$

where we have suppressed all the bigraded shifts. So $\mathbb{Y}$ is ACM because the projective dimension is four. If $\mathbb{X}$ is the set of 27 nonACM points from the last example, then

$$H_{\mathbb{X}} = H_{\mathbb{Y}} = \begin{bmatrix} 1 & 3 & 6 & 6 & \cdots \\ 3 & 9 & 18 & 18 & \cdots \\ 6 & 18 & 27 & 27 & \cdots \\ 6 & 18 & 27 & 27 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So ACM and nonACM sets of points can have the exact same Hilbert function.

We can extend the above examples to show that the converse of Theorem 3.2 fails to hold in any multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$ and with at least two n_i's greater than or equal to two.

EXAMPLE 3.5 Consider the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$. Suppose further that $n_i, n_j \geq 2$ with $i \neq j$. Set $Q_k = [1 : 0 : \cdots : 0] \in \mathbb{P}^{n_k}$ for $k \in \{1, \ldots, r\} \backslash \{i, j\}$. For any point $P = [a_1 : a_2 : a_3] \in \mathbb{P}^2$, let $P' = [a_1 : a_2 : a_3 : 0 : \cdots : 0] \in \mathbb{P}^{n_i}$, and $P'' = [a_1 : a_2 : a_3 : 0 : \cdots : 0] \in \mathbb{P}^{n_j}$. Let $\mathbb{X}$ be the set of points from Example 3.3. Consider the following set of points:

$$\mathbb{X}' = \{Q_1 \times \cdots \times P_i' \times \cdots \times P_j'' \times \cdots \times Q_r \mid P_i \times P_j \in \mathbb{X}\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}.$$

If the point $P_i \times P_j \in \mathbb{X} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ has defining ideal $(L_{i,1}, L_{i,2}, L_{j,1}, L_{j,2})$ in $k[x_0, x_1, x_2, y_0, y_1, y_2]$, then let $L'_{i,t}$, respectively $L'_{j,t}$, denote the forms we obtain replacing x_t with $x_{i,t}$, respectively, y_t with $x_{j,t}$. The defining ideal of $Q_1 \times \cdots \times P_i' \times \cdots \times P_j'' \times \cdots \times Q_r \in \mathbb{X}'$ has form

$$\begin{aligned}&(x_{1,1}, \ldots, x_{1,n_1}, \ldots, L'_{i,1}, L'_{i,2}, x_{i,3}, \ldots, x_{i,n_i}, \ldots, L'_{j,1},\\ &L'_{j,2}, x_{j,3}, \ldots, x_{j,n_j}, \ldots, x_{r,1}, \ldots, x_{r,n_r}).\end{aligned}$$

So, we have

$$R/I_{\mathbb{X}'} \cong k[x_{1,0}, x_{2,0}, \ldots, x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{j,0}, x_{j,1}, x_{j,2}, \ldots, x_{r,0}]/\tilde{I}_{\mathbb{X}}$$

where by $\tilde{I}_{\mathbb{X}}$ we mean the ideal generated by the elements of $I_{\mathbb{X}} \subseteq k[x_0, x_1, x_2, y_0, y_1, y_2]$, where we replace x_t by $x_{i,t}$ and y_t by $x_{j,t}$. The elements $x_{1,0}, \ldots, \hat{x}_{i,0}, \ldots, \hat{x}_{j,0}, \ldots, x_{r,0}$ then form a regular sequence so that

$$\frac{k[x_{1,0}, x_{2,0}, \ldots, x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{j,0}, x_{j,1}, x_{j2}, \ldots, x_{r,0}]}{(\tilde{I}_{\mathbb{X}}, x_{1,0}, \ldots, \hat{x}_{i,0}, \ldots, \hat{x}_{j,0}, \ldots, x_{r,0})} \cong k[x_0, x_1, x_2, y_0, y_1, y_2]/I_{\mathbb{X}}.$$

Then $\mathbb{X}'$ will not be ACM because

$$\operatorname{depth}(R/I_{\mathbb{X}'}) = r - 2 + \operatorname{depth}(k[x_0, x_1, x_2, y_0, y_1, y_2]/I_{\mathbb{X}}) = r - 1.$$

However, $\Delta H_{\mathbb{X}'}$ is the Hilbert function of an artinian quotient since

$$\Delta H_{\mathbb{X}'}(\underline{i}) = \begin{cases} \Delta H_{\mathbb{X}}(a, b) & \text{if } \underline{i} = ae_i + be_j = (0, \ldots, a, \ldots, b, \ldots, 0) \\ 0 & \text{otherwise} \end{cases}$$

and this function is the Hilbert function of S/I where

$$\begin{aligned} I \;=\; & (x_{i,1}, x_{i,2})^3 + (x_{j,1}, x_{j,2})^3 + (x_{i,1}, x_{i,2})^2 (x_{j,1}, x_{j,2})^2 + (x_{i,3}, \ldots, x_{i,n_i}) \\ & + (x_{j,3}, \ldots, x_{j,n_j}) + S_{e_1} + S_{e_2} + \cdots + \hat{S}_{e_i} + \cdots + \hat{S}_{e_j} + \cdots + S_{e_r}. \end{aligned}$$

where $S_{e_l} = (x_{l,1}, \ldots, x_{l,n_l})$ and $S = k[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{r,1}, \ldots, x_{r,n_r}]$.

The set of points $\mathbb{Y}$ in Example 3.4 can similarly be extended to a set of points in $\mathbb{Y}' \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ where $H_{\mathbb{X}'}$ and $H_{\mathbb{Y}'}$ are equal.

In light of the above examples, we see that to distinguish ACM sets of points from nonACM, we will need more information than just the Hilbert function of the set of points. However, as a corollary of Theorem 3.2, we can eliminate certain sets of points as being ACM directly from their Hilbert function. A similar result was also proved in [18, Theorem 4.7]. If $\underline{i} = (i_1, \ldots, i_r), \underline{j} = (j_1, \ldots, j_r) \in \mathbb{N}^r$, we set $\min\{\underline{i}, \underline{j}\} := (\min\{i_1, j_1\}, \ldots, \min\{i_r, j_r\})$.

Corollary 3.6

Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a finite set of points with Hilbert function $H_{\mathbb{X}}$. If there exists $\underline{i}, \underline{j} \in \mathbb{N}^r$ such that $H_{\mathbb{X}}(\underline{i}) = H_{\mathbb{X}}(\underline{j}) = |\mathbb{X}|$ but $H_{\mathbb{X}}(\underline{k}) \neq |\mathbb{X}|$ with $\underline{k} = \min\{\underline{i}, \underline{j}\}$, then $\mathbb{X}$ is not ACM.

Proof. For any $\underline{i} \in \mathbb{N}^r$, the Hilbert function of $\mathbb{X}$ satisfies

$$H_{\mathbb{X}}(\underline{i}) = \sum_{\underline{0} \preceq \underline{j} \preceq \underline{i}} \Delta H_{\mathbb{X}}(\underline{j}).$$

When $\mathbb{X}$ is ACM, by Theorem 3.2 we have $\Delta H_{\mathbb{X}}(\underline{j}) \geq 0$ for all $\underline{j} \in \mathbb{N}^r$. But then there exists a $\underline{k} \in \mathbb{N}^r$ such that $H_{\mathbb{X}}(\underline{i}) = |\mathbb{X}|$ if and only if $\underline{i} \succeq \underline{k}$. □

It is not presently known whether the converse of Theorem 3.2 fails to hold in multiprojective spaces of the form $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$ and with only one $n_i \geq 1$, and the rest of the n_j's equal to one. We end this section by giving partial evidence that the converse of Theorem 3.2 may hold for sets of points in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($r \geq 3$ times). This result will also allow us to give a new proof for Theorem 3.1.

Theorem 3.7

Let $\mathbb{X}$ be set of points in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($r \geq 2$ times) and suppose that $\operatorname{depth}(R/I_{\mathbb{X}}) = r - 1$. If $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$, then $\Delta H_{\mathbb{X}}$ is not the Hilbert function of an $\mathbb{N}^r$-graded artinian quotient of $k[x_{1,1}, x_{2,1}, \ldots, x_{r,1}]$.

To prove this statement, we will need the following two technical lemmas.

Lemma 3.8

Let $S = k[x_{1,1}, \ldots, x_{r-1,1}, x_{r,0}, x_{r,1}]$ be an $\mathbb{N}^r$-graded ring with $\deg x_{i,1} = e_i$ for $i = 1, \ldots, r-1$, and $\deg x_{r,0} = \deg x_{r,1} = e_r$. Let $J \subseteq S$ be any $\mathbb{N}^r$-graded ideal of S. If $H_{S/J}(\underline{i}) \leq H_{S/J}(\underline{i} + e_r)$, then $H_{S/J}(\underline{i} + e_r) = H_{S/J}(\underline{i})$ or $H_{S/J}(\underline{i}) + 1$.

Proof. We are given that $\dim_k S_{\underline{i}+e_r} - \dim_k (J)_{\underline{i}+e_r} \geq \dim_k S_{\underline{i}} - \dim_k (J)_{\underline{i}}$. Now, for any $\underline{i} = (i_1, \ldots, i_r) \in \mathbb{N}^r$, $\dim_k S_{\underline{i}} = i_r + 1$. Thus $i_r + 2 - \dim_k (J)_{\underline{i}+e_r} \geq i_r + 1 - \dim_k (J)_{\underline{i}}$. But because $\dim_k (J)_{\underline{i}+e_r} \geq \dim_k (J)_{\underline{i}}$ for all $\underline{i}$, we must have

$$\dim_k (J)_{\underline{i}} \geq \dim_k (J)_{\underline{i}+e_r} - 1 \geq \dim_k (J)_{\underline{i}} - 1.$$

So $\dim_k (J)_{\underline{i}+e_r} = \dim_k (J)_{\underline{i}}$ or $\dim_k (J)_{\underline{i}} + 1$, and thus the conclusion follows. □

Lemma 3.9

Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ *be any finite set of points such that* $\operatorname{depth}(R/I_{\mathbb{X}}) = r-1$. *Suppose* $x_{1,0}, \ldots, x_{r-1,0}$ *is a regular sequence on* $R/I_{\mathbb{X}}$, *and suppose that* $\overline{x}_{1,0}, \ldots, \overline{x}_{r,0}$ *are nonzero divisors on* $R/I_{\mathbb{X}}$. *If*

$$H_{R/(I_{\mathbb{X}},x_{1,0},\ldots,x_{r-1,0})}(\underline{i}) \leq H_{R/(I_{\mathbb{X}},x_{1,0},\ldots,x_{r-1,0})}(\underline{i}+e_r) \quad \textit{for all } \underline{i} \in \mathbb{N}^r,$$

then

$$H_{R/(I_{\mathbb{X}},x_{1,0},\ldots,x_{r,0})}(\underline{i}) = \begin{cases} 1 & \textit{if } (I_{\mathbb{X}})_{\underline{i}} = (0) \\ 0 & \textit{if } (I_{\mathbb{X}})_{\underline{i}} \neq (0). \end{cases}$$

Proof. To simplify our notation, set $J = (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r,0})$. Consider any $\underline{i} \in \mathbb{N}^r$ such that $(I_{\mathbb{X}})_{\underline{i}} = (0)$. Then $(J)_{\underline{i}} = (x_{1,0}, \ldots, x_{r,0})_{\underline{i}}$. Hence

$$(R/J)_{\underline{i}} = (R/(x_{1,0}, \ldots, x_{r,0}))_{\underline{i}} = (k[x_{1,1}, \ldots, x_{r,1}])_{\underline{i}},$$

and thus $H_{R/J}(\underline{i}) = \dim_k(k[x_{1,1}, \ldots, x_{r,1}])_{\underline{i}} = 1$.

Let $\pi_i(\mathbb{X}) = \{R_{i,1}, \ldots, R_{i,t_i}\}$ be the set of the distinct ith coordinates of the points that appear in $\mathbb{X}$. If $L_{R_{i,j}}$ is the form of degree e_i that passes through the point $R_{i,j}$, then $L_{R_{i,1}} \cdots L_{R_{i,t_i}}$ is a minimal generator of $I_{\mathbb{X}}$ of degree t_ie_i. Now $H_{\mathbb{X}}((t_i-1)e_i) = H_{\mathbb{X}}(t_ie_i) = |\pi_i(\mathbb{X})| = t_i$ (for example, see [20, Proposition 4.6]). Because $\overline{x}_{i,0}$ is a nonzero divisor, the short exact sequence

$$0 \to R/I_{\mathbb{X}}(-e_i) \xrightarrow{\times \overline{x}_{i,0}} R/I_{\mathbb{X}} \to R/(I_{\mathbb{X}}, x_{i,0}) \to 0,$$

implies that $H_{R/(I_{\mathbb{X}},x_{i,0})}(t_ie_i) = H_{\mathbb{X}}(t_ie_i) - H_{\mathbb{X}}((t-1)e_i) = 0$. In other words, $R_{t_ie_i} = (I_{\mathbb{X}}, x_{i,0})_{t_ie_i}$. So, for any $\underline{i} \succeq t_ie_i$, $R_{\underline{i}} = (I_{\mathbb{X}}, x_{i,0})_{\underline{i}} \subseteq (J)_{\underline{i}}$, and hence $H_{R/J}(\underline{i}) = 0$. So, $H_{R/J}(\underline{i}) = 0$ for all $\underline{i} \succeq t_ie_i$ and each $i = 1, \ldots, r$.

Now consider any $\underline{i} \in \mathbb{N}^r$ such that $\underline{i} \preceq (t_1-1, \ldots, t_r-1)$ and $(I_{\mathbb{X}})_{\underline{i}} \neq (0)$. Assume that $\underline{i}$ is minimal, i.e., $(I_{\mathbb{X}})_{\underline{i}} \neq (0)$, but $(I_{\mathbb{X}})_{\underline{i}-e_j} = (0)$ for $j = 1, \ldots, r$. There is then a minimal generator $F \in I_{\mathbb{X}}$ of degree $\underline{i}$ which we can write as

$$F = cx_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r,1}^{i_r} + \sum_{m \text{ a monomial of degree } \underline{i} \text{ in } (x_{1,0},\ldots,x_{r,0})} c_m m. \tag{3.1}$$

If $c \neq 0$, then because $x_{1,0}, \ldots, x_{r,0}$ and F are in $(J)_{\underline{i}}$, we then have $R_{\underline{i}} = (J)_{\underline{i}}$, from which it follows that $H_{R/J}(\underline{j}) = 0$ for all $\underline{j} \succeq \underline{i}$.

It thus remains to show that we can find a minimal generator F of degree $\underline{i}$ with form (3.1) and $c \neq 0$. Suppose not, that is, $c = 0$. Then F has the form

$$\begin{aligned} F &= x_{1,0}H_1(x_{1,0}, \ldots, x_{r,1}) + x_{1,1}^{i_1}x_{2,0}H_2(x_{2,0}, \ldots, x_{r,1}) + x_{1,1}^{i_1}x_{2,1}^{i_2}x_{3,0}H_3(x_{3,0}, \ldots, x_{r,1}) \\ &\quad + \cdots + x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}x_{r,0}H_r(x_{r,0}, x_{r,1}). \end{aligned}$$

Set $F' = x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}x_{r,0}H_r(x_{r,0}, x_{r,1})$. We first show that $F' \neq 0$. If $F' = 0$, then $F \in (x_{1,0}, \ldots, x_{r-1,0})_{\underline{i}}$, i.e.,

$$F = x_{1,0}G_1 + x_{2,0}G_2 + \cdots + x_{r-1,0}G_{r-1}. \tag{3.2}$$

But then $x_{r-1,0}G_{r-1} \in (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r-2,0})$, and since $x_{r-1,0}$ is regular on $R/(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r-2,0})$, we have $G_{r-1} \in (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r-2,0})_{\underline{i}-e_{r-1}}$. Because $(I_{\mathbb{X}})_{\underline{j}} = (0)$ for all $\underline{j} \prec \underline{i}$ we must have that $G_{r-1} \in (x_{1,0}, \ldots, x_{r-2,0})$. So $G_{r-1} = x_{1,0}G'_1 + \cdots + x_{r-2,0}G'_{r-2}$, and subbing back into (3.2) we get

$$\begin{aligned} F &= x_{1,0}G_1 + \cdots + x_{r-1,0}(x_{1,0}G'_1 + \cdots + x_{r-2,0}G'_{r-2}) \\ &= x_{1,0}G''_1 + x_{2,0}G''_2 + \cdots + x_{r-2,0}G''_{r-2} \end{aligned}$$

where $G''_i = G_i + x_{r-1,0}G'_i$. Similarly, we can show that $G''_{r-2} \in (x_{1,0}, \ldots, x_{r-3,0})$, and thus $F = x_{1,0}E_1 + \cdots + x_{r-3,0}E_{r-3}$ for some appropriate forms E_i. We can continue this process to eventually show that F is divisible by $x_{1,0}$, that is, $F = x_{1,0}F_1$. But since $x_{1,0}$ is a regular on $R/I_{\mathbb{X}}$, this implies that $F_1 \in (I_{\mathbb{X}})$, contradicting the minimality of the degree of F. So $F' \neq 0$.

Set $J' = (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r-1,0})$. We now claim that $H_{R/J'}(\underline{i}) = i_r$. Now

$$H_{R/J'}(\underline{i} - e_r) = H_{R/(x_{1,0},\ldots,x_{r-1,0})}(\underline{i} - e_r) = i_r.$$

Our hypotheses then imply that $i_r = H_{R/J'}(\underline{i} - e_r) \leq H_{R/J'}(\underline{i})$. But because

$$R/J' \cong \frac{R/(x_{1,0}, \ldots, x_{r-1,0})}{J'/(x_{1,0}, \ldots, x_{r-1,0})} \cong k[x_{1,1}, \ldots, x_{r-1,0}, x_{r,0}, x_{r,1}]/L \text{ for some ideal } L,$$

Lemma 3.8 implies $H_{R/J'}(\underline{i}) = H_{R/J'}(\underline{i} - e_r)$ or $H_{R/J'}(\underline{i} - e_r) + 1$. If $H_{R/J'}(\underline{i}) = i_r + 1$, then

$$\dim_k(J')_{\underline{i}} \;=\; (i_1+1)\cdots(i_r+1) - (i_r+1) = \dim_k(x_{1,0}, \ldots, x_{r-1,0})_{\underline{i}}.$$

This means that $(J')_{\underline{i}} = (x_{1,0}, \ldots, x_{r-1,0})_{\underline{i}}$, and hence $F \in (x_{1,0}, \ldots, x_{r-1,0})_{\underline{i}}$. But as shown above, $F' \neq 0$, so $F \notin (x_{1,0}, \ldots, x_{r-1,0})_{\underline{i}}$. Thus $H_{R/J'}(\underline{i}) = i_r$, and $\dim_k(J')_{\underline{i}} = (i_1+1)\cdots(i_r+1) - i_r$.

Let $M_{\underline{i}}$ denote the set of all monomials of degree $\underline{i}$ in R. A basis for $(J')_{\underline{i}}$ is then

$$\{F'\} \cup (M_{\underline{i}} \setminus \{x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}m \mid m = x_{r,0}^{i_r-a}x_{r,1}^{a} \text{ for } a = 0, \ldots, i_r\}).$$

To see that this is a basis, note that the elements are linearly independent and we have $\dim_k(J')_{\underline{i}}$ elements. Then, for any $b \geq 1$, the following set of elements in $(J')_{\underline{i}+be_r}$ is linearly independent:

$$\begin{aligned} \mathcal{B}_b = \{&x_{r,0}^{b}F', x_{r,0}^{b-1}x_{r,1}F', \ldots, x_{r,1}^{b}F'\} \\ &\cup \ \big(M_{\underline{i}+be_r} \setminus \{x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}m \mid m = x_{r,0}^{i_r+b-a}x_{r,1}^{a} \text{ for } a = 0, \ldots, i_r+b\}\big). \end{aligned}$$

Note that

$$|\mathcal{B}_b| \;=\; (i_1+1)\cdots(i_{r-1}+1)(i_r+b+1) - i_r.$$

Now, because $i_r = H_{R/J'}(\underline{i}) \leq H_{R/J'}(\underline{i}+be_j)$ we have

$$\dim_k R_{\underline{i}+be_j} - \dim_k(J')_{\underline{i}+be_j} \geq i_r.$$

Hence

$$|\mathcal{B}_b| = (i_1+1)\cdots(i_{r-1}+1)(i_r+b+1) - i_r = \dim_k R_{\underline{i}+be_j} - i_r$$
$$\geq \dim_k (J')_{\underline{i}+be_j} \geq |\mathcal{B}_b|.$$

It follows that $\dim_k(J')_{\underline{i}+be_j} = |\mathcal{B}_b|$, and hence the elements of $\mathcal{B}_b$ form a basis for $(J')_{\underline{i}+be_j}$.

We now pick p so that $i_r + p = t_r = |\pi_r(\mathbb{X})|$. (We have $p \geq 1$ since $\underline{i} \preceq (t_1 - 1, \ldots, t_r - 1)$, i.e., $i_r < t_r$.) As noted, $L_{R_{r,1}} \cdots L_{R_{r,t_r}} \in (I_{\mathbb{X}})_{t_r e_r}$, and furthermore, each $L_{R_{r,i}}$ has form $b_{i,0}x_{r,0} + b_{i,1}x_{r,1}$ with $b_{i,1} \neq 0$ for all i because $x_{r,0}$ is not a zero-divisor, i.e., no point $R_{r,i}$ has form $[0:1]$. Now

$$x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}L_{R_{r,1}}\cdots L_{R_{r,t_r}} \in (I_{\mathbb{X}})_{\underline{i}+pe_r} \subseteq (J')_{\underline{i}+pe_r},$$

so $x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}L_{R_{r,1}}\cdots L_{R_{r,t_r}}$ can be written as a linear combination of the elements of $\mathcal{B}_p$. But this cannot happen because $x_{1,1}^{i_1}x_{2,1}^{i_2}\cdots x_{r-1,1}^{i_{r-1}}L_{R_{r,1}}\cdots L_{R_{r,t_r}}$ contains the term $x_{1,1}^{i_1}\cdots x_{r-1,1}^{i_{r-1}}x_{r,1}^{i_r+p}$, but this term does not appear in any of our basis elements. So, if F is a minimal generator of degree $\underline{i}$, it must have the form (3.1) with $c \neq 0$. □

EXAMPLE 3.10 The hypothesis that $H_{R/(I_{\mathbb{X}},x_{1,0},\ldots,x_{r-1,0})}(\underline{i}) \leq H_{R/(I_{\mathbb{X}},x_{1,0},\ldots,x_{r-1,0})}(\underline{i}+e_r)$ in the above statement is necessary. For example, in $\mathbb{P}^1 \times \mathbb{P}^1$ consider the set of points $\mathbb{X} = \{P_{1,1}, P_{2,2}, P_{3,3}\}$ where the defining ideal of $I_{P_{i,i}} = (x_1 - ix_0, y_1 - iy_0)$. Then $\operatorname{depth}(R/I_{\mathbb{X}}) = 1$, x_0 is a regular sequence on $R/I_{\mathbb{X}}$, and $\overline{x}_0$ and $\overline{y}_0$ are nonzero-divisors on $R/I_{\mathbb{X}}$. We have

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & 3 & \cdots \\ 2 & 3 & 3 & 3 & \cdots \\ 3 & 3 & 3 & 3 & \cdots \\ 3 & 3 & 3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and thus} \quad H_{R/(I_{\mathbb{X}},x_0)} = \begin{bmatrix} 1 & 2 & 3 & 3 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The function $H_{R/(I_{\mathbb{X}},x_0)}$ fails to have $H_{R/(I_{\mathbb{X}},x_0)}(i,j) \leq H_{R/(I_{\mathbb{X}},x_0)}(i,j+e_2)$ for all $(i,j) \in \mathbb{N}^2$. Now $\dim_k(I_{\mathbb{X}})_{1,1} \neq 0$ but

$$H_{R/(I_{\mathbb{X}},x_0,y_0)} = \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is, $H_{R/(I_{\mathbb{X}},x_0,y_0)} = 1$.

Proof. (of Theorem 3.7) Let $R = k[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, \ldots, x_{r,0}, x_{r,1}]$ be the $\mathbb{N}^r$-graded coordinate ring of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Since $\operatorname{depth}(R/I_{\mathbb{X}}) = r-1$, by Theorem 2.4 we can find a regular sequence $L_1, \ldots, L_{r-1}$ on $R/I_{\mathbb{X}}$ with $\deg L_i = e_i$. Moreover, by Remark 2.5, we can assume $L_i = x_{i,0}$ for $i = 1, \dots, r-1$. Note that we can also assume that $x_{r,0}$ is a nonzero divisor of $R/I_{\mathbb{X}}$.

Set $J_0 = I_{\mathbb{X}}$, and for $i = 1, \dots, r-1$, set $J_i = (J_{i-1}, x_{i,0})$. Then, for each $i = 1, \dots, r-1$ we have a short exact sequence

$$0 \to R/J_{i-1}(-e_i) \xrightarrow{\times \overline{x}_{i,0}} R/J_{i-1} \to R/J_i \to 0.$$

We thus have

$$H_{R/J_{r-1}}(\underline{i}) = H_{R/J_{r-2}}(\underline{i}) - H_{R/J_{r-2}}(\underline{i} - e_{r-1}) \text{ for all } \underline{i} \in \mathbb{N}^r$$

with $H_{R/J_{r-2}}(\underline{i}) = 0$ if $\underline{i} \not\succeq \underline{0}$.

For all $\underline{i} \in \mathbb{N}^r$, we have the following exact sequence of vector spaces:

$$0 \to (\ker \times \overline{x}_{r,0})_{\underline{i}} \to (R/J_{r-1})_{\underline{i}} \xrightarrow{\times \overline{x}_{r,0}} (R/J_{r-1})_{\underline{i}+e_r} \to (R/(J_r, x_{r,0}))_{\underline{i}+e_r} \to 0 \quad (3.3)$$

where $\overline{x}_{r,0} \neq 0$ is a degree e_r element in $R/J_{r-1} = R/(I_{\mathbb{X}}, x_{1,0}, \dots, x_{r-1,0})$. Because $\operatorname{depth}(R/I_{\mathbb{X}}) = r-1$, then there is at least one $\underline{i} \in \mathbb{N}^r$ such that the map

$$(R/J_{r-1})_{\underline{i}} \xrightarrow{\times \overline{x}_{r,0}} (R/J_{r-1})_{\underline{i}+e_r}$$

has a non-zero kernel. This follows from the fact that $\overline{x}_{r,0}$ must be a zero divisor of R/J_{r-1}, so there exists a non-zero element $\overline{F} \in (R/J_{r-1})_{\underline{i}}$ such that $Fx_{r,0} \in J_{r-1}$.

It will now suffice to show that there exists an $\underline{i} \in \mathbb{N}^r$ such that

$$H_{R/J_{r-1}}(\underline{i}) > H_{R/J_{r-1}}(\underline{i} + e_r).$$

Indeed, because $H_{R/J_{r-1}}(\underline{i}) = H_{R/J_{r-2}}(\underline{i}) - H_{R/J_{r-2}}(\underline{i} - e_{r-1})$, the above inequality would imply

$$\begin{aligned} H_{R/J_{r-2}}(\underline{i} + e_r) - H_{J_{r-2}}(\underline{i} - e_{r-1} + e_r) - H_{R/J_{r-2}}(\underline{i}) \\ + H_{R/J_{r-2}}(\underline{i} - e_{r-1}) < 0 \Leftrightarrow \Delta H_{\mathbb{X}}(\underline{i} + e_r) < 0. \end{aligned}$$

So, not only is $\Delta H_{\mathbb{X}}$ not the Hilbert function of an artinian quotient, $\Delta H_{\mathbb{X}}$ is not the Hilbert function of *any* quotient because it has a negative entry.

Suppose, for a contradiction, that $H_{R/J_{r-1}}(\underline{i}) \leq H_{R/J_{r-1}}(\underline{i} + e_r)$ for all $\underline{i} \in \mathbb{N}^r$. Because

$$R/J_{r-1} \cong \frac{R/(x_{1,0}, \dots, x_{r-1,0})}{J_{r-1}/(x_{1,0}, \dots, x_{r-1,0})} \cong k[x_{1,1}, x_{2,1}, \dots, x_{r-1,1}, x_{r,0}, x_{r,1}]/L$$

with $L \cong J_{r-1}/(x_{1,0}, \dots, x_{r-1,0})$, $H_{R/J_{r-1}}$ has the Hilbert function of some quotient of $k[x_{1,1}, \dots, x_{r-1,1}, x_{r,0}, x_{r,1}]$ with $\deg x_{i,1} = e_i$ and $\deg x_{r,0} = \deg x_{1,1} = e_r$. It then follows by Lemma 3.8 that if $H_{R/J_{r-1}}(\underline{i}) \leq H_{R/J_{r-1}}(\underline{i} + e_r)$, then $H_{R/J_{r-1}}(\underline{i} + e_r) = H_{R/J_{r-1}}(\underline{i}) + 1$, or $H_{R/J_{r-1}}(\underline{i} + e_r) = H_{R/J_{r-1}}(\underline{i})$. We now show that both cases imply $\dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} = 0$ for all $\underline{i} \in \mathbb{N}^r$.

Case 1. If $H_{R/J_{r-1}}(\underline{i} + e_r) = H_{R/J_{r-1}}(\underline{i}) + 1$, then from the exact sequence (3.3), we have

$$\begin{aligned} \dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} &= H_{R/J_{r-1}}(\underline{i}) - H_{R/J_{r-1}}(\underline{i} + e_r) + H_{R/(J_{r-1}, x_{r,0})}(\underline{i} + e_r) \\ &= -1 + H_{R/(J_{r-1}, x_{r,0})}(\underline{i} + e_r). \end{aligned}$$

By Lemma 3.9, for any $\underline{i} \in \mathbb{N}^r$ we must have

$$H_{R/(J_{r-1},x_{r,0})}(\underline{i}) = \dim_k(R/(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{r,0}))_{\underline{i}} = 0 \text{ or } 1.$$

So, if $H_{R/J_{r-1}}(\underline{i}+e_r) = H_{R/J_{r-1}}(\underline{i}) + 1$, then $\dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} = 0$ since dimension must be nonnegative.

Case 2. If $H_{R/J_{r-1}}(\underline{i}+e_r) = H_{R/J_{r-1}}(\underline{i})$, then from (3.3) we deduce that

$$\begin{aligned}\dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} &= H_{R/J_{r-1}}(\underline{i}) - H_{R/J_{r-1}}(\underline{i}+e_r) + H_{R/(J_{r-1},x_{r,0})}(\underline{i}+e_r) \\ &= 0 + H_{R/(J_{r-1},x_{r,0})}(\underline{i}+e_r).\end{aligned}$$

Now $H_{R/J_{r-1}}(\underline{i}) = H_{R/J_{r-1}}(\underline{i}+e_r)$ can occur only if $H_{\mathbb{X}}(\underline{i}+e_r) < (i_1+1)\cdots(i_r+2)$, that is, if $(I_{\mathbb{X}})_{\underline{i}+e_r} \neq (0)$. By Lemma 3.9 we have $H_{R/(J_{r-1},x_{r,0})}(\underline{i}+e_r) = 0$, and hence $\dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} = 0$.

We now see from both Cases 1 and 2, that if $H_{R/J_{r-1}}(\underline{i}) \leq H_{R/J_{r-1}}(\underline{i}+e_r)$ for all $\underline{i} \in \mathbb{N}^r$, then we must always have $\dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} = 0$. But this contradicts the fact that since $\mathbb{X}$ is not ACM, there exists some $\underline{i} \in \mathbb{N}^r$ with $\dim_k(\ker \times \overline{x}_{r,0})_{\underline{i}} > 0$. So, $H_{R/J_{r-1}}(\underline{i}) > H_{R/J_{r-1}}(\underline{i}+e_r)$ for some $\underline{i}$, as desired. □

Combining the above result with Theorem 3.2 gives a new proof for Theorem 3.1.

Proof. (Proof of Theorem 3.1) If $\mathbb{X}$ is a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, then $\operatorname{depth}(R/I_{\mathbb{X}}) = 2$ or 1 by Theorem 2.1. If $\operatorname{depth}(R/I_{\mathbb{X}}) = 2$, then $\mathbb{X}$ is ACM, and thus by Theorem 3.2 we have that $\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded artinian quotient of $k[x_1, y_1]$. If $\operatorname{depth}(R/I_{\mathbb{X}}) = 1$, then $\mathbb{X}$ is not ACM. If we now apply Theorem 3.7 to the case $r = 2$, we have that $\Delta H_{\mathbb{X}}$ is not the Hilbert function of a bigraded artinian quotient. □

Remark 3.11 Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $\operatorname{depth}(R/I_{\mathbb{X}}) = 1, 2$, or 3. In light of Theorem 3.7, to show that the converse of Theorem 3.2 holds in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, it suffices to show that if $\operatorname{depth}(R/I_{\mathbb{X}}) = 1$, then $\Delta H_{\mathbb{X}}$ is not the $\mathbb{N}^3$-graded Hilbert function of an artinian quotient of $k[x_{1,1}, x_{2,1}, x_{3,1}]$. We are currently exploring the case of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

EXAMPLE 3.12 In the proof of Theorem 3.7, to show that $\Delta H_{\mathbb{X}}$ is not the Hilbert function of an artinian ring, we show that $\Delta H_{\mathbb{X}}$ *must* have a negative entry. This approach, however, will not work for points in $\mathbb{P}^1 \times \mathbb{P}^n$ with $\operatorname{depth}(R/I_{\mathbb{X}}) = 1$ and $n > 1$. For example let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ where $\mathbb{X}$ is the following 11 points:

$$\begin{aligned}\mathbb{X} = \{&P_1 \times Q_1, P_1 \times Q_2, P_1 \times Q_3, P_1 \times Q_4, P_2 \times Q_1, P_2 \times Q_2, P_2 \times Q_3, P_2 \times Q_5, \\ &P_3 \times Q_1, P_3 \times Q_2, P_3 \times Q_3\}\end{aligned}$$

where $P_i = [1 : i] \in \mathbb{P}^1$ for $i = 1, \ldots, 3$, and

$$Q_1 = [1 : 1 : 1], Q_2 = [1 : 1 : 2], Q_3 = [1 : 1 : 3], Q_4 = [1 : 0 : 1]$$

and $Q_5 = [1 : 0 : 2]$ in $\mathbb{P}^2$. Using `CoCoA` one finds that $R/I_{\mathbb{X}}$ is not CM. The Hilbert function of $\mathbb{X}$ is

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 3 & 5 & 5 & \cdots \\ 2 & 6 & 8 & 8 & \cdots \\ 3 & 8 & 11 & 11 & \cdots \\ 3 & 8 & 11 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad \Delta H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 2 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that $\Delta H_{\mathbb{X}}$ has no negative values, but $\Delta H_{\mathbb{X}}$ still cannot be the Hilbert function of an artinian quotient. This is because $\Delta H_{\mathbb{X}}(1,2) = 0$, so we should have $\Delta H_{\mathbb{X}}(i,j) = 0$ for all $(i,j) \succeq (1,2)$, but $\Delta H_{\mathbb{X}}(2,2) = 1$.

4. ACM sets of points and their geometry

In [10] and [19], the two authors classified ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ via the geometry of the points. In this section we revisit this classification; we show that this geometric criterion extends to a sufficient condition for ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^n$. However, we give an example to show that this criterion fails to be a necessary condition for a set of ACM points in the general case.

We begin by adapting the construction and main result of [11] to reduced points. Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_r\}$, respectively, $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_t\}$ be the set of first, respectively second, coordinates of the points in $\mathbb{X}$. For each tuple (i,j) with $1 \leq i \leq r\,, 1 \leq j \leq t$, set $P_{i,j} = P_i \times Q_j$. Then, for each such (i,j) define

$$p_{ij} := \begin{cases} 1 & \text{if } P_{i,j} \in \mathbb{X} \\ 0 & \text{otherwise.} \end{cases}$$

The set $\mathcal{S}_{\mathbb{X}}$ is then defined to be the set of t-tuples

$$\mathcal{S}_{\mathbb{X}} := \{(p_{11}, \ldots, p_{1t}), (p_{21}, \ldots, p_{2t}), \ldots, (p_{r1}, \ldots, p_{rt})\}.$$

With this notation the main result of [11] for distinct points becomes:

Theorem 4.1 ([11, Theorem 2.1])

Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathbb{X}$ is ACM if and only if the set $\mathcal{S}_{\mathbb{X}}$ is a totally ordered set with respect to $\succeq$.

We now introduce a geometric condition on a set of points in $\mathbb{P}^n \times \mathbb{P}^m$:

Definition 4.2 Let $\mathbb{X}$ be any finite set of points in $\mathbb{P}^n \times \mathbb{P}^m$. We say that $\mathbb{X}$ satisfies **property** $(\star)$ if whenever $P_1 \times Q_1$ and $P_2 \times Q_2$ are two points in $\mathbb{X}$ with $P_1 \neq P_2$ and $Q_1 \neq Q_2$, then either $P_1 \times Q_2 \in \mathbb{X}$ or $P_2 \times Q_1 \in \mathbb{X}$ (or both) are in $\mathbb{X}$.

Theorem 4.3

Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathbb{X}$ is ACM if and only if $\mathbb{X}$ satisfies property $(\star)$.

Proof. A straightforward exercise will show that the condition $(\star)$ is equivalent to the condition that $\mathcal{S}_{\mathbb{X}}$ is totally ordered. Then apply Theorem 4.1. □

EXAMPLE 4.4 The simplest example of a nonACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ are two non-collinear points. That is, $\mathbb{X} = \{P_1 \times P_1, P_2 \times P_2\}$ where P_1, P_2 are two distinct points in $\mathbb{P}^1$. Then $\mathbb{X}$ clearly does not satisfy property $(\star)$. In this case $\mathcal{S}_{\mathbb{X}} = \{(1,0),(0,1)\}$ which is not totally ordered with respect to $\succeq$.

One direction of the above result holds more generally in $\mathbb{P}^1 \times \mathbb{P}^n$:

Theorem 4.5

Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^n$. If $\mathbb{X}$ satisfies property $(\star)$, then $\mathbb{X}$ is ACM.

Proof. Let $\mathbb{X}_P$ denote the subset of points in $\mathbb{X}$ whose first coordinate is P with $P \in \pi_1(\mathbb{X})$.

Claim. There exists a point $P \in \pi_1(\mathbb{X})$ such that $\pi_2(\mathbb{X}_P) = \pi_2(\mathbb{X})$.

Proof of Claim. We always have $\pi_2(\mathbb{X}_P) \subseteq \pi_2(\mathbb{X})$. Let P be a point of $\pi_1(\mathbb{X})$ with $|\pi_2(\mathbb{X}_P)|$ maximal. We will show that this is the desired point. Suppose there is $Q \in \pi_2(\mathbb{X})\backslash\pi_2(\mathbb{X}_P)$. So, there exists a $P \neq P' \in \pi_1(\mathbb{X})$ such that $P' \times Q \in \mathbb{X}$. Let $Q' \in \pi_2(\mathbb{X}_P)$ be any point. So $P \times Q'$ and $P' \times Q$ are points in $\mathbb{X}$. By the hypotheses, $P \times Q$ or $P' \times Q'$ are in $\mathbb{X}$. But $P \times Q \notin \mathbb{X}$ (else $Q \in \pi_2(\mathbb{X}_P)$). So, for each $Q' \in \pi_2(\mathbb{X}_P)$, $P' \times Q' \in \mathbb{X}$. But this means $|\pi_2(\mathbb{X}_{P'})| > |\pi_2(\mathbb{X}_P)|$, contradicting the maximality of $|\pi_2(\mathbb{X}_P)|$. □

We now prove the statement by induction on $|\pi_1(\mathbb{X})|$. If $|\pi_1(\mathbb{X})| = 1$, then $\mathbb{X}$ is ACM. To see this, note that $I_{\mathbb{X}} = I_P + I_{\pi_2(\mathbb{X})}$ where I_P is the defining ideal of $P \in \mathbb{P}^1$ in $R_1 = k[x_0, x_1]$, but viewed as an ideal of $R = k[x_0, x_1, y_0, \ldots, y_n]$ and $I_{\pi_2(\mathbb{X})}$ is the defining ideal of $\pi_2(\mathbb{X}) \subseteq \mathbb{P}^n$ in $R_2 = k[y_0, \ldots, y_n]$, but viewed as an idea of R. So, the resolution of $R/I_{\mathbb{X}} \cong R_1/I_P \otimes_k R_2/I_{\pi_2(\mathbb{X})}$ is obtained by tensoring together the resolutions of R_1/I_P and $R_2/I_{\pi_2(\mathbb{X})}$. From this resolution we can obtain the fact that $\mathbb{X}$ is ACM.

For the induction step, set $\mathbb{Y} = \mathbb{X}\backslash\mathbb{X}_P$, where P is the point from the claim, and thus $I_{\mathbb{X}} = I_{\mathbb{X}_P} \cap I_{\mathbb{Y}}$. Note that $\mathbb{Y}$ also satisfies $(\star)$, so by induction $\mathbb{Y}$ and $\mathbb{X}_P$ are ACM. We have a short exact sequence

$$0 \to R/I_{\mathbb{X}} \to R/I_{\mathbb{X}_P} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{Y}} + I_{\mathbb{X}_P}) \to 0.$$

By induction $R/I_{\mathbb{X}_P}$ and $R/I_{\mathbb{Y}}$ are CM of dimension 2. It suffices to show that $R/(I_{\mathbb{Y}} + I_{\mathbb{X}_P})$ is CM of dimension 1. It then follows that $R/I_{\mathbb{X}}$ is CM of dimension 2, i.e., $\mathbb{X}$ is ACM.

To prove this, we use the observation from above that $I_{\mathbb{X}_P} = I_P + I_{\pi_2(\mathbb{X}_P)}$. Now any $G \in I_{\pi_2(\mathbb{X}_P)}$ is also in $I_{\mathbb{Y}}$ since for any point $P' \times Q' \in \mathbb{Y}$, $Q' \in \pi_2(\mathbb{Y}) \subseteq \pi_2(\mathbb{X}) = \pi_2(\mathbb{X}_P)$, and hence $G(P' \times Q') = 0$. Thus

$$I_{\mathbb{Y}} + I_{\mathbb{X}_P} = I_{\mathbb{Y}} + I_P.$$

Now, by change of coordinates, we can assume $I_P = (x_0)$. Also, we can assume that x_0 does not pass through any points of $\pi_1(\mathbb{Y})$. So, x_0 is a nonzero divisor of $R/I_{\mathbb{Y}}$. To finish the proof we note that by induction, $R/I_{\mathbb{Y}}$ is ACM of dimension 2, and since x_0 is a nonzero divisor of $R/I_{\mathbb{Y}}$, we have

$$R/(I_{\mathbb{Y}} + I_{\mathbb{X}_P}) = R/(I_{\mathbb{Y}}, x_0)$$

is CM of dimension 1. The desired conclusion now holds. □

Remark 4.6 By interchanging the roles of the x_i's and y_i's in the above proof, the conclusion of the previous theorem also holds for points in $\mathbb{P}^n \times \mathbb{P}^1$.

Remark 4.7 In trying to generalize the above result to points in $\mathbb{P}^m \times \mathbb{P}^n$ we ran into the following difficulty. We still have $I_{\mathbb{Y}} + I_{\mathbb{X}_P} = I_{\mathbb{Y}} + I_P$ where $P \in \mathbb{P}^m$. By changing coordinates, we can take $I_P = (x_0, \ldots, x_{m-1})$, and we can assume that x_0 does not pass through any points of $\pi_1(\mathbb{Y})$. So, x_0 is a nonzero divisor of $R/I_{\mathbb{Y}}$. So, we know that $R/(I_{\mathbb{Y}}, x_0)$ is CM of dimension 1. However, we were left with the question of whether $R/(I_{\mathbb{Y}}, x_0, \ldots, x_{m-1})$ is also CM if $R/(I_{\mathbb{Y}}, x_0)$ is CM. Computer experimentation suggests a positive answer to this question under the hypotheses of Theorem 4.5, thus suggesting Theorem 4.5 may hold more generally for sets of points in $\mathbb{P}^m \times \mathbb{P}^n$.

Corollary 4.8

Suppose $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ and $\mathbb{X}$ is not ACM. Then there exists a pair of points $P_1 \times Q_1$ and $P_2 \times Q_2 \in \mathbb{X}$ with $P_1 \neq P_2$, $Q_1 \neq Q_2$, but $P_1 \times Q_2, P_2 \times Q_1 \notin \mathbb{X}$.

While the converse of Theorem 4.5 holds in $\mathbb{P}^1 \times \mathbb{P}^1$, it fails in general.

EXAMPLE 4.9 Let $P_i = [1 : i] \in \mathbb{P}^1$ for $i = 1, \ldots, 6$, and let $P'_1, \ldots, P'_6$ be six points in general position in $\mathbb{P}^2$. Set $Q_{i,j} = P_i \times P'_j \in \mathbb{P}^1 \times \mathbb{P}^2$. Consider the following set of 27 points:

$$\begin{aligned}\mathbb{X} \;=\; \{&Q_{1,1}, Q_{1,2}, Q_{2,1}, Q_{2,2}, Q_{2,3}, Q_{2,4}, Q_{2,5}, Q_{3,1}, Q_{3,2}, Q_{3,3}, Q_{3,4}, Q_{3,5}, Q_{4,1}, Q_{4,2},\\ &Q_{4,4}, Q_{4,5}, Q_{5,1}, Q_{5,2}, Q_{5,3}, Q_{5,4}, Q_{5,6}, Q_{6,1}, Q_{6,2}, Q_{6,3}, Q_{6,4}, Q_{6,5}, Q_{6,6}\}.\end{aligned}$$

Using `CoCoA` to compute the resolution, we find

$$0 \to R^5 \to R^{13} \to R^9 \to R \to R/I_{\mathbb{X}} \to 0.$$

So $\mathbb{X}$ is ACM since the projective dimension is 3 and $\dim R = 5$, so by the Auslander-Buchsbaum formula, $\operatorname{depth}(R/I_{\mathbb{X}}) = 2$. But $\mathbb{X}$ fails property $(\star)$ since $Q_{4,5}$ and $Q_{5,3}$ are in $\mathbb{X}$, but neither $Q_{4,3}$ nor $Q_{5,5}$ are in $\mathbb{X}$.

We end this section by describing a simple construction to make sets of points that satisfy property $(\star)$.

DEFINITION 4.10 A tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers is a **partition** of an integer s if $\sum \lambda_i = s$ and $\lambda_i \geq \lambda_{i+1}$ for every i. We write $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$. To any partition $\lambda \vdash s$ we can associate the following diagram: on an $r \times \lambda_1$ grid, place λ_1 points on the first line, λ_2 points on the second, and so on. The resulting diagram is called the **Ferrer's diagram** of λ.

Construction 4.11 Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$, and let $P_1, \ldots, P_r$ be r distinct points in $\mathbb{P}^1$ and $Q_1, \ldots, Q_{\lambda_1}$ be λ_1 distinct points in $\mathbb{P}^n$. Let $\mathbb{X}_\lambda$ then be the s points of $\mathbb{P}^1 \times \mathbb{P}^n$ where

$$\begin{aligned} \mathbb{X}_\lambda \;=\; & \{P_1 \times Q_1, P_1 \times Q_2, \ldots, P_1 \times Q_{\lambda_1}, P_2 \times Q_1, \ldots, P_2 \times Q_{\lambda_2}, \ldots \\ & P_r \times Q_1, \ldots, P_r \times Q_{\lambda_r}\}. \end{aligned}$$

The set of points $\mathbb{X}_\lambda$ then resembles a Ferrer's diagram of λ and satisfies property $(\star)$. By Theorem 4.5:

Theorem 4.12

With the notation as above, $\mathbb{X}_\lambda$ is ACM.

5. ACM sets of points and their separators

In this section we study ACM set of points using the notion of a separator. Separators for points in $\mathbb{P}^n$ were first introduced by Orecchia [17] and their properties were later studied in [1, 2, 3, 14], to name but a few references. Separators were recently defined in a multigraded setting by Marino in [14, 15] for the special case of points in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, Marino classified ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ using separators; we extend some of these ideas in this section.

We begin by introducing some notation. If $S \subseteq \mathbb{N}^r$ is a subset, then $\min S$ is the set of the minimal elements of S with respect to the partial ordering $\succeq$. For any $\underline{i} \in \mathbb{N}^r$, define $D_{\underline{i}} := \{\underline{j} \in \mathbb{N}^r \mid \underline{j} \succeq \underline{i}\}$. For any finite set $S = \{\underline{s}_1, \ldots, \underline{s}_p\} \subseteq \mathbb{N}^r$, we set

$$D_S := \bigcup_{\underline{s} \in S} D_{\underline{s}}.$$

Note that $\min D_S = S$; thus D_S can be viewed as the largest subset of $\mathbb{N}^r$ whose minimal elements are the elements of S.

DEFINITION 5.1 Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and $P \in \mathbb{X}$. We say that the multihomogeneous form $F \in R$ is a **separator for** P if $F(P) \neq 0$ and $F(Q) = 0$ for all $Q \in \mathbb{X} \setminus \{P\}$. We will call F a **minimal separator** for P if there does not exist a separator G for P with $\deg G \prec \deg F$.

DEFINITION 5.2 Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Then the **degree of a point** $P \in \mathbb{X}$ is the set

$$\begin{aligned} \deg_{\mathbb{X}}(P) \;:=\; & \min\{\deg F \mid F \text{ is a separator for } P \in \mathbb{X}\} \\ =\; & \{\deg F \mid F \text{ is a minimal separator of } P \in \mathbb{X}\}. \end{aligned}$$

Here, we are using the partial order $\succeq$ on $\mathbb{N}^r$.

If $\mathbb{X} \subseteq \mathbb{P}^n$, then $\mathbb{N}$ is a totally ordered set, so we can talk about *the* degree of a point $P \in \mathbb{X}$ (as in [1, 2, 3, 17]). In the multigraded case, however, the set $\deg_{\mathbb{X}}(P) = \{\underline{\alpha}_1, \ldots, \underline{\alpha}_s\} \subseteq \mathbb{N}^r$ may have more than one element. As we will show below, if F is a separator of P with $\deg F = \underline{\alpha}_i \in \deg_{\mathbb{X}}(P)$, then the equivalence class of F in $R/I_{\mathbb{X}}$, that is, $\overline{F}$, is unique up to scalar multiplication.

Theorem 5.3

Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, and let $P \in \mathbb{X}$ be any point. If $\mathbb{Y} = \mathbb{X}\backslash\{P\}$, then there exists a finite set $S \subseteq \mathbb{N}^r$ such that

$$H_{\mathbb{Y}}(\underline{i}) = \begin{cases} H_{\mathbb{X}}(\underline{i}) & \text{if } \underline{i} \notin D_S \\ H_{\mathbb{X}}(\underline{i}) - 1 & \text{if } \underline{i} \in D_S. \end{cases}$$

Equivalently, $\dim_k(I_{\mathbb{X}})_{\underline{i}} \leq \dim_k(I_{\mathbb{Y}})_{\underline{i}} \leq \dim_k(I_{\mathbb{X}})_{\underline{i}} + 1$ *for all* $\underline{i} \in \mathbb{N}^r$.

Proof. The second statement follows from the first since the formula implies $H_{\mathbb{X}}(\underline{i})-1 \leq H_{\mathbb{Y}}(\underline{i}) \leq H_{\mathbb{X}}(\underline{i})$ for all $\underline{i} \in \mathbb{N}^r$. To prove the first statement, the short exact sequence

$$0 \to R/(I_{\mathbb{Y}} \cap I_P) \to R/I_{\mathbb{Y}} \oplus R/I_P \to R/(I_{\mathbb{Y}} + I_P) \to 0$$

implies that

$$H_{\mathbb{Y}}(\underline{i}) = H_{\mathbb{X}}(\underline{i}) - H_P(\underline{i}) + H_{R/(I_{\mathbb{Y}}+I_P)}(\underline{i}) \ \text{ for all } \underline{i} \in \mathbb{N}^r$$

since $I_{\mathbb{Y}} \cap I_P = I_{\mathbb{X}}$.

Now $R/I_P \cong k[z_1, \ldots, z_r]$, the $\mathbb{N}^r$-graded ring with $\deg z_i = e_i$. So $H_P(\underline{i}) = 1$ for all $\underline{i} \in \mathbb{N}^r$. Also,

$$R/(I_{\mathbb{Y}} + I_P) \cong (R/I_P)/((I_{\mathbb{Y}} + I_P)/(I_P)).$$

So, $(I_{\mathbb{Y}} + I_P)/I_P \cong J$, where J is an $\mathbb{N}^r$-homogeneous ideal of $k[z_1, \ldots, z_r]$. Thus $H_{R/(I_{\mathbb{Y}}+I_P)}(\underline{i}) = 0$ or 1 for all $\underline{i} \in \mathbb{N}^r$.

When $H_{R/(I_{\mathbb{Y}}+I_P)}(\underline{i}) = 0$, then $H_{R/(I_{\mathbb{Y}}+I_P)}(\underline{j}) = 0$ for all $\underline{j} \succeq \underline{i}$. The desired set is then $S = \min \mathcal{T}$ where $\mathcal{T} = \{\underline{i} \in \mathbb{N}^r \mid H_{R/(I_{\mathbb{Y}}+I_P)}(\underline{i}) = 0\}$. □

Corollary 5.4

Suppose $\deg_{\mathbb{X}}(P) = \{\underline{\alpha}_1, \ldots, \underline{\alpha}_s\} \subseteq \mathbb{N}^r$. *If F and G are any two minimal separators of P with $\deg F = \deg G = \underline{\alpha}_i$, then $G = cF + H$ for some $0 \neq c \in k$ and $H \in (I_{\mathbb{X}})_{\underline{\alpha}_i}$. Equivalently, there exists $0 \neq c \in k$ such that $\overline{G} = \overline{cF} \in R/I_{\mathbb{X}}$.*

Proof. Suppose F and G are separators of P and $\deg F = \deg G = \underline{\alpha}$ for some $\underline{\alpha} \in \deg_{\mathbb{X}}(P)$. Suppose that $G \neq cF + H$ for any nonzero scalar $c \in k$ and any $H \in (I_{\mathbb{X}})_{\underline{\alpha}}$. Then the vector space $(I_{\mathbb{X}}, F, G)_{\underline{\alpha}} \subseteq (I_{\mathbb{Y}})_{\underline{\alpha}}$ where $\mathbb{Y} = \mathbb{X} \setminus \{P\}$. Since $F \notin (I_{\mathbb{X}})_{\underline{\alpha}}$, and since $G \notin (I_{\mathbb{X}}, F)_{\underline{\alpha}}$, we must have

$$\dim_k(I_{\mathbb{Y}})_{\underline{\alpha}} \geq \dim_k(I_{\mathbb{X}}, F, G)_{\underline{\alpha}} \geq \dim_k(I_{\mathbb{X}})_{\underline{\alpha}} + 2.$$

However, this inequality contradicts the conclusion of Theorem 5.3. □

Theorem 5.5

Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, and suppose F is a separator of a point $P \in \mathbb{X}$. Then $(I_{\mathbb{X}} : F) = I_P$.

Proof. For any $G \in I_P$, $FG \in I_{\mathbb{X}}$ since FG vanishes at all points of $\mathbb{X}$. Conversely, let $G \in (I_{\mathbb{X}} : F)$. So $GF \in I_{\mathbb{X}} \subseteq I_P$. Now $F \notin I_P$, and because I_P is a prime ideal, we have $G \in I_P$, as desired. □

Corollary 5.6

With the hypotheses as in the previous theorem,

$$\dim_k(I_{\mathbb{X}}, F)_{\underline{i}} = \dim_k(I_{\mathbb{X}})_{\underline{i}} + 1 \quad \textit{for all } \underline{i} \succeq \deg F.$$

Proof. We have a short exact sequence

$$0 \to R/(I_{\mathbb{X}} : F)(-\deg F) \xrightarrow{\times F} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}}, F) \to 0.$$

By the previous theorem $R/(I_{\mathbb{X}} : F) \cong R/I_P$. So

$$H_{R/(I_{\mathbb{X}},F)}(\underline{i}) = H_{\mathbb{X}}(\underline{i}) - H_{R/I_P}(\underline{i} - \deg F) \quad \text{for all } \underline{i} \in \mathbb{N}^r.$$

Now $H_{R/I_P}(\underline{i}) = 1$ for all $\underline{i} \in \mathbb{N}^r$, and equals 0 otherwise. The conclusion follows. □

The main theorem of this section shows that every point $P \in \mathbb{X}$ has a unique degree if $\mathbb{X}$ is ACM.

Theorem 5.7

Let $\mathbb{X}$ be any ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Then for any point $P \in \mathbb{X}$ we have $|\deg_{\mathbb{X}}(P)| = 1$.

Proof. After a change of coordinates, we can assume that $x_{1,0}, \ldots, x_{r,0}$ form a regular sequence on $R/I_{\mathbb{X}}$, and in particular, for each i, $x_{i,0}$ does not vanish at any point of $\mathbb{X}$. Suppose, for a contradiction, that $P \in \mathbb{X}$ is a point with $\deg_{\mathbb{X}}(P) = \{\underline{\alpha}_1, \ldots, \underline{\alpha}_t\}$ with $t = |\deg_{\mathbb{X}}(P)| \geq 2$. If $\{F_1, \ldots, F_t\}$ are $t \geq 2$ minimal separators of P with $\deg F_i = \underline{\alpha}_i$, we can reorder and relabel the separators so that $\deg F_i \leq_{lex} \deg F_{i+1}$ for $i = 1, \ldots, t-1$ with respect to the lexicographical order.

For ease of notation, let $F = F_1$ and $G = F_2$ be the two smallest minimal separators with respect to the lexicographical order. Suppose $\deg F = \underline{a} = (a_1, \ldots, a_r)$ and $\deg G = \underline{b} = (b_1, \ldots, b_r)$. Set $s = \min\{i \mid a_i < b_i\}$; such an s exists since $\deg F \neq \deg G$ by Corollary 5.4. Also, let $p = \min\{j \mid a_j > b_j\}$. Such a p must exist; otherwise $\deg G \succeq \deg F$, contradicting the fact that F and G are minimal separators of P.

Consider $\underline{c} = (c_1, \ldots, c_r)$ where $c_i = \max\{a_i, b_i\}$. Since $\underline{c} \succeq \underline{a}$, by Corollary 5.6 we must have that $\dim_k(I_{\mathbb{X}}, F)_{\underline{c}} = \dim_k(I_{\mathbb{X}})_{\underline{c}} + 1$. So, a basis for $(I_{\mathbb{X}}, F)_{\underline{c}}$ is given by the $\dim_k(I_{\mathbb{X}})_{\underline{c}}$ basis elements of $(I_{\mathbb{X}})_{\underline{c}}$ and any other form of degree $\underline{c}$ in $(I_{\mathbb{X}}, F)_{\underline{c}} \backslash (I_{\mathbb{X}})_{\underline{c}}$. One such form is

$$x_{1,0}^{c_1 - a_1} \cdots x_{r,0}^{c_r - a_r} F = x_{s,0}^{c_s - a_s} \cdots x_{r,0}^{c_r - a_r} F.$$

Recall, we are assuming that $x_{i,0}$s form a regular sequence on $R/I_{\mathbb{X}}$, so none of the $x_{i,0}$'s vanish at any of the points. As well, $c_i = a_i = b_i$ for $i = 1, \ldots, s-1$.

A similar argument implies that $\dim_k(I_{\mathbb{X}}, G)_{\underline{c}} = \dim_k(I_{\mathbb{X}})_{\underline{c}} + 1$, so a basis for $(I_{\mathbb{X}}, G)_{\underline{c}}$ is given by the $\dim_k(I_{\mathbb{X}})_{\underline{c}}$ basis elements of $(I_{\mathbb{X}})_{\underline{c}}$ and $x_{1,0}^{c_1 - b_1} \cdots x_{r,0}^{c_r - b_r} G = x_{p,0}^{c_p - b_p} \cdots x_{r,0}^{c_r - b_r} G$ (because $c_i = b_i$ for $i = 1, \ldots, p-1$).

Since $\underline{c} \succeq \deg G$, and $\underline{c} \succeq \deg F$ we have

$$(I_{\mathbb{X}}, F)_{\underline{c}} \subseteq (I_{\mathbb{Y}})_{\underline{c}} \quad \text{and} \quad (I_{\mathbb{X}}, G)_{\underline{c}} \subseteq (I_{\mathbb{Y}})_{\underline{c}}.$$

But $\dim_k(I_{\mathbb{X}}, F)_{\underline{c}} = \dim_k(I_{\mathbb{X}})_{\underline{c}} + 1$, and since $\dim_k(I_{\mathbb{Y}})_{\underline{c}} \leq \dim_k(I_{\mathbb{X}})_{\underline{c}} + 1$, we must have $(I_{\mathbb{X}}, F)_{\underline{c}} = (I_{\mathbb{Y}})_{\underline{c}}$. A similar argument implies that $(I_{\mathbb{X}}, G)_{\underline{c}} = (I_{\mathbb{Y}})_{\underline{c}}$. Hence,

$$(I_{\mathbb{X}}, F)_{\underline{c}} = (I_{\mathbb{X}}, G)_{\underline{c}}.$$

Because $x_{p,0}^{c_p-b_p} \cdots x_{r,0}^{c_r-b_r} G \in (I_{\mathbb{X}}, G)_{\underline{c}}$, our discussion about the basis for $(I_{\mathbb{X}}, F)_{\underline{c}}$ implies

$$x_{p,0}^{c_p-b_p} \cdots x_{r,0}^{c_r-b_r} G = H + c x_{s,0}^{c_s-a_s} \cdots x_{r,0}^{c_r-a_r} F \quad \text{with } H \in (I_{\mathbb{X}})_{\underline{c}} \text{ and } 0 \neq c \in k.$$

Note that $c \neq 0$ because if $c = 0$, then the right hand side vanishes at all the points of $\mathbb{X}$, but the left hand side does not. We thus have

$$x_{p,0}^{c_p-b_p} \cdots x_{r,0}^{c_r-b_r} G - c x_{s,0}^{c_s-a_s} \cdots x_{r,0}^{c_r-a_r} F \in (I_{\mathbb{X}}).$$

Because $x_{1,0}, \ldots, x_{r,0}$ form a regular sequence on $R/I_{\mathbb{X}}$ and since $\mathbb{X}$ is ACM, any permutation of these variables is again a regular sequence on $R/I_{\mathbb{X}}$. So, we can assume there is a regular sequence whose first two elements are $x_{s,0}$ and $x_{p,0}$. So, $x_{p,0}^{c_p-b_p} \cdots x_{r,0}^{c_r-b_r} G \in (I_{\mathbb{X}}, x_{s,0})$, and since $x_{p,0}$ is regular on $R/(I_{\mathbb{X}}, x_{s,0})$ we have $x_{p+1,0}^{c_{p+1}-b_{p+1}} \cdots x_{r,0}^{c_r-b_r} G \in (I_{\mathbb{X}}, x_{s,0})$. Thus

$$x_{p+1,0}^{c_{p+1}-b_{p+1}} \cdots x_{r,0}^{c_r-b_r} G = H_1 + H_2 x_{s,0} \quad \text{with } H_1 \in I_{\mathbb{X}} \text{ and } H_2 \in R.$$

If $Q \in \mathbb{X} \backslash \{P\}$, then $G(Q) = 0$ implies that $H_2(Q) = 0$ since $H_1(Q) = 0$ and $x_{s,0}(Q) \neq 0$. On the other hand, if we evaluate both sides at P we have

$$0 \neq x_{p+1,0}^{c_{p+1}-b_{p+1}} \cdots x_{r,0}^{c_r-b_r}(P) G(P) = H_1(P) + x_{s,0}(P) H_2(P) = x_{s,0}(P) H_2(P).$$

But because $x_{s,0}(P) \neq 0$, this forces $H_2(P) \neq 0$. So, H_2 is a separator of P with $\deg H_2 = (b_1, \ldots, b_s - 1, \ldots, b_p, c_{p+1}, \ldots, c_r)$. Let F_ℓ be a minimal separator with $\deg F_\ell \preceq \deg H_2$. But then $\deg F_\ell \leq_{\text{lex}} \deg G = (b_1, \ldots, b_s, \ldots, b_r)$. But any minimal separator whose degree is smaller than G with respect to lex must have the same degree as F_1, i.e., $\deg F_1 = \deg F_\ell$. So, $\deg F_1 \preceq \deg H_2$. But this contradicts the fact that $a_p > b_p$ and hence $\deg F_1 \not\preceq \deg H_2$. This gives the desired contradiction. □

Remark 5.8 If $\mathbb{X}$ is a finite set of ACM points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, then by the above theorem we know that $|\deg_{\mathbb{X}}(P)| = 1$ for any $P \in \mathbb{X}$. So we can talk about *the* degree of a point in this situation.

In the forthcoming paper of Marino [16], it is shown that the converse of the above statement holds in $\mathbb{P}^1 \times \mathbb{P}^1$, thus giving a new characterization of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We record the precise statement here:

Theorem 5.9

Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathbb{X}$ is ACM if and only if $|\deg_{\mathbb{X}}(P)| = 1$ for all $P \in \mathbb{X}$.

However, the converse of Theorem 5.7 fails to hold in general as shown below.

EXAMPLE 5.10 Let $P_1, \ldots, P_6$ be six points in general position in $\mathbb{P}^2$. If $Q_{i,j} = P_i \times P_j \in \mathbb{P}^2 \times \mathbb{P}^2$, then let $\mathbb{X}$ be the set of points

$$\begin{aligned}\mathbb{X} &= \{Q_{1,1}, Q_{1,2}, Q_{2,1}, Q_{2,2}, Q_{3,1}, Q_{3,2}, Q_{4,1}, Q_{4,2}, Q_{5,2}, Q_{5,3},\\ &\quad Q_{5,4}, Q_{5,5}, Q_{5,6}, Q_{6,1}, Q_{6,3}, Q_{6,4}, Q_{6,5}, Q_{6,6}\}.\end{aligned}$$

Using CoCoA we found that the resolution of $R/I_{\mathbb{X}}$ is

$$0 \to R^2 \to R^{14} \to R^{33} \to R^{34} \to R^{15} \to R \to R/I_{\mathbb{X}} \to 0$$

and thus $R/I_{\mathbb{X}}$ is not Cohen-Macaulay. The bigraded Hilbert function of $\mathbb{X}$ is

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 3 & 6 & 6 & \cdots \\ 3 & 8 & 14 & 14 & \cdots \\ 6 & 14 & 18 & 18 & \cdots \\ 6 & 14 & 18 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

If we remove the point $Q_{5,2}$ and compute the Hilbert function of $\mathbb{Y} = \mathbb{X} \setminus \{Q_{5,2}\}$ we get

$$H_{\mathbb{Y}} = \begin{bmatrix} 1 & 3 & 6 & 6 & \cdots \\ 3 & 8 & 14 & 14 & \cdots \\ 6 & 14 & 17 & 17 & \cdots \\ 6 & 14 & 17 & 17 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

From the Hilbert function, it follows that $\deg_{\mathbb{X}}(Q_{5,2}) = \{(2,2)\}$ because the Hilbert function drops by one for all $\underline{i} \succeq (2,2)$. By checking all other points in a similar fashion, we have that $\deg_{\mathbb{X}}(Q_{6,1}) = \{(2,2)\}$ and if we remove any point $Q_{i,j}$ with $i \leq 4$, then $\deg_{\mathbb{X}}(Q_{i,j}) = \{(2,1)\}$, and if we remove any point of the form $Q_{i,j}$ with $j \geq 3$, then $Q_{i,j}$ has only a minimal separator of degree $(1,2)$.

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