Tautological cycles on Jacobian varieties

GIAMBATTISTA MARINI

University of Rome “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy

E-mail: marini@mat.uniroma2.it

Received July 13, 2007. Revised November 19, 2007

ABSTRACT

In this paper we study the algebraic structure of the Tautological ring of a Jacobian: by the use of hard-Lefschetz-primitive classes we construct convenient generators that allow us to list and describe all the possible structures that may occur (the explicit list is given for $g \leq 9$ and for a few special curves).

§1. Introduction

Let $\mathcal{C}$ be a complex curve of genus g and let $J(\mathcal{C})$ denote its Jacobian. We consider the group of rational cycles modulo algebraic equivalence

$$\mathcal{A}_\bullet(J(\mathcal{C}))_{\mathbb{Q}}$$

and the so called “tautological ring” $\mathcal{R}(\mathcal{C})$, that is, the subgroup of $\mathcal{A}_\bullet(J(\mathcal{C}))_{\mathbb{Q}}$ containing $\mathcal{C}$ and stable with respect to the Fourier transform, the intersection product, the Pontryagin product, pull-backs and push-forwards of multiplication maps by integers. The tautological ring has a mysterious algebraic structure, which has not been completely understood so far. Beauville proved that $\mathcal{R}(\mathcal{C})$ is finite dimensional as a $\mathbb{Q}$ -vector space [4]. He proved that the W^d ’s and their intersections generate $\mathcal{R}(\mathcal{C})$ as a vector space (this gives a rough bound for its dimension). Polishchuk found a set of equivalence relations between tautological cycles and also show compatibility with the action of the Fourier transform [19], Colombo-van Geemen [7] found relations for curves that are a m -cover of $\mathbb{P}^1$, Herbaut [12] and Kouvidakis-Van Der Geer [16] found relations for curves carrying a g_d^r . The difficulty of understanding the structure of $\mathcal{R}(\mathcal{C})$ comes from the fact that for a Jacobian, the cycles occurring in nature, such

---

Keywords: Tautological ring, Chow group, algebraic cycles, Jacobian varieties.

MSC2000: 14C25, 14C15, 14H40, 14K12.

as $\mathcal{C}$ itself or the W^d 's (i.e. the varieties parameterizing effective d -degrees line bundles on $\mathcal{C}$, or, equivalently, the products $\mathcal{C}^{*d}/d!$) are not pure with respect to Beauville's graduation [3]. From the point of view of Fourier duality, the products of components of $\mathcal{C}$, though they are pure cycles with respect to both graduations, do not have a nice behavior under the intersection with the Theta divisor. This creates trouble when trying to describe the tautological ring of the curve $\mathcal{C}$: the matrices representing the intersection product and the Pontryagin product look completely chaotic if one works with such cycles, and it is difficult to even determine an exhaustive set of relations among them.

In this paper we study the algebraic structure of $\mathcal{R}(\mathcal{C})$: by the use of hard-Lefschetz-primitive classes we construct convenient generators that allow us to “control” all the possible structures of the tautological ring (in particular, determining the matrices representing the Fourier transform, both intersection and Pontryagin products explicitly and the admissible dimensions). We also define algebraic models for $\mathcal{R}(\mathcal{C})$. The explicit list of all the possibilities that may occur is given for $g \leq 9$ and for a few special curves.

In the next section, we recall some basic results and fix the notation. Section 3 is devoted to technical results concerning the Fourier transform of a primitive pure cycle. In Section 4, we construct cycles for the tautological ring of a Jacobian, which have a nice behavior in some sense (see Lemma 10, Corollaries 13 and 15). Section 5 is specular to Section 4: we introduce abstract algebraic models for $\mathcal{R}(\mathcal{C})$. In Section 6, we apply our machinery to give an exhaustive description of $\mathcal{R}(\mathcal{C})$ for all the possibilities that may occur in the cases where $g \leq 9$. We also consider the cases where $\mathcal{C}$ (a curve of any genus) has at least a g_4^1 .

We would like to thank the referee for pointing out the connection between the cycles λ_I 's of our construction and Lefschetz theory.

§2. Preliminary Lemmas & Definitions

Throughout this section, (X, Θ) denotes a principally polarized abelian variety (p.p.a.v.) of dimension g , $\hat{X} = \text{Pic}^0(X)$ denotes the dual abelian variety of X and $\mathcal{A}_\bullet(X)_\mathbb{Q}$ denotes the group of rational algebraic cycles modulo algebraic equivalence, graded by dimension. We also set

$$\mathcal{A}^{g-d}(X)_\mathbb{Q} = \mathcal{A}_d(X)_\mathbb{Q} = \text{“subgroup of } d\text{-dimensional cycles”}.$$

The group $\mathcal{A}_\bullet(X)_\mathbb{Q}$ has two ring structures: the first one is given by the intersection product [10], the second one is given by the Pontryagin product “*”, that we recall to be defined by $\alpha * \beta = \mathcal{S}_*(\alpha \times \beta)$, where $\alpha \times \beta$ is the natural product in $X \times X$ and $\mathcal{S} : X \times X \rightarrow X$ denotes the sum map on X . The intersection product is homogeneous with respect to codimension and its unit is given by the class of X itself. The Pontryagin product is homogeneous with respect to dimension and its unit is given by $\{o\}$, where o denotes the origin of X and $\{o\}$ denotes the class of a point in $\mathcal{A}_0(X)_\mathbb{Q}$. Besides the two products mentioned above, the group $\mathcal{A}_\bullet(X)_\mathbb{Q}$ has a remarkable automorphism, namely the Fourier transform, and it has a second

graduation defined in terms of pull-backs $\text{mult}(m)^*$. Here, $\text{mult}(m)$ denotes the endomorphism of X given by the multiplication map by the integer m . The d -dimensional cycles of degree s are defined by

$$\left[\mathcal{A}_d(X)_{\mathbb{Q}}\right]_s := \left\{ W \in \mathcal{A}_d(X)_{\mathbb{Q}} \mid \text{mult}(m)^*W = m^{2g-2d-s}W, \forall m \in \mathbb{Z} \right\}$$

(we use the same notation from [3] and shall refer to this graduation as Beauville's graduation). It is worth noticing that this graduation can also be defined in terms of push-forwards $\text{mult}(m)_*$: from [3], we know that $\text{mult}(m)_*W = m^{2d+s}W$, for all $m \in \mathbb{Z}$, if and only if $\text{mult}(m)^*W = m^{2g-2d-s}W$, for all $m \in \mathbb{Z}$. The effect on Chow groups of such pull-backs and push-forwards has been studied by Beauville in [2, 3], Dehninger and Murre in [8] and Künnemann in [14]. Dimensional graduation and Beauville's graduation give rise to a direct sum decomposition which reduces to

$$\mathcal{A}^p(X)_{\mathbb{Q}} = \bigoplus_{s=p-g}^{\max\{0, p-1\}} \left[\mathcal{A}^p(X)_{\mathbb{Q}}\right]_s, \quad (1.1)$$

if we take into account the fact that we are working modulo algebraic equivalence (see [3], main theorem and proposition (4.a)). We shall use the fact that, in particular,

$$\left[\mathcal{A}^p(X)_{\mathbb{Q}}\right]_s = 0, \quad s \geq \max\{p, 1\}. \quad (1.2)$$

Clearly,

$$\begin{aligned} \mathcal{A}^0(X)_{\mathbb{Q}} &= \left[\mathcal{A}^0(X)_{\mathbb{Q}}\right]_0 = \mathbb{Q}X, \quad \mathcal{A}^1(X)_{\mathbb{Q}} = \left[\mathcal{A}^1(X)_{\mathbb{Q}}\right]_0 \cong \frac{\text{Pic}(X)}{\text{Pic}^0(X)}, \\ \mathcal{A}_0(X)_{\mathbb{Q}} &= \left[\mathcal{A}_0(X)_{\mathbb{Q}}\right]_0 = \mathbb{Q}\{o\}. \end{aligned}$$

We now recall briefly the definition and the main features of the Fourier transform. Let $\mathcal{P}$ denote the normalized Poincaré divisor on $X \times \widehat{X}$. The Fourier transform is the map

$$\begin{aligned} \mathcal{F} : \mathcal{A}_{\bullet}(X)_{\mathbb{Q}} &\longrightarrow \mathcal{A}_{\bullet}(\widehat{X})_{\mathbb{Q}}; \\ W &\mapsto e^{\mathcal{P}}(W) \end{aligned}$$

where $e^{\mathcal{P}} = X \times \widehat{X} + \mathcal{P} + \mathcal{P}^2/2 + \mathcal{P}^3/3! + \dots$ is the exponential of $\mathcal{P}$ under the intersection product and $e^{\mathcal{P}}$ acts on W as a correspondence. We identify $\mathcal{A}_{\bullet}(\widehat{X})_{\mathbb{Q}}$ with $\mathcal{A}_{\bullet}(X)_{\mathbb{Q}}$ via the natural identification of $\widehat{X}$ with X , induced by the principal polarization Θ (this gives $\mathcal{P} = \Theta \times X + X \times \Theta - \mathcal{S}^*\Theta \in \mathcal{A}^1(X \times X)_{\mathbb{Q}}$). Under this identification, we consider the Fourier transform $\mathcal{F}$ as a transform on $\mathcal{A}_{\bullet}(X)_{\mathbb{Q}}$. The main features of Fourier transform are the following (see [18, 2, 3, 4]):

$$\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha)_{\bullet} \mathcal{F}(\beta) \quad \text{and} \quad \mathcal{F}(\alpha_{\bullet} \beta) = (-1)^g \mathcal{F}(\alpha) * \mathcal{F}(\beta); \quad (2.1)$$

$$\mathcal{F} \circ \mathcal{F} = (-1)^g \text{mult}(-1)^*, \quad \text{in particular } \mathcal{F} \text{ is bijective}; \quad (2.2)$$

$$\mathcal{F} \left[\mathcal{A}_d(X)_{\mathbb{Q}} \right]_s = \left[\mathcal{A}^{d+s}(X)_{\mathbb{Q}} \right]_s; \quad (2.3)$$

$$\mathcal{F}(W) = e^{\Theta} \cdot \left[(W^- \cdot e^{\Theta}) * e^{-\Theta} \right], \quad \text{where } W^- := \text{mult}(-1)^*W. \quad (2.4)$$

Furthermore, both the intersection product and the Pontryagin product are homogeneous with respect to Beauville's graduation.

Notation 3 Let $\alpha \in \mathcal{A}_\bullet(X)_\mathbb{Q}$ be a cycle. We set

$$\alpha^k = \alpha^{*k} = 0, \quad \forall k < 0; \quad \alpha^0 = X, \quad \alpha^{*0} = \{o\}, \quad \forall \alpha \neq 0.$$

We also use the standard notation (defined for all $n \in \mathbb{Z}$, cf. [1]):

$$\binom{n}{h} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-h+1)}{h!}, \quad h > 0; \quad \binom{n}{0} = 1; \quad \binom{n}{h} = 0, \quad h < 0$$

(and $k! = 1, \forall k \leq 0$). Furthermore, throughout the paper we keep the notation

$$\Gamma := \frac{\Theta^{g-1}}{(g-1)!}, \quad \Gamma^{[*m]} := \frac{\Gamma^{*m}}{m!} \quad \text{and} \quad \Theta^{[k]} := \frac{\Theta^k}{k!},$$

thus $\Gamma^{[*m]} = \Theta^{[g-m]}$ (note that Γ and its $*$ -powers, likewise the theta divisor, lives in 0-Beauville's degree).

Remarkable well known identities are $\mathcal{F}(\Theta) = (-1)^{g-1}\Gamma$, $\mathcal{F}(\Gamma) = -\Theta$ and, more in general,

$$\mathcal{F}(\Gamma^{[*m]}) = (-1)^m \Gamma^{[*g-m]}, \quad \Gamma^{[*m]} \bullet \Theta^{[h]} = \binom{g-m+h}{h} \Gamma^{[*m-h]}. \quad (4)$$

We may also note that the identities above hold for all m and h in $\mathbb{Z}$.

§3. On the Fourier transform

This section deals with remarkable properties of primitive pure cycles, i.e. cycles having a trivial intersection with the theta divisor and that are pure dimensionally and with respect to Beauville's degree. As we already mentioned in the introduction, the structure of $\mathcal{R}(\mathcal{C})$ looks chaotic, for example it is not a priori clear how to classify all possible structures that may occur. A control on such structures turns out to be possible by using primitive classes and the spaces they generate under the Λ_Γ -operator from Lefschetz theory: for $Y \in [\mathcal{A}_d(X)_\mathbb{Q}]_s$ satisfying $Y \bullet \Theta = 0$ the $\mathbb{Q}$ -vector space with basis $\{Y * \Gamma^{*m}\}_{m=1}^{g-s-2d}$ (where Γ is as in Notation 3), is $\mathcal{F}$ -stable and turns out to be essential to our description of the tautological ring $\mathcal{R}(\mathcal{C})$.

Throughout this section (X, Θ) denotes a p.p.a.v. of dimension $g \geq 1$. By the hard Lefschetz theorem for Chow motives of abelian varieties ([15]) we have isomorphisms

$$L^{g-i} : R^i(X) \longrightarrow R^{2g-i}(X)(g-i),$$

where $i \leq g$, $R(X)$ is the Chow motive of X and L is the Lefschetz operator associated with the principal polarization Θ . These yield isomorphisms on associated cycles groups:

$$L^{g-i} : \text{Image } \pi_i|_{CH_d(X)_\mathbb{Q}} \cong \text{Image } \pi_{2g-i}|_{CH_{d+i-g}(X)_\mathbb{Q}}$$

where the π_j are the projectors associated to the decomposition of the diagonal

$$\Delta \subset X \times X, \text{ i.e. } \pi_j = \frac{1}{(2g-j)!} [\log(\Delta)]_{\text{rel}}^{*2g-j} \in [CH_g(X \times X)_{\mathbb{Q}}]_0$$

(the Chow group of cycles modulo rational equivalence), where $_{\text{rel}}^*$ is the relative Pontryagin product with respect to the projection on the first factor of $X \times X$ and the logarithm is taken with respect to such product (see [14]). We recall that π_j projects the Chow ring $CH_{\bullet}(X)_{\mathbb{Q}}$ to $\sum_d [CH_d(X)_{\mathbb{Q}}]_{2g-2d-j}$. As a consequence, for a non-trivial primitive class $Y \in [CH_d(X)_{\mathbb{Q}}]_s$ we must have $s + 2d \leq g$ (for otherwise we would have the contradiction $L^{g-i}(Y) \neq 0$ under the choice $g-i = s+2d-g$) and, denoting by Λ_{Γ} the Lambda operator from Lefschetz theory ([15, Definition 12], for $\alpha = \Gamma$), we must have $\Lambda_{\Gamma}^m(Y) \neq 0$, $\forall m \in [0, g-s-2d]$. Furthermore, $\mathcal{F}(\Lambda_{\Gamma}^m(Y)) = (-1)^m \Lambda_{\Gamma}^{g-s-2d-m}(Y)$ and $L^h \circ \Lambda_{\Gamma}^m(Y) = \binom{g-s-2d-m+h}{h} \Lambda_{\Gamma}^{m-h}(Y)$. As the operators L and Λ_{Γ} correspond respectively to the intersection with the theta divisor and the Pontryagin product with Γ , we have the following.

Lemma 5

Let (X, Θ) be a p.p.a.v. of dimension $g \geq 1$ (note that it need not to be a Jacobian). Consider $Y \in [\mathcal{A}_d(X)_{\mathbb{Q}}]_s$ and assume Y is a non-trivial primitive class, i.e. $Y \neq 0$ and $Y \bullet \Theta = 0$. Then, for all $h, m \in \mathbb{Z}$ the following hold

$$0 \leq s+2d \leq g; \tag{5.1}$$

$$Y * \Gamma^{[*m]} \neq 0 \iff 0 \leq m \leq g-s-2d. \tag{5.2}$$

$$\mathcal{F}(Y * \Gamma^{[*m]}) = (-1)^m Y * \Gamma^{[*g-s-2d-m]}; \tag{5.3}$$

$$(Y * \Gamma^{[*m]}) \bullet \Theta^{[h]} = \binom{g-s-2d-m+h}{h} Y * \Gamma^{[*m-h]}. \tag{5.4}$$

The result we have just illustrated can also be proved by a very elementary argument (namely without disturbing motives and hard lefschezt theory for Chow motives). The end of this paragraph is dedicated to such elementary proof of Lemma (5), we also give a useful formula for the Fourier transform (formula (7.1) below). First, we give the following definition: for $W \in [\mathcal{A}_d(X)_{\mathbb{Q}}]_s$ and $k, t \in \mathbb{Z}$ we define

$$\Omega_{k;t}(W) := \sum_{j=0}^d (-1)^{k+t+j} \left[(W \bullet \Theta^{[j]}) * \Gamma^{[*g-s-2d+j+t+k]} \right] \bullet \Theta^{[t]}.$$

We may note that for $W \in [\mathcal{A}_d(X)_{\mathbb{Q}}]_s$, we have $W^- := \text{mult}(-1)^* W = (-1)^s W$ and the equality (2.4), or rather its series expansion, reduces to

$$\mathcal{F}(W) = \sum_{k,t \in \mathbb{Z}} \Omega_{k;t}(W). \tag{6}$$

Lemma 7

Let (X, Θ) be a p.p.a.v. of dimension $g \geq 1$, let $W \in [\mathcal{A}_d(X)_{\mathbb{Q}}]_s$ and let

$\Omega_{k;t} = \Omega_{k;t}(W)$. Then

$$\mathcal{F}(W) = \Omega_{0;0} = \sum_{j=0}^d (-1)^j (W \bullet \Theta^{[j]}) * \Gamma^{[*g-s-2d+j]}. \quad (7.1)$$

Proof. Clearly, $\dim \Omega_{k;t} = g-d-s+k$. Thus, also in view of (6) and the fact that, by (2.3), $\mathcal{F}(W)$ is pure of dimension $g-d-s$, one has

$$\sum_{t \geq 0} \Omega_{k;t} = 0, \quad \forall k \neq 0. \quad (7.2)$$

We now prove the following,

$$\Omega_{k;t} = 0, \quad \forall k+t \geq 1. \quad (7.3)$$

First, as $\Omega_{k;t} = (\Omega_{k+t;0}) \bullet \Theta^{[t]}$, $\forall t \in \mathbb{Z}$ (both sides vanish for $t < 0$), it suffices to prove that $\Omega_{k;0} = 0$, $\forall k \geq 1$. Furthermore, as $\Omega_{k;0} \in [\mathcal{A}^{s+d-k}(X)_{\mathbb{Q}}]_s$, the cases where $k > d$ follow from (1.2) for $s > 0$, and they follow from trivial codimension reasons for $s \leq 0$. For $1 \leq k \leq d$ we proceed by descending induction on k : fix k in the range considered and assume we are done for all $k' > k$. Then, by (7.2) and the inductive hypothesis, we have $0 = \sum_{t \geq 0} \Omega_{k;t} = \sum_{t \geq 0} (\Omega_{k+t;0}) \bullet \Theta^{[t]} = \Omega_{k;0}$.

Now, we are done: $\mathcal{F}(W) = \sum_{k,t} \Omega_{k;t} = \sum_t \Omega_{0;t} = \Omega_{0;0}$, where the first equality is (6), the second one follows from (7.2) and the third one follows from (7.3). $\square$

Proof. (elementary proof of Lemma (5): First, as $Y \bullet \Theta = 0$, formulae (7.1) give $\mathcal{F}(Y) = \Omega_{0;0}(Y) = Y * \Gamma^{[*g-s-2d]}$. In particular, (5.1) holds (for otherwise the Fourier transform of Y would be trivial, contradicting the hypothesis $Y \neq 0$).

Step 1. We prove (5.4) in the special case $g-m = s+2d$. First, note that (5.4) is trivial for $h \leq 0$. We proceed by induction on h : fix $h > 0$ and assume we are done for all h' satisfying $h' < h$. From (7.3), applying (7.2) with $k = -h$ we obtain

$$0 = \sum_{t=0}^h \Omega_{-h;t}(Y) = \sum_{t=0}^h (-1)^{t-h} (Y * \Gamma^{[*g-s-2d-h+t]}) \bullet \Theta^{[t]}. \quad (5.5)$$

For $t > 0$ we can apply the inductive hypothesis to the term between brackets, i.e. we can write $Y * \Gamma^{[*g-s-2d-h+t]} = (Y * \Gamma^{[g-s-2d]}) \bullet \Theta^{[h-t]}$. Finally, from (5.5) we get

$$\begin{aligned} Y * \Gamma^{[*g-s-2d-h]} &= - \sum_{t=1}^h (-1)^t (Y * \Gamma^{[*g-s-2d-h+t]}) \bullet \Theta^{[t]} \\ &= - \sum_{t=1}^h (-1)^t (Y * \Gamma^{[g-s-2d]}) \bullet \Theta^{[h-t]} \bullet \Theta^{[t]} \\ &= - \sum_{t=1}^h (-1)^t \binom{h}{t} (Y * \Gamma^{[g-s-2d]}) \bullet \Theta^{[h]} \\ &= (Y * \Gamma^{[g-s-2d]}) \bullet \Theta^{[h]}. \end{aligned} \quad (5.6)$$

Step 2. We prove (5.3). As $\mathcal{F}(Y) = Y * \Gamma^{[*g-s-2d]}$ and $\mathcal{F}(\Gamma^{[*m]}) = (-1)^m \Theta^{[m]}$, applying (2.1) we obtain

$$\begin{aligned} \mathcal{F}(Y * \Gamma^{[*m]}) &= (-1)^m (Y * \Gamma^{[*g-s-2d]}) \cdot \Theta^{[m]} \\ &= (-1)^m Y * \Gamma^{[*g-s-2d-m]} \end{aligned} \quad (5.7)$$

where the second equality holds a priori only for $m \geq 0$ (apply (5.6) with $h = m$). This proves (5.3) for $m \geq 0$. It remains to prove the vanishing of the right hand side of (5.3) for $m < 0$. This is clear because the Fourier transform of such term is (up to a sign) the left hand side of (5.3) for $m' = g - s - 2d - m$ (which must be positive by (5.1)) and because this latter term vanishes by (5.3) itself for m' . This concludes the proof of (5.3) and it also proves (5.2): as $Y * \Gamma^{[*g-s-2d]} = \mathcal{F}(Y) \neq 0$, the product $Y * \Gamma^{*m}$ cannot vanish for m in the range $[0, g-s-2d]$.

Step 3. We conclude the proof of (5.4). The only non-trivial case is that one where $h > 0$ and $g-s-2d-m \geq 0$. Assuming these inequalities we have

$$\begin{aligned} (Y * \Gamma^{[*m]}) \cdot \Theta^{[h]} &= ((Y * \Gamma^{[*g-s-2d]}) \cdot \Theta^{[g-s-2d-m]}) \cdot \Theta^{[h]} \\ &= \binom{g-s-2d-m+h}{h} (Y * \Gamma^{[*g-s-2d]}) \cdot \Theta^{[g-s-2d-m+h]} \\ &= \binom{g-s-2d-m+h}{h} Y * \Gamma^{[*m-h]}, \end{aligned}$$

where the second equality is trivial, the first and the third equality follow from (5.6). $\square$

§4. The tautological ring $\mathcal{R}(\mathcal{C})$

In this section $\mathcal{C}$ denotes a curve of genus g and

$$\mathcal{C}_{(s)} \in [\mathcal{A}_1(X)_{\mathbb{Q}}]_s$$

denotes its s -component from Beauville's decomposition. We recall a result of Colombo-van Geemen [7] stating that, assuming $\mathcal{C}$ is a m -cover of $\mathbb{P}^1$, $\mathcal{C}_{(s)}$ vanishes for $s \geq m-1$, i.e. $\mathcal{C}_{(s)} \neq 0$ implies $0 \leq s \leq m-2$. Note that, as a consequence,

$$\mathcal{C}_{(s)} \neq 0 \quad \implies \quad 0 \leq s \leq \frac{g-1}{2}. \quad (8)$$

In fact, every curve of genus g is a m -cover of $\mathbb{P}^1$ for some $m \leq \frac{g+3}{2}$ [1]. Other remarkable vanishing for curves carrying a g_d^r are given by Herbaut [12] and Kouvidakis-Van Der Geer [16]. We may also note that $\mathcal{C}_{(0)} = \Gamma$ (cf. begin of §3). In fact, from (2.3), the cycle $\mathcal{F}(\mathcal{C}_{(0)})$ is the divisorial component of $\mathcal{F}(\mathcal{C})$ and, at the same time, the divisorial component of $\mathcal{F}(\mathcal{C})$ is $-\Theta = \mathcal{F}(\Gamma)$.

To the purpose of constructing interesting cycles (see the introduction to §3) and decomposing $\mathcal{R}(\mathcal{C})$ as the direct sum of “remarkable” subspaces (i.e. having a nice

behavior under the Fourier transform and both products), we introduce a few notations and a technical lemma.

Notation 9 We consider the family $\mathcal{J}$ of non-ordered sets of strictly positive integers $I = \{i_1, \dots, i_d\}$ and we define

$$d(I) := \text{cardinality of } I ; \quad s(I) := i_1 + \dots + i_d .$$

The empty set $\emptyset$ is also admitted. We may note that $I = \emptyset$ if and only if $d(I) = 0$, if and only if $s(I) = 0$. We shall also use the following notation: for a cycle λ_I we set

$$\lambda_I^{[m]} := \lambda_I * \Gamma^{[*m]}, \quad m \in \mathbb{Z}; \quad R_I := \left\langle \lambda_I^{[m]} \right\rangle_{m \in \mathbb{N}},$$

i.e. R_I is the vector space (over $\mathbb{Q}$) generated by the $\lambda_I^{[m]}$. Given two integers n, s and cycles λ_I defined for all $I \in \mathcal{J}$ satisfying $d(I) = n$ and $s(I) = s$, we set

$$R_{[n; s]} := \sum_{I \mid d(I)=n, s(I)=s} R_I .$$

Lemma 10

Let $\mathcal{C}$ be a curve of genus g , let (X, Θ) be its principally polarized Jacobian and let $\mathcal{R} = \mathcal{R}(\mathcal{C})$ denote its tautological ring. Then there exist cycles $\lambda_I \in [\mathcal{A}_{d(I)}(X)_{\mathbb{Q}}]_{s(I)}$, $I = \{i_1, \dots, i_d\} \in \mathcal{J}$, such that

$$\lambda_{I \bullet} \Theta = 0; \tag{10.1}$$

$$\lambda_I = \mathcal{C}_{(i_1)} * \dots * \mathcal{C}_{(i_d)}, \quad \text{modulo } \sum_{n < d(I)} R_{[n; s(I)]} \tag{10.2}$$

(in particular $\lambda_{\emptyset} = \{o\}$, the unit of $\mathcal{R}$ with respect to the Pontryagin product);

$$\sum_{n \leq t, s \in \mathbb{N}} R_{[n; s]} \supseteq \sum_{n \leq t} \mathcal{R}(\mathcal{C}) \cap \mathcal{A}_n(X)_{\mathbb{Q}}, \quad \forall t \in \mathbb{N} \tag{10.3}$$

(in particular the $\lambda_I^{[m]}$ generate $\mathcal{R}(\mathcal{C})$ as a $\mathbb{Q}$ -vector space).

Before giving the proof, we want to state a few straightforward considerations.

Remark. Consider a cycle λ_I as in Lemma (10) and let $s = s(I)$, $d = d(I)$. Assuming $\lambda_I \neq 0$, Lemma (5) applies and gives $s + 2d \leq g$, non-triviality of the $\lambda_I^{[m]}$'s for $0 \leq m \leq g - s - 2d$, and stability of the R_I 's under the Fourier transform and under the intersection with the theta divisor (also, the $\lambda_I^{[m]}$'s are stable under the Pontryagin product with Γ by definition). Furthermore, as the R_I 's have a basis of pure cycles, they are also stable with respect to pull-backs and push-forwards of multiplication maps by integers. Thus, clearly the $R_{[n; s]}$'s satisfy the same stability properties.

Remark 11 Going back to the operators L and Λ_{Γ} , namely to the intersection with Θ and the Pontryagin product with Γ , by (5.2) and (5.4) it is clear that

$$\text{Image } \Lambda_{\Gamma} = \left\langle \lambda_I^{[m]} \right\rangle_{m \geq 1}, \quad \text{Image } L = \left\langle \lambda_I^{[m]} \right\rangle_{g - s(I) - 2d(I) - m \geq 1}$$

(in particular, by (5.2) and (5.4) we find the well-known fact that L is injective on $\text{Image } \Lambda_r$ and Λ_r is injective on $\text{Image } L$).

Remark. As a consequence of (10.2), we are forced to define $\lambda_{\{s\}} = \mathcal{C}_{(s)}$, where $s \geq 1$ (recall that the λ_I 's are defined only for strictly positive I 's). Another consequence is that for $a_1, \dots, a_n \in \mathbb{N}$ satisfying $s := \sum a_i > g-2n$ we have

$$\mathcal{C}_{(a_1)} * \dots * \mathcal{C}_{(a_n)} \in \sum_{i=0}^{n-1} R_{[i, s]} \quad (12.1)$$

(we want to stress that the summation above is only up to $n-1$) indeed, assuming that all the a_i are nonzero (otherwise the claim is trivial), one has $\lambda_{\{a_1, \dots, a_n\}} = 0$ by Lemma (5), then (12.1) follows from (10.2). In dimension 1, one finds the known fact that $\mathcal{C}_{(s)} = 0$ for $s > g-2$. In dimension 2 we get (assume $i, j \neq 0$):

$$\lambda_{\{i, j\}} = 0, \quad \text{for } i + j \geq g-3; \quad (12.2)$$

$$\mathcal{C}_{(i)} * \mathcal{C}_{(j)} = \lambda_{\{i, j\}} \mod \mathbb{Q} \mathcal{C}_{(i+j)} * \mathcal{C}_{(0)}. \quad (12.3)$$

Proof. (of Lemma 10): First, we recall that $\mathcal{R}(\mathcal{C})$ is generated by the $\mathcal{C}_{(i)}$'s and their Pontryagin products ([4]). We proceed by induction on dimension. Fix $\delta \geq 0$ and assume the λ_I 's are defined for all $I \in \mathcal{J}$ satisfying $d(I) \leq \delta$. Let $\Omega := \langle \lambda_I^{[m]} \rangle_{d(I) \leq \delta, m \in \mathbb{Z}}$ and let

$$Y := \mathcal{C}_{i_1} * \dots * \mathcal{C}_{i_{\delta+1}}, \quad s := s(\{i_1, \dots, i_{\delta+1}\}).$$

According with (10.1) and (10.2), we search, in particular, for a cycle $\lambda = \lambda_{\{i_1, \dots, i_{\delta+1}\}}$ satisfying $\lambda = Y$ modulo Ω and $\lambda \cdot \Theta = 0$. Finding such a cycle λ is equivalent to proving that there exists a cycle $Z \in \Omega$ such that $Y \cdot \Theta = Z \cdot \Theta$ (define $\lambda = Y - Z$). Recall that Ω is stable under the Fourier transform and intersection with the theta divisor.

We distinguish two possibilities. First, by (2.3), assuming $g - s - \delta - 1 \leq \delta$, we have $\dim \mathcal{F}(Y) = g - s - \delta - 1 \leq \delta$, so $\mathcal{F}(Y) \in \Omega$. Thus $Y = \pm \mathcal{F}(\mathcal{F}(Y)) \in \mathcal{F}(\Omega) = \Omega$ and, defining $Z = Y$ (i.e. $\lambda = 0$), we are done.

We now consider the case when $g - s - \delta - 1 \geq \delta$. We have

$$\begin{aligned} Y \cdot \Theta &= (-1)^{g+s} \mathcal{F}(\mathcal{F}(Y \cdot \Theta)) \\ &= (-1)^{g+s} \mathcal{F} \left[\sum_{j=0}^{\delta} (-1)^j (j+1) (Y \cdot \Theta^{[j+1]}) * \Gamma^{[*g-s-2\delta+j]} \right] \\ &= \sum_{j=0}^{\delta} (j+1) \left[\mathcal{F}(Y \cdot \Theta^{[j+1]}) \right] \cdot \Theta^{[g-s-2\delta+j]} \end{aligned}$$

where the first equality follows by (2.2), for the second equality just apply Lemma (7) with $W = Y \cdot \Theta$, and the third equality follows by (2.1) and (4). Eventually, defining

$$Z := \sum_{j=0}^{\delta} \frac{j+1}{g-s-2\delta+j} \left[\mathcal{F}(Y \cdot \Theta^{[j+1]}) \right] \cdot \Theta^{[g-s-2\delta+j-1]}$$

(recall that we are under the hypothesis $g-s-2\delta \geq 1$), we have $Y_{\bullet}\Theta = Z_{\bullet}\Theta$ as required. We are left to prove that $Z \in \Omega$: from the inductive hypothesis and (10.3) we have $Y_{\bullet}\Theta^{[j+1]} \in \mathcal{R}(\mathcal{C}) \cap \mathcal{A}_{\delta-1-j}(X)_{\mathbb{Q}} \subseteq \Omega$. As Ω is stable under the Fourier transform and intersection with the theta divisor, each term from the summation which defines Z is in Ω , and we are done.

So far we have defined λ_I for $d(I) \leq \delta+1$ and such λ_I 's satisfy (10.1) and (10.2). We are left to prove (10.3) with $t = \delta+1$. This is clear. In fact,

$$\sum_{n \leq \delta+1, s \in \mathbb{N}} R_{[n;s]} \supseteq \langle \mathcal{C}_{(a_1)} * \dots * \mathcal{C}_{(a_n)} \rangle_{a_i \geq 0, n \leq \delta+1} = \sum_{n \leq \delta+1} \mathcal{R}(\mathcal{C}) \cap \mathcal{A}_n(X)_{\mathbb{Q}}$$

where equality follows by the fact that $\mathcal{R}(\mathcal{C})$ is generated by the classes $\mathcal{C}_{(i)}$. $\square$

Corollary 13

The spaces $R_{[n;s]}$ are independent.

Proof. As the spaces $[\mathcal{A}_d(X)_{\mathbb{Q}}]_s$ are independent and the $R_{[n;s]}$ are generated by homogeneous elements, it suffices to prove that does not exist a non-trivial relation $\sum \alpha_n = 0$, $\alpha_n \in R_{[n;s_0]} \cap [\mathcal{A}_{d_0}(X)_{\mathbb{Q}}]_{s_0}$, where d_0 and s_0 are fixed (we may note that $d_0 \geq n$, unless α_n is trivial). By contradiction, assume there is such a non-trivial relation and observe that each α_n must be of the form $\sum c_i \lambda_{I_i}^{[d_0-n]}$, where $d(I_i) = n$, $c_i \in \mathbb{Q}$. Then, by (5.4), for $n' = \min\{n \mid \alpha_n \neq 0\}$ we have

$$0 = \Theta^{d_0-n'} \cdot \left(\sum \alpha_n \right) = \Theta^{d_0-n'} \cdot \alpha_{n'} \neq 0.$$

This is a contradiction. $\square$

Corollary 14

Let δ and s be integers. Then, $[\mathcal{A}_{\delta}(X)_{\mathbb{Q}}]_s \cap \sum_{n < \delta} R_{[n;s]} \subseteq \text{Image } \Lambda_{\Gamma}$.

Proof. This is clear by Lemma (10) and Remark (11). $\square$

Corollary 15

The set of λ_I 's from Lemma (10) is unique.

Proof. Consider two such sets $\{\lambda_J\}$ and $\{\lambda'_J\}$ and let J satisfy $\lambda_J \neq \lambda'_J$ with $\delta := d(J)$ minimal (so that $R_{[n;s]} = R'_{[n;s]}$, $\forall n < \delta$). Then, from (10.1) and (10.2), we have $(\lambda_J - \lambda'_J)_{\bullet} \Theta = 0$ and $\lambda_J - \lambda'_J \in \sum_{n < \delta} R_{[n;s(J)]}$. This contradicts the fact that, in view of Corollary (14) and Remark (11), the intersection with the Theta divisor map is expected to be injective on $[\mathcal{A}_{\delta}(X)_{\mathbb{Q}}]_s \cap \sum_{n < \delta} R_{[n;s]}$. $\square$

Corollary 16

Let $I = \{i_1, \dots, i_d\}$. Then, $\lambda_I = 0 \Leftrightarrow \mathcal{C}_{i_1} * \dots * \mathcal{C}_{i_d} \in \sum_{n < d(I)} R_{[n;s(I)]}$. In particular,

$$\text{If } \mathcal{C}_{i_1} * \dots * \mathcal{C}_{i_d} = 0, \text{ then } \lambda_{\{i_1, \dots, i_d\}} = 0; \quad (16.1)$$

$$\text{If } \lambda_I = 0, \text{ then } \lambda_J = 0, \forall J \supseteq I. \quad (16.2)$$

Proof. This is clear by the previous corollary, by the proof of Lemma (10), and by the fact that the new set of λ 's, obtained substituting $\lambda_{\{i_1, \dots, i_d\}}$ with 0, does satisfy (10.1), (10.2) and (10.3). $\square$

Remark 17 To the purpose to inherit our cycles $\lambda_I^{[m]}$ in the setting of Lefschetz theory we may note that it is a long nonetheless straightforward computation to check that they are compatible with the projectors $p_{i,k}$ (see [15, Theorem 4.1]). In fact, $p_{i,k}$ turns out to be the projection to second factor of the direct sum decomposition

$$\mathcal{R}(\mathcal{C}) = \bigoplus_{(I,m) \notin \xi_{i,k}} \langle \lambda_I^{[m]} \rangle_{(I,m)} \oplus \bigoplus_{(I,m) \in \xi_{i,k}} \langle \lambda_I^{[m]} \rangle_{(I,m)}$$

where $\xi_{i,k} := \{(I, m) \mid i = 2g - s(I) - 2d(I) - 2m, k = g - s(I) - 2d(I) - m\}$ (we want to stress that the sum above is direct in view of Corollary (13)). Furthermore, note that

$$(I, m) \in \xi_{i,k} \quad \text{if and only if} \quad \begin{cases} \lambda_I^{[m]} \in \text{Image } \pi_i \\ k = \max \{ \kappa \mid \lambda_I^{[m]} * \Gamma^{*\kappa} \neq 0 \} \end{cases}$$

(where π_i is as in §3), equivalently, iff $\lambda_I^{[m]} \in \pi_i L^k \mathcal{F}(\text{"primitive"})$. Finally, as far it concerns Λ , the “hard Lefschetz inverse” of L ([15, formulae 15]), it is straightforward to check it acts on $\lambda_I^{[m]}$ by

$$\Lambda(\lambda_I^{[m]}) = \frac{1}{g - s - 2d - m} \lambda_I^{[m+1]}.$$

The two restrictions $\Lambda \circ L|_{\text{Image } \Lambda}$ and $L \circ \Lambda|_{\text{Image } L}$ are the identity map.

Going back to our purpose of listing all possible structures for the tautological ring, Lemma (10) and its corollaries are the main ingredient. In fact, especially in view of Corollary (13), the $\lambda_I^{[m]}$ appear to be very convenient generators to consider. To our purpose we need to discuss the matrices of the algebraic structure of our ring $\mathcal{R}(\mathcal{C})$; namely the matrices representing $\mathcal{F}$, the Pontryagin product and the intersection product, corresponding to our basis of $\lambda_I^{[m]}$'s.

The reader is advised to look at pictures drawn for the cases where $g = 6, 7, 8, 9$ at Section §6 (clearly, one can draw similar pictures for any g).

Remark 18 First, according with (5.2), non triviality of $\lambda_I^{[m]}$ depends only on I (consider m in the range from 0 to $g - s(I) - 2d(I)$). As for the matrix representing

$$\mathcal{F}, \text{ it is clear from (5.3) that it is a block matrix } \begin{pmatrix} B_1 & \ddots & 0 \\ & \ddots & \\ 0 & & B_k \end{pmatrix}, \text{ where } B_i \text{ is}$$

the matrix with $\{1, -1, \dots\}$ on the secondary diagonal (starting with +1 at the left-bottom corner) and zero elsewhere, and where each block corresponds to the elements

$\lambda_I^{[0]}, \dots, \lambda_I^{[g-s(I)-2d(I)]}$. In order to write down explicitly the matrices representing the Pontryagin product and the intersection product, it is required a further computation, at least for higher genus cases: the definition itself of the $\lambda_I^{[m]}$'s takes care of the Pontryagin product of a generator of type $\lambda_\emptyset^{[k]}$ (first column of pictures drawn at Section §6) with any other generator, while (5.4) takes care of intersection products such elements. To be very explicit, this amounts to the following

$$\begin{aligned} \lambda_\emptyset^{[k]} * \lambda_I^{[m]} &= \binom{m+k}{m} \lambda_I^{[m+k]} \\ \lambda_\emptyset^{[k]} \bullet \lambda_I^{[m]} &= \binom{2g-m-k-s-2d}{g-k} \lambda_I^{[m-g+k]}. \end{aligned} \quad (19)$$

Furthermore, in sight of the straightforward identities

$$\begin{aligned} \lambda_I^{[m]} * \lambda_J^{[h]} &= \binom{m+h}{m} \lambda_\emptyset^{[m+h]} * \lambda_I * \lambda_J \\ \lambda_I^{[t]} \bullet \lambda_J^{[k]} &= \binom{i'-t+j'-k}{i'-t} \lambda_\emptyset^{[g-i'+t-j'+k]} \bullet \lambda_I^{[i']} \bullet \lambda_J^{[j']} \end{aligned} \quad (20)$$

where $i' := g - s(I) - 2d(I)$ and $j' := g - s(J) - 2d(J)$, the products of the form $\lambda_I * \lambda_J$ and $\lambda_I^{[i']} \bullet \lambda_J^{[j']}$ determine all other products. We shall refer to these products as primitive products (to avoid long lists we shall limit ourselves to write down only primitive products when describing intersection and Pontryagin matrices). We may also note that the two equalities (20) are $\mathcal{F}$ -dual to each other (under $t = i' - m$, $k = j' - h$), in particular primitive intersection products are the $\mathcal{F}$ -dual of primitive Pontryagin products:

$$\lambda_I^{[i']} \bullet \lambda_J^{[j']} = \mathcal{F}(\lambda_I * \lambda_J). \quad (21)$$

We end this section with some calculation. Following the line of the proof of Lemma (10) we can compute explicitly the λ_I 's. To this purpose we need to know the intersection of the products $\mathcal{C}_{(i_1)} * \dots * \mathcal{C}_{(i_k)}$ with the theta divisor. Such intersections are computed in [4] and explicitly computed in [19]. In particular we have

$$(\mathcal{C}_{(i)} * \mathcal{C}_{(j)}) \bullet \Theta = - \binom{i+j+2}{i+1} \mathcal{C}_{(i+j)}, \quad i, j \geq 1 \quad (22)$$

and, for $i, j, h \geq 1$, we have

$$\begin{aligned} (\mathcal{C}_{(h)} * \mathcal{C}_{(i)} * \mathcal{C}_{(j)}) \bullet \Theta &= - \binom{i+j+2}{i+1} \mathcal{C}_{(h)} * \mathcal{C}_{(i+j)} \\ &\quad - \binom{j+h+2}{j+1} \mathcal{C}_{(i)} * \mathcal{C}_{(j+h)} \\ &\quad - \binom{h+i+2}{h+1} \mathcal{C}_{(j)} * \mathcal{C}_{(h+i)}. \end{aligned} \quad (23)$$

As already observed, $\lambda_{\{s\}} = \mathcal{C}_{(s)}$ ($s \geq 1$). As for $\lambda_{\{i,j\}}$, it vanishes for $i+j \geq g-3$ and, for arbitrary i, j , it must be equal to $\mathcal{C}_{(i)} * \mathcal{C}_{(j)} + \alpha \mathcal{C}_{(0)} * \mathcal{C}_{(i+j)}$ for some $\alpha \in \mathbb{Q}$, see (12.2) and (12.3). In order to satisfy (10.1), i.e. $\lambda_{\{i,j\}} \bullet \Theta = 0$, the coefficient

α must satisfy $(\mathcal{C}_{(i)} * \mathcal{C}_{(j)}) \bullet \Theta = -\alpha(g-i-j-2)\mathcal{C}_{(i+j)}$ (see also 5.4). This gives $\alpha = \frac{\binom{i+j+2}{i+1}}{g-i-j-2}$ (for $i+j \leq g-3$). Thus

$$\lambda_{\{i,j\}} = \begin{cases} 0, & i+j \geq g-3; \\ \mathcal{C}_{(i)} * \mathcal{C}_{(j)} + \frac{\binom{i+j+2}{i+1}}{g-i-j-2} \mathcal{C}_{(i+j)} * \mathcal{C}_{(0)}, & i+j \leq g-3. \end{cases} \quad (24)$$

In particular, for $i+j = g-3$, we have $\mathcal{C}_{(i)} * \mathcal{C}_{(j)} = -\binom{g-1}{i+1}\mathcal{C}_{(g-3)} * \mathcal{C}_{(0)}$. It should be observed that (24) can be read as a result that regards primitive products:

$$\lambda_{\{i\}} * \lambda_{\{j\}} = \lambda_{\{i,j\}} - \frac{\binom{i+j+2}{i+1}}{g-i-j-2} \lambda_{\{i+j\}}^{[1]}, \quad i+j \leq g-3 \quad (25)$$

(and such primitive product is trivial for $i+j \geq g-2$).

Corollary 26

If $\mathcal{C}_{(i)} = 0$, then $\mathcal{C}_{(s)} = 0$ for all $s \geq i$.

Proof. Let $s > i$. By formulae (22), the cycle $\mathcal{C}_{(s)}$ is a nontrivial multiple of $(\mathcal{C}_{(i)} * \mathcal{C}_{(s-i)}) \bullet \Theta$. $\square$

Similarly, by formulae (23), it is straightforward to check that for $g \geq s+5$, where $s = s(\{i, j, h\}) = i+j+h$, we must have

$$\lambda_{\{i,j,h\}} = \mathcal{C}_{(i)} * \mathcal{C}_{(j)} * \mathcal{C}_{(h)} + \alpha \lambda_{\{i,j+h\}}^{[1]} + \beta \lambda_{\{j,h+i\}}^{[1]} + \gamma \lambda_{\{h,i+j\}}^{[1]} + \tau \lambda_{\{s\}}^{[2]}, \quad (27)$$

where $\alpha = \frac{\binom{j+h+2}{j+1}}{g-s-4}$, $\beta = \frac{\binom{h+i+2}{h+1}}{g-s-4}$, $\gamma = \frac{\binom{i+j+2}{i+1}}{g-s-4}$ and $\tau = \frac{-(s+3)!}{\binom{g-s-2}{2}(i+1)!(j+1)!(h+1)!}$. We may note that $\lambda_{\{i,j,h\}}$ vanishes for $g \leq s+5$.

§5. Algebraic models for $\mathcal{R}(\mathcal{C})$

So far we have given generators for $\mathcal{R}(\mathcal{C})$ and we have collected enough pieces of information about their behavior (under intersection and Pontryagin product, the Fourier transform, pull-backs $\text{mult}(n)^*$ and push-forwards $\text{mult}(n)_*$). A priori it is not clear whether the jumble of formulas at our disposal hides some further relation (indeed, for example, we shall see that, for $g = 7$, we must have $\lambda_{\{2,1\}} = 0$). Answer to this question means to determine admissible algebraic models. This reduces to quite elementary algebra, the results are collected in this section.

DEFINITION 28 Let $g \geq 1$ and let $\mathcal{J}$ be the family from notation (9). A set $\mathcal{A}$ of multi indexes is said *g-admissible* if

$$\begin{aligned} \emptyset \in \mathcal{A} \subseteq \mathcal{J}; \quad s(I) + 2d(I) \leq g, \quad \forall I \in \mathcal{A}; \\ I \in \mathcal{A} \Rightarrow J \in \mathcal{A}, \quad \forall J \subseteq I \end{aligned}$$

We consider symbols $\lambda_I^{[m]}$. We also consider symbols $\text{mult}(n)_*$ and $\text{mult}(n)^*$ operating on the $\lambda_I^{[m]}$ as the multiplication by $n^{2d(I)+2m+s(I)}$ and $n^{2g-2d(I)-2m-s(I)}$ respectively.

Proposition 29

Consider an integer $g \geq 1$, a set $\mathcal{A}$ of g -admissible multi indexes and the vector space

$$\mathbf{V} := \bigoplus_{I \in \mathcal{A}} \bigoplus_{m=0}^{g-s(I)-2d(I)} \mathbb{Q} \lambda_I^{[m]}.$$

endowed with the two graduations “ $s(I)$ ” and “ $d(I)+m$ ”. For $I \in \mathcal{J}$ and $m \in \mathbb{Z}$ also define $\lambda_I^{[m]} = 0$ if either $I \notin \mathcal{A}$ or m is not in the range $[0, g-s(I)-2d(I)]$. Let $\lambda_I := \lambda_I^{[0]}$, consider stable (with respect to both graduations) $\mathbf{V}$ -valued commutative “primitive” products $\lambda_I * \lambda_J$ satisfying $\lambda_\emptyset * \lambda_I \equiv \lambda_I$ and define

$$\lambda_\emptyset^{[k]} * \lambda_I^{[m]} = \binom{m+k}{m} \lambda_I^{[m+k]} \quad (29.1)$$

$$\lambda_I^{[m]} * \lambda_J^{[h]} = \binom{m+h}{m} \lambda_\emptyset^{[m+h]} * (\lambda_I * \lambda_J) \quad (29.2)$$

Assuming (associativity for primitive products)

$$(\lambda_I * \lambda_J) * \lambda_K \equiv \lambda_I * (\lambda_J * \lambda_K), \quad (29.3)$$

there exist products “ $*$ ” and “ $\bullet$ ” and a endomorphism $\mathcal{F}$ satisfying

$$(2.1), (2.2), (2.3), (2.4), (5.1), (5.2), (5.3), (19), (20).$$

Furthermore, such “ $*$ ”, “ $\bullet$ ” and $\mathcal{F}$ are unique.

Proof. First, it is easy to check that (29.1) and (29.2) are compatible and that $\mathbf{V}$ is a ring under “ $*$ ”. Then, clearly one has to define the endomorphism $\mathcal{F}$ via (5.2), i.e. $\mathcal{F}(\lambda_I^{[m]}) = (-1)^m \lambda_I^{[g-s(I)-2d(I)-m]}$ and “ $\bullet$ ” via the $\mathcal{F}$ -dual of “ $*$ ”, i.e. $\lambda_I^{[m]} \bullet \lambda_J^{[h]} = (-1)^{s(I)+s(J)} \mathcal{F}[\mathcal{F}(\lambda_I^{[m]}) * \mathcal{F}(\lambda_J^{[h]})]$. Finally, is again straightforward to check that $\mathbf{V}$ is a ring under “ $\bullet$ ” and that all mentioned properties hold. Uniqueness of such “ $*$ ”, “ $\bullet$ ” and $\mathcal{F}$ is also clear. $\square$

This proposition leads to two heuristic considerations about models for $\mathcal{R}(\mathcal{C})$: for the first one, once primitive Pontryagin products are given the structure of the model is uniquely determined; the second consideration is that checking the existence of a model reduces to checking associativity for primitive products. The first non-trivial example of such verification is $(\lambda_{\{1\}} * \lambda_{\{1\}}) * \lambda_{\{2\}} = \lambda_{\{1\}} * (\lambda_{\{1\}} * \lambda_{\{2\}})$ and it occurs only for $g \geq 8$.

Remark 30 Going back to tautological cycles, we may note that further geometrical relations, such as those one given by formulae (22) and formulae (23), give rise to further restrictions for admissible models. Indeed, as such formulas force the definition of the λ_I ’s as polynomials in the $\mathcal{C}_{(i)}$ ’s, they turn out to reduce to conditions on primitive Pontryagin products. In case $g < s(I) + 2d(I)$ (so λ_I is “a priori” trivial), such formulas gives relations (almost always trivial ones):

$$\lambda_{\{2,1\}} = 0, \text{ for } g = 7; \quad (30.1)$$

$$3\lambda_{\{2,2\}} + 10\lambda_{\{3,1\}} = 0, \text{ for } g = 8; \quad (30.2)$$

$$3\lambda_{\{2,2\}} + 10\lambda_{\{3,1\}} = 0, \lambda_{\{1,4\}} = 0, \lambda_{\{2,3\}} = 0, \text{ for } g = 9. \quad (30.3)$$

Proof. On one hand, for reason of dimension and graduations, also in view of the fact that the $\lambda_I^{[m]}$'s vanish for $s(I) + 2d(I) + m > g$, we know “a priori” the following:

$$\begin{cases} \mathcal{C}_{(1)}^{*3} \in \langle \lambda_{\{3\}}^{[2]} \rangle, & \text{for } g = 7; & \mathcal{C}_{(1)}^{*2} * \mathcal{C}_{(2)} = 0, & \text{for } g = 8; \\ \mathcal{C}_{(1)}^{*4} \in \langle \lambda_{\{4\}}^{[3]} \rangle, & \mathcal{C}_{(1)}^{*2} * \mathcal{C}_{(3)} = 0 \text{ and } \mathcal{C}_{(1)} * \mathcal{C}_{(2)}^{*2} = 0, & \text{for } g = 9. \end{cases} \quad (*)$$

On the other hand, substituting formulae (25) into formulae (23) we get

$$\begin{aligned} (\mathcal{C}_{(1)}^{*3}) \cdot \Theta &= -18 \lambda_{\{2,1\}} + \frac{180}{g-5} \lambda_{\{3\}}^{[1]}, & \text{for } g \geq 6; \\ (\mathcal{C}_{(1)}^{*2} * \mathcal{C}_{(2)}) \cdot \Theta &= -6 \lambda_{\{2,2\}} - 20 \lambda_{\{1,3\}} + \frac{420}{g-6} \lambda_{\{4\}}^{[1]}, & \text{for } g \geq 7; \\ (\mathcal{C}_{(1)}^{*2} * \mathcal{C}_{(3)}) \cdot \Theta &= -30 \lambda_{\{1,4\}} - 6 \lambda_{\{2,3\}}, & \text{mod } \langle \lambda_{\{5\}}^{[1]} \rangle, & \text{for } g \geq 8; \\ (\mathcal{C}_{(1)} * \mathcal{C}_{(2)}^{*2}) \cdot \Theta &= -20 \lambda_{\{1,4\}} - 20 \lambda_{\{2,3\}}, & \text{mod } \langle \lambda_{\{5\}}^{[1]} \rangle, & \text{for } g \geq 8; \\ (\mathcal{C}_{(1)}^{*4}) \cdot \Theta &= \frac{720}{g-8} \lambda_{\{1,3\}}^{[1]} + \frac{216}{g-8} \lambda_{\{2,2\}}^{[1]}, & \text{mod } \langle \lambda_{\{1,1,2\}}, \lambda_{\{4\}}^{[2]} \rangle, & \text{for } g \geq 9. \end{aligned}$$

Thus, taking intersection with the Θ -divisor of relations (*) and comparing with the set of relations just above, also in view of Corollary (13), we obtain the vanishing at (30.1), (30.2), (30.3). $\square$

We conclude this paragraph observing that Corollary (26) can be stated as

$$\{i\} \in \mathcal{A} \Rightarrow \{j\} \in \mathcal{A}, \quad \forall j \leq i$$

(by the very definition of the model $\mathbf{V}$, clearly $\{i\} \in \mathcal{A}$ if and only if $\lambda_{\{i\}} \neq 0$).

§6. $\mathcal{R}(\mathcal{C})$ in low genus and other examples

In this section we describe the possible structures for $\mathcal{R}(\mathcal{C})$ for $g \leq 9$. We want to stress that such classification uses previous general results, thus it can be written for any genus g . To our purpose, as generators of $\mathcal{R}(\mathcal{C})$, we consider the $\lambda_I^{[m]}$. For $g \leq 5$ the situation is very simple because only the $\lambda_{\emptyset}^{[m]}$ (that are the powers of the theta divisor) and the $\lambda_{\{s\}}^{[m]}$ may occur. In fact, $\lambda_{\{i,j\}}$ may occur only for $g \geq i+j+2d \geq 6$. For the sake of completeness we also discuss this cases (anyway, for $g \leq 2$, $\mathcal{R}(\mathcal{C})$ has dimension $g+1$ and it is just generated by the powers of the theta divisor). Once for all we recall that $\Gamma = \frac{\Theta^{g-1}}{(g-1)!} = \mathcal{C}_{(0)}$ and $\lambda_{\{s\}} = \mathcal{C}_{(s)}$, $s \geq 1$. We also observe the following.

Remark 31 The matrix representing the Fourier transform $\mathcal{F}$ is as described in Remark (18). Therefore, in the pictures below (drawn for $g = 6, 7, 8, 9$), $\mathcal{F}$ acts as the vertical symmetry (up to a sign, see either (5.3) or Remark (18)). The intersection with the Theta divisor and the Pontryagin product with $\mathcal{C}_{(0)}$ correspond respectively to a step downwards and a step upwards (up to a non-zero coefficient, see (5.4) and

the definition of $\lambda_I^{[m]}$ from notation (9)). We want to stress that, in view of Corollary (13) and property (5.2), the possible structures that may occur are very a few (in the pictures drawn below, columns are independent and non-trivial columns have non-trivial elements; therefore, the elements from non-trivial columns are a basis of the tautological ring). As for the matrices representing the Pontryagin and the intersection products (again, see Remark (18)), they are governed by primitive products, (19), (20) and (21).

Case $g = 3$.

Other than the powers of the theta divisor (namely the column of $\lambda_\emptyset$), by property (8) only $\lambda_{\{1\}}$ ($= \mathcal{C}_{(1)}$) may occur, and it is nontrivial for the general Jacobian by a result of Ceresa (see [5]). Namely, this two possibilities are as follows.

- a) $\mathcal{R}(\mathcal{C}) = R_\emptyset$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 4$)
- b) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 5$).

In particular, the admissible dimensions for $\mathcal{R}(\mathcal{C})$ are 4 and 5, corresponding respectively to cases where $\mathcal{C}_{(1)} = 0$ and $\mathcal{C}_{(1)} \neq 0$.

Case $g = 4$.

Again by property (8), $\mathcal{C}_{(s)} = 0$ for $s \geq 2$ and the situation is similar to the previous one:

- a) $\mathcal{R}(\mathcal{C}) = R_\emptyset$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 5$)
- b) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 7$)

so, the admissible dimensions for $\mathcal{R}(\mathcal{C})$ are 5 and 7, corresponding respectively to cases where $\mathcal{C}_{(1)} = 0$ and $\mathcal{C}_{(1)} \neq 0$.

Case $g = 5$.

Again, by (8), we have $\mathcal{C}_{(s)} = 0$ for $s \geq 3$. In case $\mathcal{C}_{(1)} = 0$, we must have also $\mathcal{C}_{(2)} = 0$ by Corollary (26), and the tautological ring has dimension 6 (case “a” below). Furthermore, assuming $\mathcal{C}_{(1)} \neq 0$ and $\mathcal{C}_{(2)} = 0$, the tautological ring $\mathcal{R}(\mathcal{C})$ has dimension 9. In case $\mathcal{C}_{(1)} \neq 0$ and $\mathcal{C}_{(2)} \neq 0$, $\mathcal{R}(\mathcal{C})$ has dimension 11. So, the possibilities that may occur are the following:

- a) $\mathcal{R}(\mathcal{C}) = R_\emptyset$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 6$)
- b) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 9$)
- c) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 11$).

We may note that this three possibilities are characterized by the following properties, respectively: $\mathcal{C}_{(1)} = 0$ ($\Rightarrow \mathcal{C}_{(2)} = 0$); $\mathcal{C}_{(1)} \neq 0$ and $\mathcal{C}_{(2)} = 0$; $\mathcal{C}_{(1)} \neq 0$ and $\mathcal{C}_{(2)} \neq 0$.

In view of Remark (31), as to describe the structure of $\mathcal{R}(\mathcal{C})$, we are left to write down primitive Pontryagin products. Reminding that $\lambda_\emptyset * \lambda_I \equiv \lambda_I$, we are only left to compute $(\lambda_{\{1\}})^{*2}$ in case (c): in view of (25) we have $(\lambda_{\{1\}})^{*2} = -6\lambda_{\{2\}}^{[1]}$. Explicitly, in view of (21), the corresponding primitive intersection product is $(\lambda_{\{1\}}^{[2]})^2 = 6\lambda_{\{2\}}$.

Case $g = 6$.

Again by property (8), we have $\mathcal{C}_{(s)} = 0$ for $s \geq 3$. Thus, the admissible λ_I 's, namely those one satisfying $g - s(I) - 2d(I) \geq 0$, are $\lambda_\emptyset (= \{o\})$, $\lambda_{\{1\}} (= \mathcal{C}_{(1)})$ and $\lambda_{\{2\}} (= \mathcal{C}_{(2)})$, $\lambda_{\{1,1\}} (= \mathcal{C}_{(1)} * \mathcal{C}_{(1)} + 3\mathcal{C}_{(2)} * \mathcal{C}_{(0)})$ by formulae (23)). Then, a set of generators for $\mathcal{R}(\mathcal{C})$, is given by the cycles in the following picture:

Picture “g=6”

We recall that, as explained in Remark (31), the Fourier transform acts as the vertical symmetry (up to a sign, see either (5.3) or Remark (18)) and the matrices representing the Pontryagin and the intersection products are governed by primitive Pontryagin products, (19), (20) and (21). As for primitive Pontryagin products, we must have $\lambda_\emptyset * \lambda_I \equiv \lambda_I$ and the following relation (it follows by formulae (25)):

$$\lambda_{\{1\}} * \lambda_{\{1\}} = -3\lambda_{\{2\}}^{[1]} + \lambda_{\{1,1\}}.$$

All other primitive products are trivial because they would land outside of the picture (in other terms, for reason of graduation). In case $\mathcal{C}_{(1)} = 0$, we clearly have $\mathcal{C}_{(2)} = \lambda_{\{1,1\}} = 0$ (see also Corollary 26). Consequently, the possibilities that may occur (in the sense that they are algebraically admissible as explained in §5) are the following :

- a) $\mathcal{R}(\mathcal{C}) = R_\emptyset$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 7$)
- b) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 11$)
- c) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{1,1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 12$)
- d) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 14$)
- e) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 15$).

In particular, the admissible dimensions are 7, 11, 12, 14, 15.

We may note that in this case ($g = 6$) the list of all products is not so long so we can be more explicit: beside the products we wrote, all the non-trivial products are $\lambda_{\emptyset}^{[m]} * \lambda_I^{[h]} = \binom{m+h}{m} \lambda_I^{[m+h]}$, for all cases; $\lambda_{\{1\}}^{[1]} * \lambda_{\{1\}} = -6\lambda_{\{2\}}^{[2]}$, in cases (d) and (e). Furthermore,

$$\lambda_{\emptyset}^{[6-m]} \bullet \lambda_I^{[6-h]} = \binom{m+h}{m} \lambda_I^{[6-m-h]}, \quad \lambda_{\{1\}}^{[3]} * \lambda_{\{1\}}^{[3]} = 3\lambda_{\{2\}}^{[1]} + \lambda_{\{1,1\}},$$

$$\lambda_{\{1\}}^{[3]} * \lambda_{\{1\}}^{[2]} = 6\lambda_{\{2\}}^{[2]};$$

all other intersection products are trivial.

Case $g = 7$.

Again by property (8), we have $\mathcal{C}_{(s)} = 0$ for $s \geq 4$. Thus, a priori, the admissible λ_I 's, namely those one satisfying $g - s(I) - 2d(I) \geq 0$, are $\lambda_{\{\emptyset\}}, \lambda_{\{1\}}, \lambda_{\{2\}}, \lambda_{\{3\}}, \lambda_{\{1,1\}}, \lambda_{\{2,1\}}$. We have seen, vanishing (30.1), that $\lambda_{\{2,1\}} = 0$. This reduces the set of generators for $\mathcal{R}(\mathcal{C})$ to the following set of cycles:

$J(\mathcal{C})$				
Θ				
$\Gamma^{[*5]}$	$\lambda_{\{1\}}^{[4]}$			
$\Gamma^{[*4]}$	$\lambda_{\{1\}}^{[3]}$	$\lambda_{\{2\}}^{[3]}$		
$\Gamma^{[*3]}$	$\lambda_{\{1\}}^{[2]}$	$\lambda_{\{2\}}^{[2]}$	$\lambda_{\{3\}}^{[2]}$	$\lambda_{\{1,1\}}^{[1]}$
$\Gamma^{[*2]}$	$\lambda_{\{1\}}^{[1]}$	$\lambda_{\{2\}}^{[1]}$	$\lambda_{\{3\}}^{[1]}$	$\lambda_{\{1,1\}}$
Γ	$\lambda_{\{1\}}$	$\lambda_{\{2\}}$	$\lambda_{\{3\}}$	
$\{o\}$				

Picture “g=7”

In the picture above, as usual, the Fourier transform acts as the vertical symmetry (up to a sign, see either (5.3) or Remark (18)) and the matrices representing the Pontryagin and the intersection products are governed by primitive Pontryagin products, (19), (20) and (21). As for primitive Pontryagin products, we must have the following relations:

- $\lambda_{\emptyset} * \lambda_I = \lambda_I, \quad \forall I;$
- $\lambda_{\{1\}} * \lambda_{\{1\}} = -2\lambda_{\{2\}}^{[1]} + \lambda_{\{1,1\}};$
- $\lambda_{\{1\}} * \lambda_{\{2\}} = -5\lambda_{\{3\}}^{[1]};$
- $\lambda_{\{1\}} * \lambda_{\{1,1\}} = 70\lambda_{\{3\}}^{[2]}, \quad (\text{therefore, } \lambda_{\{1,1\}} = 0 \text{ implies } \lambda_{\{3\}} = 0);$
- all other primitive products are trivial.

The first one is clear, the second one and the third one follow by formulae (25), the fourth one is as in the argument at Remark (30). Triviality of those primitive

product we did not mention explicitly follows because such products must be linear combinations of cycles of type $\lambda_I^{[m]}$ satisfying $2d(I) + s(I) + m > 7$. Consequently, also in view of Corollary (26), the structures for $\mathcal{R}(\mathcal{C})$ that may occur are the following:

- a) $\mathcal{R}(\mathcal{C}) = R_\emptyset$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 8$)
- b) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 13$)
- c) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{1,1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 15$)
- d) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 17$)
- e) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 19$)
- f) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{3\}} \oplus R_{\{1,1\}}$ ($\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 22$).

In particular, the admissible dimensions are 8, 13, 15, 17, 19, 22. Furthermore, all the cases above are algebraically admissible in the sense of Section §5. We are left to prove the following claim.

Case $g = 8$.

Again by property (8), we have $\mathcal{C}_{(s)} = 0$ for $s \geq 4$. Thus, a priori, the admissible λ_I 's, namely those one satisfying $g - s(I) - 2d(I) \geq 0$, are $\lambda_{\{\emptyset\}}, \lambda_{\{1\}}, \lambda_{\{2\}}, \lambda_{\{3\}}, \lambda_{\{1,1\}}, \lambda_{\{2,1\}}, \lambda_{\{3,1\}}, \lambda_{\{2,2\}}$. We have seen, vanishing (30.2), that $3\lambda_{\{2,2\}} + 10\lambda_{\{3,1\}} = 0$. This reduces the set of generators for $\mathcal{R}(\mathcal{C})$ to the set of cycles given in the following picture:

$J(\mathcal{C})$						
Θ						
$\Gamma^{[*6]}$	$\lambda_{\{1\}}^{[5]}$					
$\Gamma^{[*5]}$	$\lambda_{\{1\}}^{[4]}$	$\lambda_{\{2\}}^{[4]}$				
$\Gamma^{[*4]}$	$\lambda_{\{1\}}^{[3]}$	$\lambda_{\{2\}}^{[3]}$	$\lambda_{\{3\}}^{[3]}$	$\lambda_{\{1,1\}}^{[2]}$		
$\Gamma^{[*3]}$	$\lambda_{\{1\}}^{[2]}$	$\lambda_{\{2\}}^{[2]}$	$\lambda_{\{3\}}^{[2]}$	$\lambda_{\{1,1\}}^{[1]}$	$\lambda_{\{2,1\}}^{[1]}$	
$\Gamma^{[*2]}$	$\lambda_{\{1\}}^{[1]}$	$\lambda_{\{2\}}^{[1]}$	$\lambda_{\{3\}}^{[1]}$	$\lambda_{\{1,1\}}$	$\lambda_{\{2,1\}}$	$\lambda_{\{3,1\}}$
Γ	$\lambda_{\{1\}}$	$\lambda_{\{2\}}$	$\lambda_{\{3\}}$			
$\{o\}$						

Picture “g=8”

In the picture above, as usual, the Fourier transform acts as the vertical symmetry (up to a sign, see either (5.3) or Remark (18)) and the matrices representing the Pontryagin and the intersection products are governed by primitive Pontryagin products, (19), (20) and (21). As for primitive Pontryagin products, we must have the following relations:

- $\lambda_\emptyset * \lambda_I = \lambda_I, \quad \forall I;$

- $\lambda_{\{1\}} * \lambda_{\{1\}} = -\frac{3}{2}\lambda_{\{2\}}^{[1]} + \lambda_{\{1,1\}} ;$
- $\lambda_{\{1\}} * \lambda_{\{2\}} = -\frac{10}{3}\lambda_{\{3\}}^{[1]} + \lambda_{\{2,1\}} ;$
- $\lambda_{\{1\}} * \lambda_{\{3\}} = \lambda_{\{3,1\}} ;$
- $\lambda_{\{2\}} * \lambda_{\{2\}} = -\frac{10}{3}\lambda_{\{3,1\}} \left(= \lambda_{\{2,2\}} \right) ;$
- $\lambda_{\{1\}} * \lambda_{\{1,1\}} = 20\lambda_{\{3\}}^{[2]} - \frac{33}{2}\lambda_{\{2,1\}}^{[1]} ;$
- all other primitive products are trivial.

Up to the fifth one, these relations follow by formulae (25), following the line of the argument at Remark (30) the sixth one is straightforward, the triviality of those primitive products we did not write explicitly follows by the fact that such products must be linear combinations of cycles of type $\lambda_I^{[m]}$ satisfying $2d(I) + s(I) + m > 8$.

Furthermore, as a consequence of these relations and also in view of Corollary (13), it is clear that the vanishing of $\lambda_{\{1,1\}}$ implies the vanishing of both $\lambda_{\{3\}}$ and $\lambda_{\{2,1\}}$, furthermore, the vanishing of $\lambda_{\{3\}}$ implies the vanishing of $\lambda_{\{3,1\}}$ (equivalently of $\lambda_{\{2,2\}}$). Eventually, also in view of Corollary (26), the structures for $\mathcal{R}(\mathcal{C})$ that may occur are the following:

- a) $\mathcal{R}(\mathcal{C}) = R_\emptyset \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 9)$
- b) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 15)$
- c) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{1,1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 18)$
- d) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 20)$
- e) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 23)$
- f) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{2,1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 25)$
- g) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{3\}} \oplus R_{\{1,1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 27)$
- h) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{3\}} \oplus R_{\{1,1\}} \oplus R_{\{2,1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 29)$
- i) $\mathcal{R}(\mathcal{C}) = R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{3\}} \oplus R_{\{1,1\}} \oplus R_{\{2,1\}} \oplus R_{\{3,1\}} \quad (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 30).$

In particular, the admissible dimensions are 9, 15, 18, 20, 23, 25, 27, 29, 30. Furthermore, all the cases above are algebraically admissible in the sense of Section §5.

Case $g = 9$.

Again, by property (8), we have $\mathcal{C}_{(s)} = 0$ for $s \geq 5$. Thus, a priori, the admissible λ_I 's, namely those one satisfying $g - s(I) - 2d(I) \geq 0$, are $\lambda_{\{\emptyset\}}, \lambda_{\{1\}}, \lambda_{\{2\}}, \lambda_{\{3\}}, \lambda_{\{4\}}, \lambda_{\{1,1\}}, \lambda_{\{2,1\}}, \lambda_{\{3,1\}}, \lambda_{\{2,2\}}, \lambda_{\{4,1\}}, \lambda_{\{3,2\}}, \lambda_{\{1,1,1\}}$. (For convenience of the reader we recall that, aside formula (24) defining the $\lambda_{\{i,j\}}$'s, we have $\lambda_{\{1,1,1\}} = \mathcal{C}_{(1)}^{*3} + 9\mathcal{C}_{(2)} * \mathcal{C}_{(1)} * \mathcal{C}_{(0)} + 15\mathcal{C}_{(3)} * \mathcal{C}_{(0)}^2$, formulas (24) and (27)). We have seen that we have $\lambda_{\{4,1\}} = \lambda_{\{3,2\}} = 0$ and $3\lambda_{\{2,2\}} + 10\lambda_{\{3,1\}} = 0$, vanishing (30.3). This reduces

the set of generators for $\mathcal{R}(\mathcal{C})$ to the set of cycles given in the following picture:

$J(\mathcal{C})$									
Θ									
$\Gamma^{[*7]}$	$\lambda_{\{1\}}^{[6]}$								
$\Gamma^{[*6]}$	$\lambda_{\{1\}}^{[5]}$	$\lambda_{\{2\}}^{[5]}$							
$\Gamma^{[*5]}$	$\lambda_{\{1\}}^{[4]}$	$\lambda_{\{2\}}^{[4]}$	$\lambda_{\{3\}}^{[4]}$		$\lambda_{\{1,1\}}^{[3]}$				
$\Gamma^{[*4]}$	$\lambda_{\{1\}}^{[3]}$	$\lambda_{\{2\}}^{[3]}$	$\lambda_{\{3\}}^{[3]}$	$\lambda_{\{4\}}^{[3]}$	$\lambda_{\{1,1\}}^{[2]}$	$\lambda_{\{2,1\}}^{[2]}$			
$\Gamma^{[*3]}$	$\lambda_{\{1\}}^{[2]}$	$\lambda_{\{2\}}^{[2]}$	$\lambda_{\{3\}}^{[2]}$	$\lambda_{\{4\}}^{[2]}$	$\lambda_{\{1,1\}}^{[1]}$	$\lambda_{\{2,1\}}^{[1]}$	$\lambda_{\{3,1\}}^{[1]}$	$\lambda_{\{1,1,1\}}$	
$\Gamma^{[*2]}$	$\lambda_{\{1\}}^{[1]}$	$\lambda_{\{2\}}^{[1]}$	$\lambda_{\{3\}}^{[1]}$	$\lambda_{\{4\}}^{[1]}$	$\lambda_{\{1,1\}}$	$\lambda_{\{2,1\}}$	$\lambda_{\{3,1\}}$		
Γ	$\lambda_{\{1\}}$	$\lambda_{\{2\}}$	$\lambda_{\{3\}}$	$\lambda_{\{4\}}$					
$\{o\}$									

Picture “g=9”

In the picture above, as usual, the Fourier transform acts as the vertical symmetry (up to a sign, see either (5.3) or Remark (18)) and the matrices representing the Pontryagin and the intersection products are governed by primitive Pontryagin products, (19), (20) and (21). As for primitive Pontryagin products, we must have the following relations:

- $\lambda_{\emptyset} * \lambda_I = \lambda_I$, $\forall I$;
- $\lambda_{\{1\}} * \lambda_{\{1\}} = -\frac{6}{5}\lambda_{\{2\}}^{[1]} + \lambda_{\{1,1\}}$;
- $\lambda_{\{1\}} * \lambda_{\{2\}} = -\frac{5}{2}\lambda_{\{3\}}^{[1]} + \lambda_{\{2,1\}}$;
- $\lambda_{\{1\}} * \lambda_{\{1,1\}} = \lambda_{\{1,1,1\}} + 9\lambda_{\{3\}}^{[2]} - \frac{39}{5}\lambda_{\{2,1\}}^{[1]}$;
- $\lambda_{\{1\}} * \lambda_{\{3\}} = -5\lambda_{\{4\}}^{[1]} + \lambda_{\{3,1\}}$;
- $\lambda_{\{1\}} * \lambda_{\{1,2\}} = 45\lambda_{\{4\}}^{[2]} + \frac{5}{2}\lambda_{\{3,1\}}^{[1]}$;
- $\lambda_{\{1\}} * \lambda_{\{1,1,1\}} = -1080 \lambda_{\{4\}}^{[3]}$;
- $\lambda_{\{2\}} * \lambda_{\{2\}} = -\frac{20}{3}\lambda_{\{4\}}^{[1]} - \frac{10}{3}\lambda_{\{3,1\}}$;
- $\lambda_{\{2\}} * \lambda_{\{1,1\}} = -4\lambda_{\{3,1\}}^{[1]} + 54\lambda_{\{4\}}^{[2]}$;
- $\lambda_{\{1,1\}} * \lambda_{\{1,1\}} = -\frac{10368}{5}\lambda_{\{4\}}^{[3]}$;
- all other primitive products are trivial.

The computation of primitive products above is very similar to the one given for lower genus: it follows by the formulae we saw before ((25), (27), (30)), and by triviality of those primitive products that must be linear combinations of cycles of type $\lambda_I^{[m]}$ satisfying $2d(I) + s(I) + m > 9$. Similarly to the previous case, as a consequence of these relations and Corollary (13), we obtain implications as follows:

$$\begin{aligned}
\lambda_{\{1,1,1\}} = 0 &\Rightarrow \lambda_{\{4\}} = 0; \\
\lambda_{\{2,1\}} = 0 &\Rightarrow \lambda_{\{4\}} = \lambda_{\{3,1\}} = 0; \\
\lambda_{\{3\}} = 0 &\Rightarrow \lambda_{\{3,1\}} = \lambda_{\{4\}} = 0; \\
\lambda_{\{1,1\}} = 0 &\Rightarrow \lambda_{\{3\}} = \lambda_{\{2,1\}} = \lambda_{\{1,1,1\}} = 0 \quad (\Rightarrow \lambda_{\{4\}} = \lambda_{\{3,1\}} = 0); \\
\lambda_{\{2\}} = 0 &\Rightarrow \lambda_{\{3\}} = \lambda_{\{2,1\}} = \lambda_{\{3,1\}} = \lambda_{\{4\}} = 0.
\end{aligned}$$

Eventually, also in view of Corollary (26), the structures for $\mathcal{R}(\mathcal{C})$ that may occur are the following:

$$\begin{aligned}
\text{a) } \mathcal{R}(\mathcal{C}) &= R_\emptyset & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 10) \\
\text{b) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 17) \\
\text{d) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 21) \\
\text{e) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{1,1\}} \oplus R_{\{1,1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 22) \\
\text{c) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 23) \\
\text{f) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 27) \\
\text{i) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{1,1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 28) \\
\text{h) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{2,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 30) \\
\text{m) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{2,1\}} \oplus R_{\{1,1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 31) \\
\text{g) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 32) \\
\text{l) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{1,1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 33) \\
\text{j) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{2,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 35) \\
\text{n) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{2,1\}} \oplus R_{\{1,1,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 36) \\
\text{k) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{2,1\}} \oplus R_{\{3,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 37) \\
\text{p) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{2,1\}} \oplus R_{\{1,1,1\}} \oplus R_{\{3,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 38) \\
\text{o) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{2,1\}} \oplus R_{\{1,1,1\}} \oplus R_{\{4\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 40) \\
\text{q) } \mathcal{R}(\mathcal{C}) &= R_\emptyset \oplus R_{\{1\}} \oplus R_{\{2\}} \oplus R_{\{1,1\}} \oplus R_{\{3\}} \oplus R_{\{2,1\}} \oplus R_{\{1,1,1\}} \oplus R_{\{4\}} \oplus R_{\{3,1\}} & (\Rightarrow \dim \mathcal{R}(\mathcal{C}) = 42).
\end{aligned}$$

In particular, the admissible dimensions are 10, 17, 21, 22, 23, 27, 28, 30, 31, 32, 33, 35, 36, 37, 38, 40, 42. Furthermore, all the cases above are algebraically admissible in the sense of Section §5.

$\mathcal{R}(\mathcal{C})$ for trigonal curves

The tautological ring of a trigonal curve was already discussed by Beauville in [4]. In terms of our models the situation is very simple: as $\mathcal{C}_{(s)} = 0$ for $s \geq 2$, there exist an integer k such that the non-trivial λ_I 's are those one having $I = \{1, \dots, 1\}$ and $d(I) \leq k$. We may note that as $g - s(I) - 2d(I) \geq 0$ we have $k \leq \frac{g}{3}$. It is also clear that primitive Pontryagin products are $\lambda_I * \lambda_J = \lambda_{I \cup J}$, I and J as above. In

particular,

$$\begin{aligned}\dim \mathcal{R}(\mathcal{C}) &= (g+1) + (g-2) + (g-5) + \dots + (g+1-3k) \\ &= (k+1) \left(g+1 - \frac{3}{2}k \right) .\end{aligned}$$

$\mathcal{R}(\mathcal{C})$ for curves with a g_4^1

In this case, $\mathcal{C}_{(s)} = 0$ for $s \geq 3$. Thus, there exist integers $k_0 \geq k_1 \geq \dots \geq k_m$ such that the non-trivial λ_I 's, namely those one having $I = \{2, \dots, 2, 1, \dots, 1\}$, where “2” is repeated h -times ($h \leq m$) and “1” is repeated up to k_h -times. As $s(I) = 2h + k_h$ and $d(I) = h + k_h$, admissibility of multi indexes reduces to the conditions

$$4h + 3k_h \leq g, \quad 0 \leq h \leq m,$$

i.e. $3k_0 \leq g$, $4 + 3k_1 \leq g$, $8 + 3k_2 \leq g$ ... etcetera. As $s(I)$ and $d(I)$ determine I , indeed “2” is repeated $s(I) - d(I)$ times and “1” is repeated $2d(I) - s(I)$ times, also in view of Corollary (13), the spaces $\left\langle \dots, \lambda_I^{[t]}, \dots \right\rangle_{t=0}^{g-s(I)-2d(I)}$ are independent, therefore

$$\dim \mathcal{R}(\mathcal{C}) = \sum_{h=0}^m (k_h + 1) \left(g + 1 - 4h - \frac{3}{2}k_h \right) .$$

Remark. It is straightforward to check that all the structures described above satisfies all Polishchuk [19] relations. On the other hand we should also admit that (for a fixed genus) the general structure does not satisfy further relations than Polishchuk ones. Thus, Polishchuk conjecture ([19, p. 879]) stating that his set of relations is complete remains open. We may note that looking at our classification it is clear that such conjecture is equivalent to state that for the general curve of genus g the following property holds: $\lambda_{\{2\}} \neq 0$ ($g = 5$); $\lambda_{\{2\}} \neq 0$ and $\lambda_{\{1,1\}} \neq 0$ ($g = 6$); $\lambda_{\{3\}} \neq 0$ ($g = 7$); $\lambda_{\{3,1\}} \neq 0$ ($g = 8$); $\lambda_{\{4\}} \neq 0$ and $\lambda_{\{3,1\}} \neq 0$ ($g = 9$). It is not known whether all of the described structures actually occur. They are known the non vanishing of $\lambda_{\{1\}}$ for the generic curve of genus $g \geq 3$ (Ceresa [5]), the non vanishing of $\lambda_{\{2\}}$ for the generic curve of genus $g \geq 11$ (Fackhruddin [9]) and the non vanishing of $\lambda_{\{k\}}$ for some plane curves of degree $d \geq k + 4$ (Ikeda [13]).

References

1. E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, *Geometry of Algebraic Curves, I*, Springer-Verlag, New York, 1985.
2. A. Beauville, Quelques remarques sur la transformation de Fourier dans l’anneau de Chow d’une variété abélienne, *Algebraic Geometry (Tokyo/Kyoto, 1982)*, 238–260, Lecture Notes in Math. **1016**, Springer, Berlin, 1983.
3. A. Beauville, Sur l’anneau de Chow d’une variété abélienne, *Math. Ann.* **273** (1986), 647–651.
4. A. Beauville, Algebraic cycles on Jacobian varieties, *Compositio Math.* **140** (2004), 683–688.

5. G. Ceresa, C is not algebraically equivalent to C^- in its Jacobian, *Ann. of Math. (2)* **117** (1983), 285–291.
6. A. Collino, Poincaré’s formulas and hyperelliptic curves, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **109** (1975), 89–101.
7. E. Colombo and B. van Geemen, Note on curves in a Jacobian, *Compositio Math.* **88** (1993), 333–353.
8. Ch. Deninger and J. Murre, Motivic decomposition of abelian schemes and the Fourier transform, *J. Reine Angew. Math.* **422** (1991), 201–219.
9. N. Fakhruddin, Algebraic cycles on generic abelian varieties, *Compositio Math.* **100** (1996), 101–119.
10. W. Fulton, *Intersection Theory*, Ergebnisse de Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1984.
11. M. Green, J. Murre, and C. Voisin, *Algebraic Cycles and Hodge Theory*, Lecture Notes in Mathematics **1594**, Springer-Verlag, Berlin, 1994.
12. F. Herbaut, *Algebraic cycles on the Jacobian of a curve with a g_d^r* , arXiv:math AG/0606140.
13. A. Ikeda, Algebraic cycles and infinitesimal invariants on Jacobian varieties, *J. Algebraic Geom.* **12** (2003), 573–603.
14. K. Künnemann, On the Chow motive of an abelian scheme, *Motives (Seattle, WA, 1991)*, 189–205, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994.
15. K. Künnemann, A Lefschetz decomposition for Chow motives of abelian schemes, *Invent. Math.* **113** (1993), 85–102.
16. A. Kouvidakis and G. Van Der Geer, *Cycles relations on Jacobian varieties*, arXiv:math AG/0608606.
17. G. Marini, Algebraic cycles on abelian varieties and their decomposition, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat (8)*, **7** (2004), 231–240.
18. S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its applications to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153–175.
19. A. Polishchuk, Universal algebraic equivalences between tautological cycles on Jacobians of curves, *Math. Z.* **251** (2005), 875–897.