Boundary observability of a numerical approximation scheme of Maxwell's system in a cube

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ABSTRACT

We consider a numerical space discretization of Maxwell's system by the so-called Yee's scheme. For such a scheme, we first show that the boundary observability estimate is not uniform with respect to the mesh size. Using a discrete multiplier method, we prove an observability estimate that separates the low and high frequency contributions. We then describe one of its consequence to control problem, namely the Tychonoff regularization technique that shows that the discrete system with a discrete boundary control with additional internal controls (tending to zero as the mesh size goes to zero) converges to the continuous system with a boundary control.

1. Introduction

Recently boundary controllability and stability results of Maxwell's equations have been obtained using different methods, like the multiplier method, the microlocal analysis, the frequency domain method or a combination of them [31, 13, 14, 16, 12, 25, 28, 29, 33, 26, 27, 4]. Usually these results are based on an observability estimate guaranteeing that the total energy of solutions can be estimated in terms of some local measurements. A similar strategy leads to the internal controllability or stability of Maxwell's equations, see for instance [28, 29, 35, 36].

More generally, important progresses have been made on the understanding of observability and controllability of (continuous) wave phenomena [18, 15, 39, 38]. On the other hand, similar results at a discrete level, namely when the continuous operator

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is replaced by a discrete one (obtained by a finite difference, a finite element or a finite volume scheme, for instance) are less developed. In their pioneer works, R. Glowinski et al. in [6, 7, 5, 8] considered the numerical approximation of the wave equation using the so-called HUM method. They observed a bad behavior of the numerical approximation of the control, due to the spurious modes that the numerical scheme introduces at high frequencies. In these papers different remedies were proposed, like a Tychonoff regularization [7, 8], a bi-grid algorithm [5] or a mixed finite element method [6], providing good numerical results. For the wave equation, rigorous convergence analyses of these cures for some finite differences schemes (in space and in time), finite element methods and mixed finite element methods are made in [10, 37, 32, 23, 39, 38, 1, 2, 22] (for the beam equation, see [17]). The main idea is that, in general, any discrete scheme generates spurious high frequency oscillations that do not exist at the continuous level. Therefore the discretization and the control processes do not commute; in other words, the controls of the discrete model do not necessarily converge to the control of the continuous model. Some remedies are then necessary to restore the convergence of the discrete control to the continuous one.

Since there exist some analogies between Maxwell's equations and the wave equation, we may expect similar phenomena for the numerical approximation schemes of Maxwell's equations too. The goal of the present paper is to answer to such questions for Maxwell's system set on the unit cube approximated by a well-known scheme, the so-called Yee's scheme. For such a scheme, we first show that the boundary observability estimate is not uniform with respect to the mesh size. In a second step, using a discrete multiplier method (a discrete version of the multiplier method of the continuous Maxwell system initially proposed by V. Komornik [14], inspired from the multiplier method for standard hyperbolic equations, like the wave equation, proposed by L. F. Ho in [9] and by J.-L. Lions in [18]), we prove an appropriate observability estimate and deduce that the Tychonoff regularization technique allows to recover the convergence of the discrete control to the continuous one.

Note that Yee's scheme for Maxwell's system is not a direct generalization of standard finite difference schemes of the wave equation (like the 7 points scheme for the Laplace operator), because it involves different nodal points for the approximation of the different components of the electric and magnetic fields. Therefore the extension of the above mentioned results obtained for the wave equation is not fully direct for Maxwell's system (even if expectable). More precisely the spectral analysis as well as the statement and proof of the multiplier identity require much more complex developments.

The rest of the paper is organized as follows: In Section 2 we describe the so-called Yee's scheme for Maxwell's equations in the unit cube and present a three-dimensional discrete differential calculus. Then in Section 3 we show that the boundary observability estimate is not uniform with respect to the mesh size. Section 4 is concerned with the discrete multiplier identity and the appropriate observability estimate. The Tychonoff regularization method is shortly described in Section 5. Some open problems are stated in Section 6 and finally the one-dimensional discrete differential calculus is presented as an appendix.

2. Discretization by Yee's scheme of Maxwell's system

Let $\Omega = (0, 1)^3$ be the unit cube of $\mathbb{R}^3$, and let Γ be its boundary.

In Ω we consider the homogeneous Maxwell's equations with a perfect conducting boundary:

$$E' - \operatorname{curl} H = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (1)$$

$$H' + \operatorname{curl} E = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (2)$$

$$\operatorname{div} E = \operatorname{div} H = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (3)$$

$$E \times \nu = 0, \quad H \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty) \quad (4)$$

$$E(0) = E_0 \quad \text{and} \quad H(0) = H_0 \quad \text{in } \Omega, \quad (5)$$

where E, H are three-dimensional vector-valued functions of t , $x = (x_1, x_2, x_3)$; which represent the electric and magnetic fields respectively and ν is the unit outward normal vector along Γ .

The energy $\mathcal{E}(t)$ of this system (1)-(5) at time t is defined by

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} (|E(t)|^2 + |H(t)|^2) dx. \quad (6)$$

Figure 1: The unit cell of Yee's grid.

The (space) Yee's scheme consists in approaching the different components of E and H in different nodal even/odd points so that the divergence properties (3) hold at the discrete level. More precisely, each component of E (resp. H) is placed in the center of four circulating components of H (resp. E) as indicated by Figure 1. To be more

precise, given $N \in \mathbb{N}$ and setting $h = \frac{1}{N+1}$, the vector field H will be approximated by

$$\begin{aligned} &H_{h\kappa}(2i_1 + 1 - \delta_{\kappa,1}, 2i_2 + 1 - \delta_{\kappa,2}, 2i_3 + 1 - \delta_{\kappa,3}), \\ &i_l = 0, \dots, N - 1 + 2\delta_{\kappa,l}, \end{aligned} \quad (7)$$

for all $\kappa, l = 1, 2, 3$, where $\delta_{\kappa,l} = 1$ if $\kappa = l$ and 0 otherwise. By analogy, the vector field E will be approximated by

$$E_{h\kappa}(2i_1 + \delta_{\kappa,1}, 2i_2 + \delta_{\kappa,2}, 2i_3 + \delta_{\kappa,3}), i_l = 0, \dots, N + 1 - 2\delta_{\kappa,l}, \quad (8)$$

for all $\kappa, l = 1, 2, 3$. Here and below $H_{h\kappa}(l, m, n)$ (resp. $E_{h\kappa}(l, m, n)$) is an approximation of $H_\kappa(l\delta, m\delta, n\delta)$ (resp. $E_\kappa(l\delta, m\delta, n\delta)$), when $\delta = h/2 = \frac{1}{2(N+1)}$.

Now for the function H_h , we define $\partial_{\kappa h} H_h$ as follows: for $i = 1, 2, 3$,

$$\begin{aligned} \partial_{ih} H_{h\kappa}(l, m, n) &= \frac{1}{h} (H_{h\kappa}(l + \delta_{i,1}, m + \delta_{i,2}, n + \delta_{i,3}) \\ &\quad - H_{h\kappa}(l - \delta_{i,1}, m - \delta_{i,2}, n - \delta_{i,3})). \end{aligned} \quad (9)$$

We define similarly $\partial_{\kappa h} E_h$. Obviously if $H_{h\kappa}(l, m, n)$ is defined for even (resp. odd) indices l , then $\partial_{1h} H_{h\kappa}(l, m, n)$ is defined for odd (resp. even) indices l . With this definition, we can define the discrete curl operator:

$$\text{curl}_h H_h = \begin{pmatrix} \partial_{2h} H_{h3} - \partial_{3h} H_{h2} \\ \partial_{3h} H_{h1} - \partial_{1h} H_{h3} \\ \partial_{1h} H_{h2} - \partial_{2h} H_{h1} \end{pmatrix}. \quad (10)$$

It is easy to check that $\text{curl}_h H_h$ is defined on the same indices as E_h and reciprocally $\text{curl}_h E_h$ is defined on the same indices as H_h .

We can now describe the approximation of the continuous Maxwell system (1)-(5) by Yee's scheme: Let E_h, H_h be the solution of

$$E'_h - \text{curl}_h H_h = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (11)$$

$$H'_h + \text{curl}_h E_h = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (12)$$

$$\text{div}_h E_h = \text{div}_h H_h = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (13)$$

$$\gamma E_h \times \nu = 0, \quad \gamma H_h \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty) \quad (14)$$

$$E_h(0) = E_{h0} \quad \text{and} \quad H_h(0) = H_{h0} \quad \text{in } \Omega. \quad (15)$$

Here and in the sequel when saying that a discrete or semi-discrete system holds in a domain, it means that it holds for all the indices (l, m, n) such that $(l\delta, m\delta, n\delta)$ is in that domain.

As previously said Yee's scheme is characterized by the fact that the different components of the unknowns E_h and H_h are not defined on the same nodal points. This renders the spectral analysis and the proof of the multiplier identity of the next sections more delicate than for the wave equation approximated by standard finite difference schemes.

Let us further notice that we easily check that

$$\text{div}_h \text{curl}_h E_h = 0 \quad \text{and} \quad \text{div}_h \text{curl}_h H_h = 0 \quad \text{in } \Omega.$$

Therefore if the initial data E_{h0} and H_{h0} are discrete divergence free then the solution E_h and H_h of (11)-(15) will be discrete divergence free at all times thanks to (11) and (12).

Finally we introduce the discrete integral

$$\int_{\Omega} H_{h1} = h^3 \sum_{i=1}^N \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} H_{h1}(2i, 2j+1, 2k+1).$$

We proceed similarly for the other components according to the context. Then by Green's formula (60) we get for any $\kappa = 1, 2, 3$:

$$\int_{\Omega} H_{h1} \partial_{\kappa h} Y = - \int_{\Omega} \partial_{\kappa h} H_{h1} Y + \int_{\partial\Omega} \gamma H_{h1} \gamma Y \nu_{\kappa}. \quad (16)$$

Here Y is defined on appropriate indices according to the value of κ , γH_{h1} is the trace of H_{h1} on the boundary, namely (according to the convention used in the Appendix):

$$\begin{aligned} H_{h1}(2i, 0, 2k+1) &= H_{h1}(2i, 1, 2k+1), \\ H_{h1}(2i, 2N+2, 2k+1) &= H_{h1}(2i, 2N+1, 2k+1), \\ H_{h1}(2i, 2j+1, 0) &= H_{h1}(2i, 2j+1, 1), \\ H_{h1}(2i, 2j+1, 2N+2) &= H_{h1}(2i, 2j+1, 2N+1), \end{aligned}$$

the trace of Y being defined similarly. Finally, we set

$$\begin{aligned} \int_{\partial\Omega} X &= \sum_{m,n} (X_{0,m,n} + X_{2N+2,m,n}) \\ &+ \sum_{l,n} (X_{l,0,n} + X_{l,2N+2,n}) + \sum_{l,m} (X_{l,m,0} + X_{l,m,2N+2}). \end{aligned}$$

A direct consequence of this three-dimensional Green formula is

$$\int_{\Omega} \operatorname{curl}_h E_h \cdot H_h = \int_{\Omega} E_h \cdot \operatorname{curl}_h H_h - \int_{\partial\Omega} \gamma E_h \times \nu \cdot \gamma H_h. \quad (17)$$

The well-posedness of problem (11)-(15) is direct since by skipping the condition (13), this problem is equivalent to the ordinary differential equation

$$U'_h = A_h U_h, U_h(0) = U_{h0},$$

where A_h is a square matrix. Therefore its solution is simply given by

$$U_h(t) = e^{tA_h} U_{h0}.$$

Note that this problem is conservative because the (discrete) energy (compare with the energy (6) of the continuous system)

$$\mathcal{E}_h(t) = \frac{1}{2} \int_{\Omega} (|E_h(t)|^2 + |H_h(t)|^2) \quad (18)$$

is constant. Indeed its derivative is given by

$$\mathcal{E}'_h(t) = \int_{\Omega} (E'_h(t) \cdot E_h(t) + H'_h(t) \cdot H_h(t)).$$

By (11) and (12), we then get

$$\mathcal{E}'_h(t) = \int_{\Omega} (E_h(t) \cdot \operatorname{curl}_h H_h(t) - \operatorname{curl}_h E_h(t) \cdot H_h(t)),$$

and by (17) we find

$$\mathcal{E}'_h(t) = - \int_{\partial\Omega} \gamma E_h \times \nu \cdot \gamma H_h = 0,$$

thanks to the boundary condition satisfied by E_h . Note that this proof is exactly the discrete counterpart of the proof of the conservation of energy for the continuous problem.

The second order convergence of this scheme to the continuous problem is well-known, we refer to [34, 21].

3. The nonuniform discrete boundary observability estimate

Inspired in [37, 39] we here show that the discrete boundary observability estimate is not uniform with respect to the mesh size.

We introduce the standard multiplier $m(x) = x$ and set

$$\begin{aligned} \Gamma_0 &= \{x \in \Gamma : m(x) \cdot \nu(x) > 0\} \\ &= \{x = (x_1, x_2, x_3) \in \Gamma : x_1 = 1 \text{ or } x_2 = 1 \text{ or } x_3 = 1\}. \end{aligned}$$

The continuous boundary observability estimate is as follows [14, 15]: For $T > 0$ large enough, there exists $C(T) > 0$ such that the solution (E, H) of (1)-(5) satisfies

$$\mathcal{E}(0) \leq C(T) \int_0^T \int_{\Gamma_0} |H(x, t)|^2 d\Gamma(x) dt, \quad (19)$$

where we recall that the energy $\mathcal{E}(t)$ of system (1)-(5) at time t is defined by (6).

The discrete counterpart of the observability estimate (19) is: To find $T > 0$ and $C_h(T) > 0$ such that the solution (E_h, H_h) of (11)-(15) satisfies

$$\mathcal{E}_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt. \quad (20)$$

Inequality (20) holds for all $T > 0$ but with a constant $C_h(T)$ that blows up as h goes to zero. Let us now show that this estimate is indeed not uniform with respect to the mesh size h .

We start with a characterization of a part of the spectrum of the operator A_h and of the associated eigenvectors. First we recall the explicit expression of the spectrum of the discrete Laplace operator on $(0, 1)$ (with mesh size h) with Dirichlet boundary

conditions: a discrete function $(\beta_i)_{i=0,\dots,N+1}$ is an eigenvector of the discrete Laplace operator if it satisfies

$$-(\Delta_h \beta)_i = \lambda^2 \beta_i, i = 1, \dots, N, \quad (21)$$

$$\beta_0 = \beta_{N+1} = 0, \quad (22)$$

where $(\Delta_h \beta)_i := [\beta_{i+1} + \beta_{i-1} - 2\beta_i]/h^2$.

The eigenvalues and eigenvectors of this problem are explicitly known [11, 38]. The eigenvalues are

$$\lambda_{hk}^2 = \frac{4}{h^2} \sin^2 \left(\frac{k\pi h}{2} \right), k = 1, \dots, N,$$

and the corresponding eigenvectors are

$$\beta_{k,j} = \sin(k\pi jh), k, j = 1, \dots, N.$$

Observe that for a fixed $k \geq 1$, we have

$$\lambda_{hk}^2 \rightarrow k^2 \pi^2 \text{ as } h \rightarrow 0, \quad (23)$$

the k -th eigenvalue of the continuous Laplace operator with Dirichlet boundary condition at 0 and 1. On the other hand, the greatest eigenvalue blows up as $h \rightarrow 0$ (see [38] for more details), since

$$\lambda_{hN}^2 h^2 = 4 \cos^2 \left(\frac{\pi h}{2} \right) \rightarrow 4 \text{ as } h \rightarrow 0.$$

This means that the convergence (23) is not uniform in k . This non-uniform property and the behavior of the eigenvectors associated with high frequencies are the reason of the non-observability results for the discretization by finite differences of the 1-d wave equation. Similar phenomena appear for our Maxwell system (see below).

Lemma 3.1

Let λ^2 be an eigenvalue of the discrete Laplace operator on $(0, 1)$ and $(\beta_i)_{i=0,\dots,N+1}$ its associated eigenvector. Define an odd extension of β by setting

$$\beta_{-1} := -\beta_1, \beta_{N+2} := -\beta_N.$$

Denote by $\Phi = (\Phi_1, \Phi_2, 0)^\top$ the following vector:

$$\begin{aligned} \Phi_1(2i+1, 2j, 2k) &= \beta_{2i+1}^N \beta_{2j}^D \beta_{2k}^D, i = 0, \dots, N-1; j, k = 0, \dots, N+1, \\ \Phi_2(2i, 2j+1, 2k) &= -\beta_{2i}^D \beta_{2j+1}^N \beta_{2k}^D, j = 0, \dots, N-1; i, k = 0, \dots, N+1, \end{aligned}$$

where

$$\beta^N = \partial_h \beta^D,$$

and

$$\beta_{2i}^D = \beta_i, \forall i = 0, \dots, N+1.$$

If $\kappa = \sqrt{3}\lambda$, then the pair $(\Phi, \Psi) := (\Phi, -\frac{1}{i\kappa} \text{curl}_h \Phi)$ is an eigenvector of the operator A_h of eigenvalue $i\kappa$, i.e., it satisfies

$$(\text{curl}_h \Psi, -\text{curl}_h \Phi) = i\kappa(\Phi, \Psi). \quad (24)$$

Note further that the discrete functions Φ and Ψ are discrete divergence free, i.e., satisfy

$$\text{div}_h \Phi = \text{div}_h \Psi = 0. \quad (25)$$

Proof. We see that (24) will be satisfied if and only if

$$\text{curl}_h \text{curl}_h \Phi = \kappa^2 \Phi. \quad (26)$$

Using the following identity

$$\text{curl}_h \text{curl}_h = \nabla_h \text{div}_h - \Delta_h^{(3d)},$$

which is a discrete version of the continuous one and which is easy to check, where

$$\Delta_h^{(3d)} = \sum_{\kappa=1}^3 \partial_{\kappa h} \partial_{\kappa h},$$

we deduce that (26) is equivalent to

$$-\Delta_h^{(3d)} \Phi = \kappa^2 \Phi. \quad (27)$$

Now direct calculations yield

$$\begin{aligned} (\Delta_h^{(3d)} \Phi)_1(2i+1, 2j, 2k) &= (\partial_h \partial_h \beta^N)_{2i+1} \beta_{2j}^D \beta_{2k}^D \\ &\quad + \beta_{2i+1}^N (\partial_h \partial_h \beta^D)_{2j} \beta_{2k}^D + \beta_{2i+1}^N \beta_{2j}^D (\partial_h \partial_h \beta^D)_{2k}, \\ (\Delta_h^{(3d)} \Phi)_2(2i, 2j+1, 2k) &= -(\partial_h \partial_h \beta^N)_{2i} \beta_{2j+1}^N \beta_{2k}^D \\ &\quad - \beta_{2i}^N (\partial_h \partial_h \beta^N)_{2j+1} \beta_{2k}^D - \beta_{2i}^N \beta_{2j+1}^N (\partial_h \partial_h \beta^D)_{2k}. \end{aligned}$$

Then (27) holds if

$$-\partial_h \partial_h \beta^D = \lambda^2 \beta^D \quad \text{in } (0, 1), \quad (28)$$

$$-\partial_h \partial_h \beta^N = \lambda^2 \beta^N \quad \text{in } (0, 1). \quad (29)$$

The first identity is direct since

$$(\partial_h \partial_h \beta^D)_{2j} = \Delta_h \beta_j.$$

For the second one, we remark that, by the definition of β^N , we have

$$\partial_h \partial_h \beta^N = \partial_h \partial_h \partial_h \beta^D,$$

so we automatically obtain (29) from (28) for the indices $i = 3, \dots, 2N-3$. For $i = 1$ and $i = 2N-1$, the above extension of β actually implies that

$$(\partial_h \beta^N)_0 = (\partial_h \beta^N)_{2N+2} = 0,$$

which is quite natural because β^N plays the role of a Neumann eigenvector. Consequently, we have

$$(\partial_h \partial_h \beta^N)_1 = \frac{1}{h}(\partial_h \beta^N)_2 = \frac{1}{h^3}(\beta_2 + \beta_0 - 2\beta_1) = -\frac{\lambda^2}{h}\beta_1 = -\lambda^2\beta_1^N.$$

We get the same identity at $2N - 1$ and consequently (29) holds. $\square$

Now we can prove our result on the lack of uniform observability inequality as the mesh size tends to zero:

Theorem 3.2

For any $T > 0$, we have

$$\lim_{h \rightarrow 0} \sup_{(E_h, H_h) \text{ sol. of (11)-(15)}} \frac{\mathcal{E}_h(0)}{\int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt} = +\infty. \quad (30)$$

Proof. We consider a particular solution of (11)-(15) built with the help of one particular eigenvector described in the previous lemma. Namely we fix $\lambda^2 = \lambda_{hN}^2$ and then take

$$\beta_j = \sin(N\pi jh), j = 1, \dots, N.$$

We note that

$$\int_{(0,1)} |\beta^D|^2 = h \sum_{j=1}^N (\sin(N\pi jh))^2 \sim 1, \quad (31)$$

where for shortness, $a \sim b$ means here and below that there exist two positive constants c_1 and c_2 independent on a , b and the mesh size h such that

$$c_1 b \leq a \leq c_2 b.$$

Now we take

$$(E_h, H_h) = e^{i\kappa t}(\Phi, \Psi),$$

where the pair (Φ, Ψ) is the eigenvector associated with A_h of eigenvalue $i\kappa = i\sqrt{3}\lambda$.

With this definition, (E_h, H_h) automatically satisfies (11)-(15) since we recall that Φ satisfies (25).

Now by definition of (E_h, H_h) its energy $\mathcal{E}_h(0)$ will be

$$\mathcal{E}_h(0) = \frac{1}{2} \int_{\Omega} \left(|\Phi|^2 + \frac{1}{\kappa^2} |\text{curl}_h \Phi|^2 \right).$$

By direct calculations, we obtain

$$\begin{aligned} \mathcal{E}_h(0) &\sim h^3 \sum_{i=0}^{N-1} \sum_{j,k=1}^N (\beta_{2i+1}^N)^2 (\beta_{2j}^D)^2 (\beta_{2k}^D)^2 \\ &\quad + \frac{h^3}{\kappa^2} \sum_{i=1}^N \sum_{j,k=0}^{N-1} (\beta_{2i+1}^N)^2 (\beta_{2i}^D)^2 (\beta_{2k+1}^N)^2. \end{aligned}$$

Using (31) we may write

$$\mathcal{E}_h(0) \sim h \sum_{i=0}^{N-1} (\beta_{2i+1}^N)^2 + \frac{h^2}{\kappa^2} \left(\sum_{i=1}^{N-1} (\beta_{2i+1}^N)^2 \right)^2. \quad (32)$$

Now we have

$$\begin{aligned} \int_{\Gamma_0} |\gamma H|^2 &= h^2 \sum_{m,n} \sum_{\kappa=2,3} |H_\kappa(2N+1, m, n)|^2 \\ &+ h^2 \sum_{l,n} \sum_{\kappa=1,3} |H_\kappa(l, 2N+1, n)|^2 + h^2 \sum_{l,m} \sum_{\kappa=1,2} |H_\kappa(l, m, 2N+1)|^2. \end{aligned}$$

Similar calculations as before yield

$$\int_{\Gamma_0} |\gamma H|^2 \sim \frac{h}{\kappa^2} (\beta_{2N+1}^N)^2 \sum_{i=0}^{N-1} (\beta_{2i+1}^N)^2. \quad (33)$$

Using the two equivalences (32) and (33), we have

$$\frac{\mathcal{E}_h(0)}{\int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt} \sim \frac{\kappa^2}{T} \frac{1 + \frac{h}{\lambda^2} (\sum_i^{N-1} (\beta_{2i+1}^N)^2)}{(\beta_{2N+1}^N)^2}.$$

This last quotient blows up because we have chosen $\lambda^2 = \lambda_{hN}^2$, and then

$$\lambda^2 h^2 \rightarrow 4 \text{ as } h \rightarrow 0,$$

and

$$\beta_{2N+1}^N = (\partial_h \beta^D)_{2N+1} = -\frac{1}{h} \beta_N \sim -\pi.$$

Furthermore using Lemma 1.1 of [10] (see also Lemma 5.1 of [39]), we have

$$h \sum_i^{N-1} (\beta_{2i+1}^N)^2 = \frac{2}{4 - \lambda^2 h^2} \left(\frac{\beta_N}{h} \right)^2 \sim \frac{1}{h^2}.$$

These equivalences lead to

$$\frac{\mathcal{E}_h(0)}{\int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt} \sim \frac{1}{Th^2},$$

which goes to infinity as h goes to zero. $\square$

Remark 3.3 Note that the ratio in (30) remains uniformly bounded as $h \rightarrow 0$ for other branches of eigenvectors. Namely the choice $\Phi = (0, 0, \Phi_3)^\top$ with

$$\Phi_3(2i, 2j, 2k+1) = \beta_{2i}^D \beta_{2j}^D,$$

and the same definition for (E_h, H_h) leads to a ratio $\frac{\mathcal{E}_h(0)}{\int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt}$ which is of order $\frac{1}{T}$.

Remark 3.4 In order to restore the uniformity of the observability estimate (20) with respect to h there are two possibilities: either restrict the class of solutions of (11)-(15) (the so-called filtering technique) or reinforce the right-hand side of (20) (leading to a Tychonoff regularization). For the first remedy, the explicit knowledge of the spectrum is necessary, which is far from being the case in our situation. For the second remedy, internal reinforcements are used for the wave equation [7, 8, 39]. On the other hand for the beam equation, a boundary reinforcement can be used owing to the zero speed of propagation of the waves [17]. Hence our natural choice is to use an internal reinforcement.

4. The discrete observability estimate

We first establish a discrete multiplier identity and then deduce a discrete observability estimate, where we separate the low and high frequency contributions.

4.1 A discrete multiplier identity

Our proof relies on an adaptation of the continuous multiplier method. Let us recall (see for instance [15, 29]) that for the continuous problem, the proof of (19) is based on a multiplier identity whose starting point is to consider the expression

$$\int_0^T \int_{\Omega} \frac{d}{dt} (E \times H)(x, t) \cdot m(x) dx dt = \int_0^T \int_{\Omega} (E' \times H + E \times H') \cdot m(x) dx dt,$$

with the standard multiplier $m(x) = x$. The right-hand side of this identity is transformed using (1)-(2) and then applying some integrations by parts in space. We will reproduce this method at the discrete level with the necessary adaptations.

The first problem is that E_h and H_h do not live on the same indices, so that $E_h \times H_h$ has no meaning. Therefore, we proceed as follows: We remark that if $E_h \times H_h$ would have a meaning, then its first component $(E_h \times H_h)_1$ would be

$$(E_h \times H_h)_1 = E_{h2}H_{h3} - E_{h3}H_{h2}.$$

For the first term, E_{h2} is defined on indices $(2i, 2j+1, 2k)$, while H_{h3} is defined on indices $(2i+1, 2j+1, 2k)$, therefore a left shift in the first index yields the same indices $(2i, 2j+1, 2k)$. So the solution is to consider the product

$$(E_{h2}S_1^+H_{h3})(2i, 2j+1, 2k) = E_{h2}(2i, 2j+1, 2k)H_{h3}(2i+1, 2j+1, 2k),$$

which then has a meaning. Unfortunately the “product” $E_{h3}S_1^+H_{h2}$ is defined on the indices $(2i, 2j, 2k)$, which are different from $(2i, 2j+1, 2k)$. At the end this means that we consider for the first component, the two terms:

$$(E_{h2}S_1^+H_{h3})(2i, 2j+1, 2k+1),$$

and

$$(E_{h3}S_1^+H_{h2})(2i, 2j, 2k+1),$$

that we multiply by $m_1^{even}(2i)$, the (discrete) even multiplier m_1^{even} being defined by (corresponding to the discretization of the first component of the multiplier m at the even indices)

$$m_1^{even}(2i) := 2i\delta = ih.$$

The even index indicates that the function is considered at the even indices and therefore $m_1^{even}(0) = 0$ and $m_1^{even}(2N+2) = 1$. Now we take the time derivative of the above expressions, take the difference and integrate the results in time between 0 and T and in “space”. This yields

$$I_1 = \int_0^T \int_{\Omega} \frac{d}{dt} (E_{h2} S_1^+ H_{h3}) m_1^{even} dt - \int_0^T \int_{\Omega} \frac{d}{dt} (E_{h3} S_1^+ H_{h2}) m_1^{even} dt. \quad (34)$$

By integrating this expression in time, we get

$$I_1 = \left[\int_{\Omega} E_{h2} S_1^+ H_{h3} m_1^{even} - \int_{\Omega} E_{h3} S_1^+ H_{h2} m_1^{even} \right]_0^T. \quad (35)$$

Cauchy-Schwarz’s inequality then yields

$$I_1 \leq \sum_{t=0, T} \int_{\Omega} \left(|E_{h2}(\cdot, t)|^2 + |E_{h3}(\cdot, t)|^2 + |H_{h2}(\cdot, t)|^2 + |H_{h3}(\cdot, t)|^2 \right). \quad (36)$$

Lemma 4.1

Let (E_h, H_h) be the solution of (11)-(15). Then they satisfy the following discrete multiplier identity

$$\begin{aligned} I_1 = & - \int_0^T \int_{\Omega} (H_{h1})^2 dt + \frac{1}{2} \int_0^T \int_{\Omega} |H_h|^2 dt \\ & - \int_0^T \int_{\Omega} (E_{h1})^2 dt + \frac{1}{2} \int_0^T \int_{\Omega} |E_h|^2 dt \\ & + \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} (S_1^+ \partial_{1h} E_{h\kappa})^2 m_1^{\kappa} dt - \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} (\partial_{1h} H_{h\kappa})^2 m_1^{\kappa} dt \\ & + \frac{h}{2} \int_0^T \int_{\Omega} S_1^+ E_{h1} \partial_{1h} E_{h1} dt - \frac{h}{2} \int_0^T \int_{\Omega} H_{h1} \partial_{1h} (S_1^+ H_{h1}) dt \\ & - \frac{1}{2} \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 \nu_1 dt + \frac{1}{2} \int_0^T \Sigma_1(E_h) dt. \end{aligned} \quad (37)$$

Proof. By (34) and Leibniz’s rule (in time), we have

$$\begin{aligned} I_1 = & \int_0^T \int_{\Omega} \left(\frac{d}{dt} E_{h2} S_1^+ H_{h3} + E_{h2} S_1^+ \frac{d}{dt} H_{h3} \right) m_1^{even} dt \\ & - \int_0^T \int_{\Omega} \left(\frac{d}{dt} E_{h3} S_1^+ H_{h2} + E_{h3} S_1^+ \frac{d}{dt} H_{h2} \right) m_1^{even} dt. \end{aligned}$$

Using (11) and (12), we obtain

$$\begin{aligned}
I_1 &= \int_0^T \int_{\Omega} ((\partial_{3h} H_{h1} - \partial_{1h} H_{h3}) S_1^+ H_{h3} - E_{h2} S_1^+ (\partial_{1h} E_{h2} - \partial_{2h} E_{h1})) m_1^{even} dt \\
&\quad - \int_0^T \int_{\Omega} ((\partial_{1h} H_{h2} - \partial_{2h} H_{h1}) S_1^+ H_{h2} - E_{h3} S_1^+ (\partial_{3h} E_{h1} - \partial_{1h} E_{h3})) m_1^{even} dt \\
&= \sum_{\kappa=2,3} \int_0^T \int_{\Omega} (\partial_{\kappa h} H_{h1} S_1^+ H_{h\kappa} - \partial_{1h} H_{h\kappa} S_1^+ H_{h\kappa}) m_1^{even} dt \\
&\quad + \sum_{\kappa=2,3} \int_0^T \int_{\Omega} (S_1^+ \partial_{\kappa h} E_{h1} E_{h\kappa} - S_1^+ \partial_{1h} E_{h\kappa} E_{h\kappa}) m_1^{even} dt.
\end{aligned}$$

As for the continuous method, in this right-hand side, we want to introduce the four terms corresponding to $\kappa = 1$ (terms whose sum is zero). As terms like $\partial_{1h} H_{h1} S_1^+ H_{h1}$ live on the odd indices i , we here need to introduce the odd multiplier m_1^{odd} defined by

$$m_1^{odd}(2i+1) = (2i+1)\delta, i = 0, \dots, N-1.$$

By introducing the notation

$$m_1^l = \begin{cases} m_1^{odd} & \text{if } l = 1, \\ m_1^{even} & \text{if } l = 2, 3, \end{cases}$$

the above expression is equivalent to

$$I_1 = \int_0^T (I_1(H_h) - I_2(H_h) + I_1(E_h) - I_2(E_h)) dt, \quad (38)$$

where

$$\begin{aligned}
I_1(H_h) &= \sum_{\kappa=1}^3 \int_{\Omega} \partial_{\kappa h} H_{h1} S_1^+ H_{h\kappa} m_1^{\kappa}, \\
I_2(H_h) &= \sum_{\kappa=1}^3 \int_{\Omega} \partial_{1h} H_{h\kappa} S_1^+ H_{h\kappa} m_1^{\kappa}, \\
I_1(E_h) &= \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ \partial_{\kappa h} E_{h1} E_{h\kappa} m_1^{\kappa}, \\
I_2(E_h) &= \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ \partial_{1h} E_{h\kappa} E_{h\kappa} m_1^{\kappa}.
\end{aligned}$$

Now we will transform these four integrals using Green's formula in space: Namely using (16), we have

$$I_1(H_h) = - \sum_{\kappa=1}^3 \int_{\Omega} H_{h1} \partial_{\kappa h} (S_1^+ H_{h\kappa} m_1^{\kappa}) + \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma H_{h1} \gamma (S_1^+ H_{h\kappa} m_1^{\kappa}) \nu_{\kappa}.$$

By the boundary conditions satisfied by H_h , we readily check that

$$\sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma H_{h1} \gamma (S_1^+ H_{h\kappa} m_1^{\kappa}) \nu_{\kappa} = 0,$$

and therefore applying Leibniz's rule (58), we obtain

$$I_1(H_h) = - \sum_{\kappa=1}^3 \int_{\Omega} H_{h1} (S_{\kappa}^{-} S_1^{+} H_{h\kappa} \partial_{\kappa h} m_1^{\kappa} + \partial_{\kappa h} (S_1^{+} H_{h\kappa}) S_{\kappa}^{+} m_1^{\kappa}).$$

As $\partial_{\kappa h} m_1^{\kappa} = \delta_{1\kappa}$, we get

$$\begin{aligned} I_1(H_h) &= - \int_{\Omega} |H_{h1}|^2 - \sum_{\kappa=1}^3 \int_{\Omega} H_{h1} \partial_{\kappa h} (S_1^{+} H_{h\kappa}) S_{\kappa}^{+} m_1^{\kappa} \\ &= - \int_{\Omega} |H_{h1}|^2 \\ &\quad - \int_{\Omega} H_{h1} \left(\partial_{1h} (S_1^{+} H_{h1}) S_1^{+} m_1^{odd} + \sum_{\kappa=2,3} \partial_{\kappa h} (S_1^{+} H_{h\kappa}) m_1^{even} \right). \end{aligned}$$

Since $S_1^{+} m_1^{odd} = m_1^{even} + \frac{h}{2}$, the above identity becomes

$$\begin{aligned} I_1(H_h) &= - \int_{\Omega} |H_{h1}|^2 - \int_{\Omega} H_{h1} S_1^{+} (\operatorname{div}_h H_h) m_1^{even} \\ &\quad - \frac{h}{2} \int_{\Omega} H_{h1} \partial_{1h} (S_1^{+} H_{h1}). \end{aligned}$$

As H_h is discrete divergence free, we finally arrive at

$$I_1(H_h) = - \int_{\Omega} |H_{h1}|^2 - \frac{h}{2} \int_{\Omega} H_{h1} \partial_{1h} (S_1^{+} H_{h1}). \quad (39)$$

The continuous counterpart of this identity is

$$\int_{\Omega} \nabla H_1 \cdot H x_1 dx = - \int_{\Omega} |H_1| dx.$$

At the discrete level, the extra term $-\frac{h}{2} \int_{\Omega} H_{h1} \partial_{1h} (S_1^{+} H_{h1})$ comes from the use of the shift operator S_1^{+} . Similar extra terms appear later on and furnish the high frequency contribution in the multiplier identity.

In a similar manner, by (16), we have

$$\begin{aligned} I_2(H_h) &= - \sum_{\kappa=1}^3 \int_{\Omega} H_{h\kappa} \partial_{1h} (S_1^{+} H_{h\kappa} m_1^{\kappa}) \\ &\quad + \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma H_{h\kappa} \gamma (S_1^{+} H_{h\kappa} m_1^{\kappa}) \nu_1. \end{aligned} \quad (40)$$

By the boundary conditions satisfied by H_h and the properties of the multiplier m_1^{κ} , we see that

$$\begin{aligned} \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma H_{h\kappa} \gamma (S_1^{+} H_{h\kappa} m_1^{\kappa}) \nu_1 &= h^2 \sum_{m,n} H_{h2}(2N+1, m, n) H_{h2}(2N+3, m, n) \\ &\quad + h^2 \sum_{m,n} H_{h3}(2N+1, m, n) H_{h3}(2N+3, m, n). \end{aligned}$$

We now proceed with an even extension of H_{h2} and H_{h3} , namely we set

$$\begin{aligned} H_{h2}(2N+3, m, n) &:= H_{h2}(2N+1, m, n); \\ H_{h3}(2N+3, m, n) &:= H_{h3}(2N+1, m, n). \end{aligned}$$

Then we find that

$$\begin{aligned} \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma H_{h\kappa} \gamma (S_1^+ H_{h\kappa} m_1^\kappa) \nu_1 &= h^2 \sum_{m,n} |H_{h2}(2N+1, m, n)|^2 \\ &+ |H_{h3}(2N+1, m, n)|^2. \end{aligned}$$

Since $H_{h1}(2N+2, m, n) = 0$, we obtain (the continuous counterpart is

$$\begin{aligned} \int_{\Gamma} |H|^2 x_1 \nu_1 d\Gamma(x) &= \int_{\Gamma_0} |H \times \nu|^2 \nu_1 d\Gamma(x) : \\ \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma H_{h\kappa} \gamma (S_1^+ H_{h\kappa} m_1^\kappa) \nu_1 &= \int_{\Gamma_0} |\gamma H_h|^2 \nu_1. \end{aligned}$$

Inserting this identity in (40) and using Leibniz's rule (58), we obtain

$$\begin{aligned} I_2(H_h) &= - \sum_{\kappa=1}^3 \int_{\Omega} H_{h\kappa} \{ H_{h\kappa} \partial_{1h} m_1^\kappa + \partial_{1h} (S_1^+ H_{h\kappa}) S_1^+ m_1^\kappa \} \\ &+ \int_{\Gamma_0} |\gamma H_h|^2 \nu_1 \\ &= - \int_{\Omega} |H_h|^2 - \sum_{\kappa=1}^3 \int_{\Omega} H_{h\kappa} \partial_{1h} (S_1^+ H_{h\kappa}) S_1^+ m_1^\kappa \\ &+ \int_{\Gamma_0} |\gamma H_h|^2 \nu_1 \\ &= - \int_{\Omega} |H_h|^2 - \sum_{\kappa=1}^3 \int_{\Omega} S_1^- H_{h\kappa} \partial_{1h} H_{h\kappa} m_1^\kappa \\ &+ \int_{\Gamma_0} |\gamma H_h|^2 \nu_1 \\ &= - \int_{\Omega} |H_h|^2 - I_2(H_h) - \sum_{\kappa=1}^3 \int_{\Omega} (S_1^+ - S_1^-) H_{h\kappa} \partial_{1h} H_{h\kappa} m_1^\kappa \\ &+ \int_{\Gamma_0} |\gamma H_h|^2 \nu_1. \end{aligned}$$

This finally leads to

$$I_2(H_h) = -\frac{1}{2} \int_{\Omega} |H_h|^2 + \frac{h}{2} \sum_{\kappa=1}^3 \int_{\Omega} (\partial_{1h} H_{h\kappa})^2 m_1^\kappa + \frac{1}{2} \int_{\Gamma_0} |\gamma H_h|^2 \nu_1. \quad (41)$$

In a similar manner, we have

$$I_1(E_h) = - \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ E_{h1} \partial_{\kappa h} (E_{h\kappa} m_1^\kappa) + \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma (S_1^+ E_{h1}) \gamma (E_{h\kappa} m_1^\kappa).$$

By the boundary conditions satisfied by E_h , we see that

$$\begin{aligned} \sum_{\kappa=1}^3 \int_{\partial\Omega} \gamma(S_1^+ E_{h1}) \gamma(E_{h\kappa} m_1^\kappa) &= h^2 \sum_{m,n} |E_{h1}(2N+1, m, n)|^2 (1-h) \\ &\quad - h^3 \sum_{m,n} |E_{h1}(1, m, n)|^2. \end{aligned}$$

By setting, for shortness,

$$\Sigma_1(E_h) := h^2 \sum_{m,n} |E_{h1}(2N+1, m, n)|^2 (1-h),$$

and using Leibniz's rule, we obtain

$$\begin{aligned} I_1(E_h) &= - \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ E_{h1} (S_\kappa^+ \partial_{\kappa h} m_1^\kappa + \partial_{\kappa h} E_{h\kappa} S_\kappa^- m_1^\kappa) \\ &\quad + \Sigma_1(E_h) - h^3 \sum_{m,n} |E_{h1}(1, m, n)|^2 \\ &= - \int_{\Omega} (S_1^+ E_{h1})^2 - \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ E_{h1} \partial_{\kappa h} E_{h\kappa} S_\kappa^- m_1^\kappa \\ &\quad + \Sigma_1(E_h) - h^3 \sum_{m,n} |E_{h1}(1, m, n)|^2 \\ &= - \int_{\Omega} (E_{h1})^2 \\ &\quad - \int_{\Omega} S_1^+ E_{h1} \left(\partial_{1h} E_{h1} S_1^- m_1^{odd} + \sum_{\kappa=2,3} \partial_{\kappa h} E_{h\kappa} m_1^{even} \right) + \Sigma_1(E_h) \\ &= - \int_{\Omega} (E_{h1})^2 - \int_{\Omega} S_1^+ E_{h1} \operatorname{div}_h E_h m_1^{even} \\ &\quad + \int_{\Omega} S_1^+ E_{h1} \partial_{1h} E_{h1} (m_1^{even} - S_1^- m_1^{odd}) + \Sigma_1(E_h). \end{aligned}$$

As $m_1^{even} - S_1^- m_1^{odd} = \frac{h}{2}$ and E_h is discrete divergence free, we finally obtain

$$I_1(E_h) = - \int_{\Omega} (E_{h1})^2 + \frac{h}{2} \int_{\Omega} S_1^+ E_{h1} \partial_{1h} E_{h1} + \Sigma_1(E_h). \quad (42)$$

As before using Green's formula and the boundary conditions satisfied by E_h , we have

$$I_2(E_h) = - \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ E_{h\kappa} \partial_{1h} (E_{h\kappa} m_1^\kappa) + \Sigma_1(E_h) - h^3 \sum_{m,n} |E_{h1}(1, m, n)|^2.$$

Again Leibniz's rule yields

$$I_2(E_h) = - \int_{\Omega} |E_h|^2 - \sum_{\kappa=1}^3 \int_{\Omega} S_1^+ E_{h\kappa} \partial_{1h} E_{h\kappa} S_1^- m_1^\kappa + \Sigma_1(E_h).$$

The same trick as for $I_2(H_h)$ finally leads to

$$I_2(E_h) = -\frac{1}{2} \int_{\Omega} |E_h|^2 - \frac{h}{2} \sum_{\kappa=1}^3 \int_{\Omega} (S_1^+ \partial_{1h} E_{h\kappa})^2 m_1^\kappa + \frac{1}{2} \Sigma_1(E_h). \quad (43)$$

Inserting (39), (41), (42) and (43) into (38), we arrive at the multiplier identity (37). $\square$

In comparison with the continuous multiplier identity, the four terms in (37) with the factor h appear at a discrete level, their presence comes from the use of the shift operators and yield the high frequency contribution in the discrete observability estimate (the second term in (44) below).

4.2 The discrete observability estimate

Theorem 4.2

Let (E_h, H_h) be the solution of (11)-(15). There exists a minimal time $T_0 \leq 4 + h$ such that for all $T > T_0$, there exists a constant $C(T) > 0$ (independent on h) such that

$$\begin{aligned} \mathcal{E}_h(0) &\leq C(T) \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt \\ &\quad + C(T) h \sum_{\kappa, \kappa'=1}^3 \int_0^T \int_{\Omega} ((\partial_{\kappa'h} E_{h\kappa})^2 + (\partial_{\kappa'h} H_{h\kappa})^2) dt. \end{aligned} \quad (44)$$

Proof. By the discrete multiplier identity (37), the estimate (36) and the fact that $\Sigma_1(E_h)$ is nonnegative, we obtain

$$\begin{aligned} & - \int_0^T \int_{\Omega} (H_{h1})^2 dt + \frac{1}{2} \int_0^T \int_{\Omega} |H_h|^2 dt - \int_0^T \int_{\Omega} (E_{h1})^2 dt \\ & + \frac{1}{2} \int_0^T \int_{\Omega} |E_h|^2 dt \\ & - \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} (\partial_{1h} H_{h\kappa})^2 m_1^\kappa dt + \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} (S_1^+ \partial_{1h} E_{h\kappa})^2 m_1^\kappa dt \\ & + \frac{h}{2} \int_0^T \int_{\Omega} S_1^+ E_{h1} \partial_{1h} E_{h1} dt - \frac{h}{2} \int_0^T \int_{\Omega} H_{h1} \partial_{1h} (S_1^+ H_{h1}) dt \\ & - \frac{1}{2} \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 \nu_1 dt \\ & \leq \sum_{t=0, T} \int_{\Omega} (|E_{h2}(\cdot, t)|^2 + |E_{h3}(\cdot, t)|^2 + |H_{h2}(\cdot, t)|^2 + |H_{h3}(\cdot, t)|^2). \end{aligned}$$

By symmetry with respect to the indices 1, 2, 3, a similar estimate holds for the indices 2 and 3, summing these estimates, and using the conservation of energy, we then arrive

at

$$\begin{aligned}
& \frac{1}{2} \int_0^T \int_{\Omega} |H_h|^2 dt + \frac{1}{2} \int_0^T \int_{\Omega} |E_h|^2 dt \\
& + \frac{h}{2} \sum_{\kappa, \kappa'=1}^3 \int_0^T \int_{\Omega} (S_{\kappa'}^+ \partial_{\kappa'h} E_{h\kappa})^2 m_{\kappa'}^{\kappa} dt - \frac{h}{2} \sum_{\kappa, \kappa'=1}^3 \int_0^T \int_{\Omega} (\partial_{\kappa'h} H_{h\kappa})^2 m_{\kappa'}^{\kappa} dt \\
& + \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} S_{\kappa}^+ E_{h\kappa} \partial_{\kappa h} E_{h\kappa} dt - \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} H_{h\kappa} \partial_{1h} (S_{\kappa}^+ H_{h\kappa}) dt \\
& - \frac{1}{2} \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt \leq 4\mathcal{E}_h(0).
\end{aligned}$$

In view of the definition of the energy, there exists $C' > 0$ such that

$$\begin{aligned}
(T-4)\mathcal{E}_h(0) & \leq C'h \sum_{\kappa, \kappa'=1}^3 \int_0^T \int_{\Omega} (\partial_{\kappa'h} H_{h\kappa})^2 dt \\
& + \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} |S_{\kappa}^+ E_{h\kappa}| |\partial_{\kappa h} E_{h\kappa}| dt \\
& + \frac{h}{2} \sum_{\kappa=1}^3 \int_0^T \int_{\Omega} |H_{h\kappa}| |\partial_{\kappa h} (S_{\kappa}^+ H_{h\kappa})| dt \\
& + C'h \sum_{\kappa, \kappa'=1}^3 \int_0^T \int_{\Omega} (S_{\kappa'}^+ \partial_{\kappa'h} E_{h\kappa})^2 dt \\
& + \frac{1}{2} \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt.
\end{aligned}$$

By Young's inequality, we finally arrive at the discrete observability estimate (44). $\square$

Remark 4.3 From the above considerations, we see that the choice of the multiplier is not the canonical one $m_i = ih$. Again the disparity of the approximated nodes forces us to make a more tricky choice and to use even and odd multipliers.

Remark 4.4 The minimal time $T_0 \leq 4$ is exactly the minimal time that can be obtained by the multiplier method for the continuous Maxwell's system [14]. Recall that microlocal analysis arguments yields as optimal minimal time $T_0 = 2\sqrt{2}$ for the continuous Maxwell's system [28].

For further uses, we transform the previous discrete observability estimate into a more appropriate one.

Corollary 4.5

Under the assumption of Theorem 4.2, there exists a minimal time $T_0 > 0$ such that for all $T > T_0$, there exists a constant $C(T) > 0$ (independent on h) such that

$$\mathcal{E}_h(0) \leq C(T) \int_0^T \int_{\Gamma_0} |\gamma H_h|^2 dt + C(T)h \int_0^T \int_{\Omega} (|E_h'|^2 + |H_h'|^2) dt. \quad (45)$$

Proof. By Green's formula and the boundary conditions satisfied by E_h and H_h , we have (see Lemma 2.2 of [3])

$$\begin{aligned} \sum_{\kappa, \kappa'=1}^3 \int_{\Omega} (\partial_{\kappa'h} E_{h\kappa})^2 &= \int_{\Omega} (|\operatorname{curl}_h E_h|^2 + |\operatorname{div}_h E_h|^2), \\ \sum_{\kappa, \kappa'=1}^3 \int_{\Omega} (\partial_{\kappa'h} H_{h\kappa})^2 &= \int_{\Omega} (|\operatorname{curl}_h H_h|^2 + |\operatorname{div}_h H_h|^2). \end{aligned}$$

As E_h and H_h are discrete divergence free and using (11)-(12), we directly get

$$\sum_{\kappa, \kappa'=1}^3 \int_{\Omega} ((\partial_{\kappa'h} E_{h\kappa})^2 + (\partial_{\kappa'h} H_{h\kappa})^2) = \int_{\Omega} (|E'_h|^2 + |H'_h|^2). \quad \square$$

Remark that in the right-hand side of (45) the extra internal term

$$h \int_0^T \int_{\Omega} (|E'_h|^2 + |H'_h|^2)$$

can be seen as a regularization term. It does not contribute for low frequency solutions, but it is relevant for high frequency ones corresponding to eigenvalues $\lambda^2 \gtrsim \frac{1}{h}$. But, according to Theorem 3.2, in the right-hand side of (45) both terms are necessary in order to have a uniform observability constant.

5. The exact controllability result by adding internal controls

For shortness, we denote by $\ell^2(\Omega)^3$ the space of discrete vectors defined as, for instance, E_h (or H_h according to the context) equipped with the natural norm

$$\left(\int_{\Omega} |E_h|^2 \right)^{1/2}.$$

We define similarly $\ell^2(\Gamma_0)^3$.

Using the Hilbert Uniqueness Method of J.-L. Lions [18] and the observability estimate (45), we deduce the next controllability result:

Theorem 5.1

Assume that the initial data E_{h0} and H_{h0} are discrete divergence free and fix $T > T_0$ as in Corollary 4.5. Then there exist $\psi_h \in L^2(0, T; \ell^2(\Gamma_0)^3)$ and $g_{h1}, g_{h2} \in H^{-1}(0, T; \ell^2(\Omega)^3)$ such that the solution (E_h, H_h) of

$$E'_h - \operatorname{curl}_h H_h = hg_{h1} \quad \text{in } \Omega \times (0, +\infty), \quad (46)$$

$$H'_h + \operatorname{curl}_h E_h = hg_{h2} \quad \text{in } \Omega \times (0, +\infty), \quad (47)$$

$$\operatorname{div}_h E_h = \operatorname{div}_h H_h = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (48)$$

$$\gamma H_h \times \nu = \psi_h \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (49)$$

$$\gamma H_h \times \nu = 0 \quad \text{on } (\Gamma \setminus \Gamma_0) \times (0, +\infty), \quad \gamma E_h \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty), \quad (50)$$

$$E_h(0) = E_{h0} \quad \text{and} \quad H_h(0) = H_{h0} \quad \text{in } \Omega, \quad (51)$$

is at rest a time T , i.e., satisfies

$$E_h(T) = 0 \quad \text{and} \quad H_h(T) = 0 \quad \text{in} \quad \Omega.$$

Furthermore, there exists a constant $C > 0$ (independent of h) such that

$$\|\psi_h\|_{L^2(0,T;\ell^2(\Gamma_0)^3)}^2 + h \sum_{i=1,2} \|g_{hi}\|_{H^{-1}(0,T;\ell^2(\Omega)^3)}^2 \leq C \int_{\Omega} (|E_{h0}|^2 + |H_{h0}|^2). \quad (52)$$

Moreover if the right-hand side of (52) is bounded uniformly as $h \rightarrow 0$, then

$$hg_{hi} \rightarrow 0 \quad \text{in} \quad \mathcal{D}'(\Omega \times (0, T)) \quad \text{as} \quad h \rightarrow 0,$$

and there exists $\tilde{\psi}_h \in L^2(0, T; L^2(\Gamma_0)^3)$ associated with ψ_h (see below) such that (up to a subsequence)

$$\tilde{\psi}_h \rightarrow v \quad \text{in} \quad L^2(0, T; L^2(\Gamma_0)^3),$$

where v is a control of the continuous Maxwell's system.

Proof. The proof is quite standard, let us shortly describe it. We first use the standard Hilbert Uniqueness Method of J.-L. Lions: Given φ_{h0}, ψ_{h0} , consider (φ_h, ψ_h) solution of

$$\begin{aligned} \varphi'_h + \text{curl}_h \psi_h &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\ \psi'_h - \text{curl}_h \varphi_h &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\ \gamma \varphi_h \times \nu &= 0, \quad \gamma \psi_h \cdot \nu = 0 \quad \text{on} \quad \Gamma \times (0, +\infty), \\ \varphi_h(0) &= \varphi_{h0} \quad \text{and} \quad \psi_h(0) = \psi_{h0} \quad \text{in} \quad \Omega. \end{aligned}$$

By Corollary 4.5, we have

$$\begin{aligned} \mathcal{E}_h(\varphi_h, \psi_h)(0) &\leq C(T) \int_0^T \int_{\Gamma_0} |\gamma \psi_h|^2 dt \\ &\quad + C(T)h \int_0^T \int_{\Omega} \rho(t) (|\varphi'_h|^2 + |\psi'_h|^2) dt, \end{aligned} \quad (53)$$

where the energy of (φ_h, ψ_h) at time t is defined by

$$\mathcal{E}_h(\varphi_h, \psi_h)(t) = \frac{1}{2} \int_{\Omega} (|\varphi_h(t)|^2 + |\psi_h(t)|^2)$$

and ρ is a fixed smooth function equal to 1 on $(\epsilon, T - \epsilon)$ and such that $\rho(0) = \rho(T) = 0$ (ϵ being sufficiently small such that $T - 2\epsilon > T_0$).

Now we consider (E_h, H_h) solution of

$$\begin{aligned} E'_h - \text{curl}_h H_h &= h(\rho \psi'_h)' \quad \text{in} \quad H^{-1}(0, T; \ell^2(\Omega)^3), \\ H'_h + \text{curl}_h E_h &= -h(\rho \varphi'_h)' \quad \text{in} \quad H^{-1}(0, T; \ell^2(\Omega)^3), \\ \gamma H_h \times \nu &= \psi_h \quad \text{on} \quad \Gamma_0 \times (0, T), \\ \gamma H_h \times \nu &= 0 \quad \text{on} \quad (\Gamma \setminus \Gamma_0) \times (0, T), \quad \gamma E_h \cdot \nu = 0 \quad \text{on} \quad \Gamma \times (0, T), \\ E_h(T) &= 0 \quad \text{and} \quad H_h(T) = 0 \quad \text{in} \quad \Omega. \end{aligned}$$

The two first identities of the above system imply that

$$\begin{aligned} & -h \int_0^T \int_{\Omega} \rho(t)(|\psi'_h|^2 + |\varphi'_h|^2) dt \\ &= \int_0^T \int_{\Omega} ((E'_h - \operatorname{curl}_h H_h) \cdot \psi_h - (H'_h + \operatorname{curl}_h E_h) \cdot \varphi_h) dt. \end{aligned}$$

Using integration by parts in time, Green's formula (17) and the initial and final boundary conditions satisfied by (φ_h, ψ_h) and (E_h, H_h) , we obtain

$$\begin{aligned} & \int_{\Omega} (E_h(0) \cdot \psi_{h0} - H_h(0) \cdot \varphi_{h0}) \\ &= \int_0^T \int_{\Gamma_0} |\gamma \psi_h|^2 dt + h \int_0^T \int_{\Omega} \rho(t)(|\varphi'_h|^2 + |\psi'_h|^2) dt. \end{aligned} \tag{54}$$

By the estimate (53), we then deduce that the mapping

$$\Lambda : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H} : (\psi_{h0}, \varphi_{h0}) \rightarrow (E_h(0), -H_h(0))$$

is an isomorphism, where $\mathcal{H} = \{\varphi_h \in \ell^2(\Omega)^3 : \operatorname{div}_h \varphi_h = 0\}$. Therefore given (E_{h0}, H_{h0}) , we first solve the first system above in (φ_h, ψ_h) with initial data $(\psi_{h0}, \varphi_{h0}) = \Lambda^{-1}(E_{h0}, -H_{h0})$ and then consider the solution (E_h, H_h) of the second one. This solution satisfies (46)-(51) with $g_{h1} = (\rho \psi'_h)'$ and $g_{h2} = -(\rho \varphi'_h)'$ and it is at rest a time T .

Now from identity (54), Cauchy-Schwarz's inequality and estimate (53), we have

$$\int_0^T \int_{\Gamma_0} |\gamma \psi_h|^2 dt + h \int_0^T \int_{\Omega} \rho(t)(|\varphi'_h|^2 + |\psi'_h|^2) dt \leq C(T) \mathcal{E}_h(0).$$

This proves the estimate (52) because by an integration by part in time, we have

$$\|g_{h1}\|_{H^{-1}(0,T;\ell^2(\Omega)^3)} = - \sup_{\chi \in H_0^1(0,T;\ell^2(\Omega)^3)} \frac{\int_0^T \int_{\Omega} \rho \psi'_h \cdot \chi' dt}{\|\chi\|_{H_0^1(0,T;\ell^2(\Omega)^3)}} \leq C \|\rho^{1/2} \psi'_h\|_{L^2(0,T;\ell^2(\Omega)^3)}.$$

From the estimate (52) if $\int_{\Omega} (|E_{h0}|^2 + |H_{h0}|^2)$ remains bounded as h tends to 0, then for $i = 1$ or 2 :

$$h g_{hi} \rightarrow 0 \text{ in } \mathcal{D}'(\Omega \times (0, T)) \text{ as } h \rightarrow 0.$$

Consequently if E_{h0} (resp. H_{h0}) tends to the continuous initial datum E_0 (resp. H_0) (which is always possible for smooth continuous initial data, as one easily checks), then the discrete internal controls tend to zero. Furthermore the discrete boundary control tends to a continuous boundary control (up to a subsequence) in the following sense: For almost all $t \in (0, T)$, on the face $\Gamma_{01} = \{1\} \times (0, 1)^2$, we introduce the function

$$\tilde{\psi}_h(\cdot, t) = \begin{pmatrix} 0 \\ \tilde{\psi}_{h2} \\ \tilde{\psi}_{h3} \end{pmatrix} \text{ in } L^2(\Gamma_{01})^3 \text{ as follows: the nodal points } (1, (2j+1)\delta, 2k\delta) \text{ induce a}$$

triangulation of Γ_{01} into rectangles. We then define $\tilde{\psi}_{h2}$ as the unique function which

is piecewise $\mathbb{Q}_1$ with respect to this triangulation and takes the values

$$\begin{aligned}\tilde{\psi}_{h2}(1, (2j+1)\delta, 2k\delta) &= \psi_{h2}(2N+1, 2j+1, 2k), \forall 0 \leq j \leq N-1, 0 \leq k \leq N+1, \\ \tilde{\psi}_{h2}(1, 1, 2k\delta) &= 0, \forall 0 \leq k \leq N+1, \\ \tilde{\psi}_{h2}(1, (2j+1)\delta, 0) &= 0, \forall 0 \leq j \leq N-1, \\ \tilde{\psi}_{h2}(1, 0, 0) &= 0.\end{aligned}$$

We proceed similarly for $\tilde{\psi}_{h3}$ and the other faces of Γ_0 using appropriate nodal points. The construction is made in order to guarantee

$$\int_{\Gamma_0} |\tilde{\psi}_h(x, t)|^2 d\Gamma(x) \sim \int_{\Gamma_0} |\gamma\psi_h(\cdot, t)|^2.$$

Therefore by the estimate (52), the sequence $(\tilde{\psi}_h)_h$ is bounded in $L^2(0, T; L^2(\Gamma_0)^3)$ and then admits a subsequence that tends to a control of the continuous problem. $\square$

In conclusion, a Tychonoff regularization procedure allows to control uniformly the semi-discrete problem in such a way that the discrete controls tend to the appropriate continuous ones.

Remark 5.2 As underlined for finite element schemes in [30], the factor h in front of the internal controls (that may be regarded as viscosity terms) is closely related to the order two of convergence of Yee's scheme.

6. Some open problems

Standard Yee's scheme also involves a time discretization. As for the wave equation, for the full discretization we may expect a non uniform boundary observability property and therefore the scheme has also to be corrected by one of the remedies described in the introduction. The Tychonoff regularization method has recently been made with success for the 1D-wave equation in [22, 24] using a discrete version of Ingham's inequality. The extension to higher dimension and to Maxwell's system seems to be feasible but requires more investigations.

Instead of using a finite difference scheme, we can consider a finite element approximation using Nédélec elements for instance [20]. Similar phenomena probably occur with similar remedies. A full investigation has to be done, one obstacle should be the multiplier identity because the relationship between Nédélec elements and finite differences is not clear at all.

In the present paper we have used uniform meshes but we believe that all presented results still hold for non uniform meshes but this has to be checked.

As already mentioned, for the wave equation an alternative remedy is to use a mixed finite element method [6, 1, 2]. A similar scheme exists for Maxwell's system [19], where a different approximation space is used to approximate E and H . Its stability with respect to control problems would merit to be investigated.

7. Appendix: A 1 - d discrete differential calculus

Given $N \in \mathbb{N}$, we define $h = \frac{1}{N+1}$, $\delta = h/2 = \frac{1}{2(N+1)}$ and set $n = 2(N+1)$. We consider the mesh

$$x_j = j\delta, j = 0, \dots, n, \quad (55)$$

which furnishes a subdivision of the interval $[0, 1]$ into n intervals. Note that the number of points x_j in $I := (0, 1)$ is always odd.

We now introduce the space L_e (resp. L_o) of vectors U (resp. V) living on even (resp. odd) points. Namely $U \in L_e$ and $V \in L_o$ if and only if

$$\begin{aligned} U &= (u_{2j})_{j=0}^{N+1} && \text{with } u_{2j} \in \mathbb{R}, \\ V &= (v_{2j+1})_{j=0}^N && \text{with } v_{2j+1} \in \mathbb{R}. \end{aligned}$$

For even indices, we further introduce L_e^{int} , as follows: $U \in L_e^{int}$ if and only if

$$U = (u_{2j})_{j=1}^N \quad \text{with } u_{2j} \in \mathbb{R}.$$

For $U \in L_e$, its trace on the boundary is defined by

$$\gamma U = \{u_0, u_n\}.$$

On the contrary for $V \in L_o$, we introduce its trace on the boundary γV by extension, namely

$$\gamma V = \{v_0, v_n\} \text{ where we set } v_0 = v_1, v_n = v_{n-1}.$$

Now for $U \in L_e$, we define its discrete gradient $\partial_h U \in L_o$ as follows:

$$\partial_h U = ((\partial_h U)_{2j+1})_{j=0}^N \text{ with } (\partial_h U)_{2j+1} = \frac{u_{2j+2} - u_{2j}}{h}. \quad (56)$$

Similarly for $V \in L_o$, its discrete gradient $\partial_h V \in L_e^{int}$ is given by

$$\partial_h V = ((\partial_h V)_{2j})_{j=1}^N \text{ with } (\partial_h V)_{2j} = \frac{u_{2j+1} - u_{2j-1}}{h}. \quad (57)$$

With these definitions, we can state the so-called Leibniz's rule:

$$\partial_h (AB) = S^+ A \partial_h B + \partial_h A S^- B = S^- A \partial_h B + \partial_h A S^+ B, \forall A, B \in L_e, \quad (58)$$

$$\partial_h (AB) = S^+ A \partial_h B + \partial_h A S^- B = S^- A \partial_h B + \partial_h A S^+ B, \forall A, B \in L_o, \quad (59)$$

where S^+ (resp. S^-) is the shift to the right (resp. to the left), namely

$$(S^+ A)_l = A_{l+1}, \quad (S^- A)_l = A_{l-1}.$$

The verification of these identities is direct and is left to the reader.

Now for $U \in L_e$ (or $\in L_e^{int}$) and $V \in L_o$, we introduce its discrete integral by

$$\int_I U = h \sum_{j=1}^N u_{2j}, \quad \int_I V = h \sum_{j=0}^N v_{2j+1}.$$

For the sake of simplicity, we have used the same notation for the integral, since according to the context no confusion is possible.

Now we can state discrete Green's formula:

Lemma 7.1

For all $U \in L_e$ and $V \in L_o$, we have

$$\int_I U \partial_h V = - \int_I V \partial_h U + \int_{\partial I} \gamma U \gamma V \nu, \quad (60)$$

where

$$\int_{\partial I} w = w_0 + w_n,$$

and the outward normal vector ν takes the two values: $\nu_0 = -1, \nu_n = 1$.

Proof. By definition, we have

$$\int_I U \partial_h V = \sum_{j=1}^N u_{2j} (v_{2j+1} - v_{2j-1}).$$

By a discrete change of variables, we get

$$\begin{aligned} \int_I U \partial_h V &= \sum_{j=1}^N u_{2j} v_{2j+1} - \sum_{k=0}^{N-1} u_{2k+2} v_{2k+1} \\ &= \sum_{j=0}^N v_{2j+1} (u_{2j} - u_{2j+2}) + u_{2N+2} v_{2N+1} - u_0 v_1. \end{aligned}$$

We conclude by using the definition of the trace of V . □

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