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Deforming syzygies of liftable modules and generalised Knörrer functors

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Abstract

Maps between deformation functors of modules are given which generalise the maps induced by the Knörrer functors. These maps become isomorphisms after introducing certain equations in the target functor restricting the Zariski tangent space. Explicit examples are given on how the isomorphisms extend results about deformation theory and classification of MCM modules to higher dimensions.

1. Introduction

In this article we show that the mini-versal deformation space of a module on a hypersurface section (and more generally a complete intersection) of a singularity under certain conditions is given as the intersection of hypersurfaces in the mini-versal deformation space of another module on the ambient singularity.

A fundamental idea is to characterise singularities by properties of their module categories. In general this seems to be difficult. The question of which singularities have finite CM type is despite much study still not settled, cf. [34]. Similar questions: Which singularities have modular families of indecomposable MCM modules, does such families appear for infinitely many ranks, is the dimension of the parameter spaces unbounded (the "geometrically wild" case) or bounded (by 1; the "tame" case)? How are properties of the singularities of the mini-versal deformation spaces of MCM modules on an isolated singularity X related to the properties of X?

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Results in this direction are typically obtained for special classes in low dimension by explicit calculation. Then one may try to extend the results to higher dimensions. H. Knörrer introduced in 1987 a functor H which gives an equivalence between the stable categories of MCM modules on the hypersurface singularities $f(\mathbf{x})$ and $f(\mathbf{x})+uv$, see [26]. It followed that all simple hypersurface singularities are of finite CM type since this was known in dimension 1 and 2.

For curve singularities, the tame/wild dichotomy is well understood by the work of Y. Drozd and G.-M. Greuel [10, 11]. For normal surface singularities the finiteness of rank one MCM modules (resp. finite CM type) characterises the rational (resp. the quotient) singularities [29, 16, 3]. The tame/wild dichotomy has been confirmed in the minimally elliptic case, see [12] which applies results of C. Kahn [25, 24].

In general one has to expect a richer picture to emerge. In [9] R.-O. Buchweitz and G. Leuschke prove that there is no finite dimensional variety parameterising rank two MCM modules on the determinant of the generic $n \times n$-matrix for $n \geqslant 3$, while the rank one MCM modules are infinitesimally rigid by [8] and [21].

For MCM modules on isolated singularities there exists versal deformation spaces, see [33], but specific knowledge about these spaces is hard to come by. A. Ishii has constructed certain natural resolutions of components of the (reduced) versal deformation space of a (not necessarily indecomposable) MCM module on a rational surface singularity [23]. These resolutions are described for the cone over a rational normal curve of degree m in $\mathbb{P}^m$, see [17].

The main contribution in this article is to provide tools which may extend knowledge about the classification and the deformation theory of modules in low dimensions to modules of higher dimensions. Suppose Y is a complete intersection of n hypersurfaces in a singularity X. A deformation of a finitely generated module M on Y induces a deformation of the n^{th} syzygy module $N = \Omega^n_{\mathcal{O}_X} M$ on X. Theorem 1 in particular shows that if mini-versal deformation spaces S_Y and S_X exist for M and N, then S_Y is the intersection of hypersurface sections in S_X. The essential condition in Theorem 1 is the existence of a (non-flat) *lifting* of M to the double structure $\mathcal{O}_X/I_Y^2$ of Y in X. By a result of M. Auslander, S. Ding and Ø. Solberg [4], M is a direct summand of N restricted to Y, which is crucial in the proof. Without the lifting condition the result is wrong, see Example 2.

In Theorem 2 the lifting condition is substituted with the (stronger) requirement that $\mathcal{O}_X/I_Y^2 \twoheadrightarrow \mathcal{O}_Y$ should split. The conclusion is that there is an isomorphism of the deformation functor Def^Y_M with a naturally defined subfunctor of Def^X_N for all M. Invoking the Kodaira-Spencer map, we define a modular family of modules, and Corollary 1 says that X has such a family of dimension d if Y has. Corollary 2 says that the hypersurface singularities $F = f(\mathbf{x}) + y_1^{n_1} + \cdots + y_r^{n_r}$ with $n_i \geqslant 3$ are geometrically wild if f is.

In Theorem 3 an intermediate singularity (i.e. $Y \leftarrow Z \rightarrow X$) is included in the setup, and the syzygy is to be taken of the pullback M' of M to this space. Theorem 3 in particular covers Knörrer's functor H which for MCM M gives an isomorphism of deformation functors $\mathrm{Def}^Y_M \cong \mathrm{Def}^X_{H(M)}$, see Corollary 4. This result was first published by G. Pfister and D. Popescu (in a slightly more restricted situation), see [31], but was already proved by the author in his (unpublished) Master's thesis [18].

For Theorem 3 we have to prove a splitting result in our situation, and a tensor product of free resolutions with *Eisenbud systems* which adds regular sequences (see Definition 8) is an essential part of the proof.

In the last section the splitting condition in Theorem 2 is reformulated and in particular considered for cones over smooth projective varieties. In Corollary 5 we give two explicit examples of (restricted) versal deformation spaces of MCMs on the cone of the rational normal scroll in $\mathbb{P}^4$.

Some conventions and definitions used throughout the article: *A local k-algebra* A is a Henselian k-algebra (in particular local and Noetherian) where k is a field. In connection with existence of versal deformations we assume that A is *algebraic*, i.e. A is the Henselisation of a k-algebra of finite type. *An A-module* M is (if clearly not otherwise) a finitely generated A-module. For a Noetherian k-algebra A, let A_S be the Henselisation of $A \otimes_k S$ in the ideal $A \otimes_k \mathfrak{m}_S$ where S is an object in the category Hens_k of Henselian k-algebras with residue field k. A *deformation* of an A-module M to S is an A_S-module M_S, flat as S-module, together with an A_S-linear map $\pi : M_S \to M$ inducing an isomorphism $\pi \otimes_S k : M_S \otimes_S k \xrightarrow{\simeq} M$. Two deformations are equivalent if they are isomorphic over M. The *deformation functor* $\mathrm{Def}^A_M : \mathsf{Hens}_k \to \mathsf{Sets}$ associates to S the set of equivalence classes of deformations M_S of M to S. Maps are induced by tensorisation.

Most of the results in this article will suitably adapted hold in the graded case as well.

2. Deforming higher syzygies of a liftable module

In Theorem 1 the syzygy gives a formally smooth map between deformation functors of modules on different rings. Theorem 2 shows that this is an isomorphism for all modules if the double structure splits. Corollary 1 and 2 are applications to questions about modular families. In Proposition 1 a grade condition implies that the syzygy map is an isomorphism of deformation functors.

Let A be a local ring, then the (first) *syzygy* of an A-module M is the A-module $\Omega_A M = \operatorname{im} d_1$ where $F = (F, d)$ is a minimal A-free resolution $\ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$ of M (suppressing the augmentation map). We have that $\Omega_A M$ is uniquely defined up to a non-unique isomorphism. Let $\tau : A \to B$ be a map of local rings, N a B-module (hence also an A-module via τ) and let $\pi : M \to N$ be a map of A-modules. Define $\Omega\pi : \Omega_A M \to \Omega_B N$ to be any choice of lifting of π to a map of the first syzygies after fixing minimal resolutions. The n^{th} syzygy, denoted $\Omega^n_A M$, is defined by induction.

The following two lemmas and definitions are vital prerequisites for the rest of the article.

Lemma 1

Suppose A is a local k-algebra and M a finitely generated A-module, then there is a map

$$\mathrm{Def}^A_M \longrightarrow \mathrm{Def}^A_{\Omega_A M}$$

defined by sending $\pi : M_S \to M$ to $\Omega\pi : \Omega_{A_S} M_R \to \Omega_A M$. The map is functorial for isomorphisms in M and in particular independent of the choice of minimal resolution.

The proof is straight forward checking and is left to the interested reader.

Lemma 2

Suppose $C \to A$ is a map of local k-algebras and N is a finitely generated C-module, let $M = N \otimes_C A$. If $\mathrm{Tor}_1^C(N, A) = 0$ then there is a map $\mathrm{Def}_N^C \to \mathrm{Def}_M^A$ given by $[N_S] \mapsto [N_S \otimes_{C_S} A_S]$.

Proof. The map respects the equivalence relation, we have to show that $M_S := N_S \otimes_{C_S} A_S$ is S-flat. By the local criterion of flatness it is sufficient to show that $\mathrm{Tor}_1^S(M_S, k) = 0$. This follows from the 5-term exact sequence of the spectral sequence

$$\mathrm{E}^2_{pq} = \mathrm{Tor}_p^S(\mathrm{Tor}_q^{C_S}(N_S, A_S), k) \Rightarrow \mathrm{Tor}_{p+q}^C(N, A). \qquad \square$$

Definition 1 Suppose $C \to A$ is a surjective map of rings with kernel I and M is an A-module. Then M has a *lifting* to C if there is a C-module N and a C-linear map $\pi : N \to M$ such that $\mathrm{Tor}_1^C(N, A) = 0$ and $\pi \otimes A : N \otimes_C A \to M$ is an isomorphism. There exists an obstruction class

$$\mathrm{o}(C/I^2, M) \in \mathrm{Ext}_A^2(M, M \otimes_A I/I^2) \tag{1}$$

such that $\mathrm{o}(C/I^2, M) = 0$ if and only if M has a lifting to C/I^2, see [22, IV 3.1] (cf. also [27] and [20, Theorem 1]).

The Zariski tangent space $\mathrm{Def}_M^A(k[\varepsilon])$ is naturally a k-vector space and there is a canonical isomorphism $\mathrm{Def}_M^A(k[\varepsilon]) \cong \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M}, \mathrm{M})$ of k-vector spaces. If S in Hens_k satisfies $\mathfrak{m}_S^2 = 0$, there is also a natural isomorphism $\mathrm{Def}_M^A(S) \cong \mathrm{Def}_M^A(k[\varepsilon]) \otimes_k \mathfrak{m}_S$. In particular; to a finite dimensional k-vector subspace $W \subseteq \mathrm{Def}_M^A(k[\varepsilon])$, there is a canonical deformation M_W in $W \otimes_k W^* \subseteq \mathrm{Def}_M^A(k[W^*])$ where $k[W^*] = k \oplus W^*$. By standard obstruction calculus there is a complete k-algebra $H = H(W)$, which is a quotient of the power series ring $T = (\mathrm{Sym}_k W^*)\hat{}$, and a pro-object $\hat{M}_H = (M_n)$ with M_n a deformation of M to $H/\mathfrak{m}^{n+1}$ which induces M_{n-1} for all n, obtained by lifting $M_1 = M_W$ successively to maximal quotients of $T/\mathfrak{m}^{n+1}$, see [27, 32, 20]. Restricting to the full subcategory Art_k of finite length objects in Hens_k, the pair $(H, \hat{M}_H)$ defines a map $h_W : \mathrm{Hom}_{\mathsf{Art}_k}(H, -) \to \mathrm{Def}_M^A$ of functors.

Definition 2 Let V be a k-vector subspace in $\mathrm{Def}_M^A(k[\varepsilon])$. Then $\mathrm{Def}^A_{(M,V)}$ is defined as the subfunctor of Def_M^A of deformations M_S with the property that $M_S \otimes_S S/\mathfrak{m}^{n+1}$ is in the image of h_W for some finite dimensional k-vector subspace W of V for each n.

Definition 3 Let F and G be functors $F, G : \mathsf{Hens}_k \to \mathsf{Sets}$ with $\#F(k) = \#G(k) = 1$. A map $\varphi : F \to G$ is *smooth* if $f_\varphi : F(R) \to F(S) \times_{G(S)} G(R)$ is surjective for all surjections $\pi : R \to S$ in Hens_k, and φ is *formally smooth* if f_φ is surjective for all surjections $R \to S$ in Art_k.

If S is algebraic and $\xi_S \in F(S)$, then (S, ξ_S) is a (formally) *versal family* for F if the induced map $\mathrm{Hom}_{\mathsf{Hens}_k}(S, -) \to F$ is (formally) smooth, and a (formally) *mini-versal* family if the map in addition is an isomorphism at $k[\varepsilon]$. See [2].

Theorem 1

Let $\pi : C \to A$ be a surjective map of local k-algebras. Set $I = \ker \pi$, and assume I is generated by a regular sequence of length n and M is a finitely generated A-module which has a lifting to C/I^2. Then the following holds:

(1) *There is a map of deformation functors*

$$\sigma : \mathrm{Def}_M^A \longrightarrow \mathrm{Def}^C_{(\Omega^n_C M, V)},$$

where

$$V = \mathrm{im}\left\{\mathrm{Def}^A_M(k[\varepsilon]) \to \mathrm{Def}^C_{\Omega^n_C M}(k[\varepsilon])\right\},$$

which is formally smooth and an isomorphism at tangent spaces.

(2) *If (S, M_S) is a versal family for Def^A_M, then $(S, \sigma(M_S))$ is a versal family for $\mathrm{Def}^C_{(\Omega^n_C M, V)}$.*

Remark 1 If M is locally free on the complement of the closed point in $\mathrm{Spec}\, A$, then, using [2] and [15, Theorem 3], it is shown in [33] that there exists a versal family for Def^A_M. I.e. there exists a deformation M_S for an algebraic S such that the map of functors $\mathrm{Hom}_{\mathsf{Hens}_k}(S, -) \to \mathrm{Def}^A_M$ induced by M_S is smooth. It follows that there exists a finitely generated k-algebra S^{ft} with a k-point t_0 and an $A \otimes S^{\mathrm{ft}}$-module $M_{S^{\mathrm{ft}}}$ flat as S^{ft}-module inducing M_S at t_0. By openness of versality [2, 4.4 and 3.7] (cf. [23, 2.13]) we may assume that it is versal at all k-points in $\mathrm{Spec}\, S^{\mathrm{ft}}$.

Example 1 If L is an A-module, set $N = \Omega^m_C L$ for $m \geqslant n$. Then $M = N \otimes_C A$ satisfies the conditions of Theorem 1 since $\mathrm{Tor}^C_1(N, A) = \mathrm{Tor}^C_{m+1}(L, A) = 0$ which implies that $\mathrm{Tor}^{C/I^2}_1(N \otimes_C C/I^2, A) = 0$.

Let N be an MCM C-module, then $\mathrm{Tor}^C_1(N, A) = 0$ since I is N-regular (cf. [14, 21.9]). Hence $M = N \otimes_C A$ is liftable to C/I^2 and satisfies the conditions of Theorem 1.

Example 2 Theorem 1 is not true without the existence of a lifting of M to C/I^2. If $C = k[x]^{\mathrm{h}}$, $f = x^2$, $A = C/(f)$ and $M = k$, then $\Omega_C k \cong C$, hence is (infinitesimally) rigid as C-module while $\mathrm{Def}^A_M \cong \mathrm{Hom}(A, -)$ by Proposition 1 and in particular has non-trivial Zariski tangent space. By Lemma 7 (or direct calculation) $\mathrm{o}(C/(f^2), M) \neq 0$.

The deformation functors may also have isomorphic Zariski tangent spaces, but differ in the obstructions: Let $0 \neq f \in \mathfrak{m}^2_P$ where P is a local, regular k-algebra with k as residue field. If $A = P/(f)$, $M = k$ and $C = P[v]^{\mathrm{h}}/(f + v) \cong P$, then again $\mathrm{Def}^A_M \cong \mathrm{Hom}(A, -)$ and in particular M has obstruction given by f. But $\mathrm{Def}^C_{\Omega_C M} \cong \mathrm{Def}^C_k$ if $\dim P \geqslant 3$ by Proposition 1, hence $\mathrm{Def}^C_{\Omega_C M}$ is smooth. By Lemma 7, $\mathrm{o}(C/(f^2), M) \neq 0$.

Lemma 3

If $C \to A$ is a surjective map of local rings and the kernel I is generated by a regular sequence, then for any $n \geqslant 0$ there is a map of A-modules

$$\mathrm{Ext}^2_A(M, M \otimes_A I/I^2) \longrightarrow \mathrm{Ext}^2_A(\Omega^n_A M, \Omega^n_A M \otimes_A I/I^2) \tag{2}$$

which takes $\mathrm{o}(C/I^2, M)$ to $\mathrm{o}(C/I^2, \Omega^n_A M)$.

Proof. For all $i > 0$ there are quite generally natural syzygy maps

$$\omega^i : \operatorname{Ext}^i_A(M, M) \to \operatorname{Ext}^i_A(\Omega_A M, \Omega_A M) \tag{3}$$

obtained by composing the connecting map $\operatorname{Ext}^i_A(M, M) \to \operatorname{Ext}^{i+1}_A(M, \Omega_A M)$ with the inverse of the connecting isomorphism

$$\operatorname{Ext}^i_A(\Omega_A M, \Omega_A M) \xrightarrow{\simeq} \operatorname{Ext}^{i+1}_A(M, \Omega_A M).$$

Since I/I^2 is A-free of finite rank,

$$\operatorname{Ext}^2_A(M, M \otimes_A I/I^2) \cong \operatorname{Ext}^2_A(M, M) \otimes_A I/I^2.$$

The map in the lemma is ω^2 iterated n times tensored with I/I^2. In the Yoneda complex this is simply to chop off the first n maps. $\square$

Remark 2 Let M_S be a deformation of M in $\operatorname{Def}^A_M(S)$ and $\pi : R \to S$ a *small* surjection (i.e. $\mathfrak{m}_R \cdot \ker \pi = 0$), then there is a an obstruction class $\mathrm{o}_A(\pi, M_S) \in \operatorname{Ext}^2_A(M, M) \otimes_k \ker \pi$ which vanish if and only if there exists a deformation M_R of M to R such that $M_R \otimes_R S$ is equivalent to M_S, see [27]. Since $- \otimes_k \ker \pi$ may be taken outside the Ext^2, it follows as in Lemma 3 that

$$\omega^2 \otimes \mathrm{id}_{\ker \pi}(\mathrm{o}_A(\pi, M_S)) = \mathrm{o}_A(\pi, \Omega_{A_S} M_S) \in \operatorname{Ext}^2_A(\Omega_A M, \Omega_A M) \otimes_k \ker \pi.$$

Proof of Theorem 1 Since both functors are locally of finite presentation, it follows from [2, 3.7] that an algebraic family is versal if it is formally versal. The second part of the theorem therefore follows from the first since the composition

$$\operatorname{Hom}_{\mathsf{Hens}_k}(S, -) \to \operatorname{Def}^A_M \to \operatorname{Def}^C_{(\Omega^n_C M, V)}$$

of two formally smooth maps is formally smooth.

A deformation of M as A-module is also a deformation of M as C-module, which gives a map $\operatorname{Def}^A_M \to \operatorname{Def}^C_M$. By Lemma 1 there is a map $\operatorname{Def}^C_M \to \operatorname{Def}^C_{\Omega^n_C M}$, and by Lemma 2 there is a map $\operatorname{Def}^C_{\Omega^n_C M} \to \operatorname{Def}^A_{\Omega^n_C M \otimes_C A}$ since $\operatorname{Tor}^C_1(\Omega^n_C M, A) = \operatorname{Tor}^C_{n+1}(M, A) = 0$ by assumption. The resulting map $\operatorname{Def}^A_M \to \operatorname{Def}^C_{\Omega^n_C M}$ factors via the inclusion through a map $\sigma : \operatorname{Def}^A_M \to \operatorname{Def}^C_{(\Omega^n_C M, V)}$. By [4, 3.6] $\Omega^n_C M \otimes_C A$ contains M as a direct summand if M is liftable to C/I^2 with the additional assumption that $\operatorname{Tor}^{C/I^2}_i(N, A) = 0$ for all $i > 0$. However we claim that $\operatorname{Tor}^{C/I^2}_1(N, A) = 0 \Rightarrow \operatorname{Tor}^{C/I^2}_i(N, A) = 0$ for all $i > 0$. If (F, d) is an A-free resolution of M with differential d, let $(\widetilde{F}, \tilde{d})$ be a lifting of (F, d) to a map $\tilde{d}$ of a graded module $\widetilde{F}$ which is C/I^2-free in each degree. Then $(\tilde{d})^2$ is induced by a cocycle $\varphi \in \operatorname{Hom}^2_A(F, F \otimes_A I/I^2)$. Since I/I^2 is A-free we have

$$\mathrm{o}(C/I^2, M) = [\varphi] \in \mathrm{H}^2 \operatorname{Hom}_A(F, F) \otimes_A I/I^2 = \operatorname{Ext}^2_A(M, M) \otimes_A I/I^2.$$

Since $\mathrm{o}(C/I^2, M) = 0$, there is a $\psi \in \operatorname{Hom}^1_A(F, F) \otimes_A I/I^2$ with $\partial \psi = \varphi$. Adjusting $\tilde{d}$ by ψ gives a differential $\tilde{d}'$ on $\widetilde{F}$, i.e. $(\tilde{d}')^2 = 0$, hence $F \otimes_A I/I^2 \xrightarrow{\iota} \widetilde{F} \xrightarrow{\pi} F$ is a short exact sequence of complexes and by the long exact homology sequence, $(\widetilde{F}, \tilde{d}')$ is a C/I^2-free resolution of N. Tensoring $(\widetilde{F}, \tilde{d}')$ with A gives (F, d) and hence $\mathrm{Tor}_i^{C/I^2}(N, A) = 0$ for all $i > 0$.

We have obtained a natural map

$$\tau : \mathrm{Def}_M^A \to \mathrm{Def}_{(M\oplus Y, V')}^A; \quad M_S \mapsto \tau M_S := \Omega_{C_S}^n M_S \otimes_{C_S} A_S \tag{4}$$

where $\Omega_C^n M \otimes_C A \cong M \oplus Y$ for some finitely generated A-module Y, and $V' = \mathrm{im}(\mathrm{id}, \eta^1)$ where

$$(\mathrm{id}, \eta^i) : \mathrm{Ext}_A^i(M, M) \hookrightarrow \mathrm{Ext}_A^i(M\oplus Y, M\oplus Y) \quad \text{for } i > 0 \tag{5}$$

is the composition of $\mathrm{Ext}_A^i(M, M) \to \mathrm{Ext}_C^i(M, M)$, the n^{th} iterate $(\omega^i)^n$ of (3), and the natural map $\mathrm{Ext}_C^i(\Omega, \Omega) \to \mathrm{Ext}_A^i(\overline{\Omega}, \overline{\Omega})$ obtained by tensorisation and the collapse of the spectral sequence $\mathrm{E}_2^{pq} = \mathrm{Ext}_A^p(\mathrm{Tor}_q^C(\Omega, A), \overline{\Omega}) \Rightarrow \mathrm{Ext}_C^{p+q}(\Omega, \overline{\Omega})$ (where $\Omega = \Omega_C^n M$ and $\overline{\Omega} = \Omega \otimes_C A$).

Formal smoothness of $\sigma : \mathrm{Def}_M^A \to \mathrm{Def}_{(\Omega_C^n M, V)}^C$ follows: If $\pi : R \to S$ is a small surjection we have

$$(\mathrm{id}, \eta^2)(\mathrm{o}_A(\pi, M_S)) = \mathrm{o}_C(\pi, \sigma M_S) \otimes_C A = \mathrm{o}_A(\pi, \tau M_S)$$

and since (id, η^2) is injective $\mathrm{o}_C(\pi, \sigma M_S) = 0 \Rightarrow \mathrm{o}_A(\pi, M_S) = 0$ (see Remark 2). $\square$

A stronger condition gives a stronger conclusion than in Theorem 1:

Theorem 2

Let $\pi : C \to A$ be a surjective map of local k-algebras. Set $I = \ker \pi$, assume I is generated by a regular sequence of length n, and assume the induced map $C/I^2 \to A$ has a section $A \to C/I^2$. Then for all finitely generated A-modules M there is an isomorphism of deformation functors

$$\sigma : \mathrm{Def}_M^A \xrightarrow{\simeq} \mathrm{Def}_{(\Omega_C^n M, V)}^C$$

where

$$V = \mathrm{im}\left\{\mathrm{Def}_M^A(k[\varepsilon]) \to \mathrm{Def}_{\Omega_C^n M}^C(k[\varepsilon])\right\}.$$

Remark 3 The existence of a splitting $A \to C/I^2$ implies that $\mathrm{o}(C/I^2, M) = 0$ for all A-modules M since $N = C/I^2 \otimes_A M$ gives a lifting of M to C/I^2.

Criteria for the existence of a splitting are discussed in the last section. For now we remark that a splitting exists if the equations $(\mathbf{F})$ defining C are given as "deformations" of the equations $(\mathbf{f})$ defining A by parameters $(\mathbf{v}) \subseteq I$ which only occur in expressions of $\mathbf{v}$-degree $\geqslant 2$ as in $F = f + v_1 v_2 g$. More precisely suppose $A = P/(\mathbf{f})$ for a local ring P and suppose there is a map $\sigma : P \to C$ commuting with the maps to A. Then σ induces a splitting if $\sigma(\mathbf{f}) \subseteq I^2$. For a class of examples, see Theorem 3 (with $p = 0$).

Proof. For the extension of the surjectivity assured in Theorem 1 to all deformations we proceed as follows: Given a deformation L_S in $\mathrm{Def}_{(\Omega_C^n M, V)}^C(S)$. By formal smoothness of σ there in particular exists a compatible system of deformations M_i of M to $S_i = S/\mathfrak{m}_S^{i+1}$ and isomorphisms $\varphi_i : \sigma M_i \xrightarrow{\simeq} L_i := L_S \otimes_{C_S} C_{S_i}$ for all $i > 0$. By induction and [28, 22.1] we get an $\hat{S}$-flat $\hat{A}_{\hat{S}} := A\hat{\otimes}_k\hat{S}$-module $\hat{M}_{\hat{S}}$ and an isomorphism $\hat{\varphi} : \Omega^n_{\hat{C}_{\hat{S}}}\hat{M}_{\hat{S}} \xrightarrow{\simeq} \hat{L}_{\hat{S}}$.

Let $L = L_S \otimes_{C_S} A_S$, and let $\hat{L} = L \otimes_{A_S} \hat{A}_S$ be the completion of L. Observe that the splitting $A \to C/I^2$ induces a splitting $\hat{A}_{\hat{S}} \to \hat{C}_{\hat{S}}/\hat{I}^2_{\hat{S}}$. Via the isomorphism induced from $\hat{\varphi}$ and the splitting $\Omega^n_{\hat{C}_{\hat{S}}}\hat{M}_{\hat{S}} \otimes_{\hat{C}_{\hat{S}}} \hat{A}_{\hat{S}} = \hat{M}_{\hat{S}} \oplus Y$ assured by [4, 3.6], there is a map $L \to \hat{M}_{\hat{S}}$. Let M_S be defined as the image of L under this map. Then M_S is a finitely generated A_S-module, and the completion of M_S is $\hat{M}_{\hat{S}}$. From [28, 7.11] it follows that there exists a map $\varphi_S : \sigma M_S \to L_S$ inducing φ_1. By [28, 22.5] φ_S is injective and $\operatorname{coker}\varphi_S$ is S-flat. Since $\varphi_S \otimes_S k$ is an isomorphism, it follows that $\operatorname{coker}\varphi_S = 0$ and φ_S is an isomorphism. Hence σM_S is equivalent to the deformation L_S and σ is surjective.

To get injectivity of σ we prove injectivity of τ in (4). Assume $\varphi : \tau M_S \xrightarrow{\simeq} \tau M'_S$. Restricting φ to the direct summand M_S and composing with the projection $\tau M'_S \to M'_S$ gives a map $\psi : M_S \to M'_S$ compatible with the structure maps to M. By [28, 22.5] ψ is an isomorphism as above, hence τ is injective and so is σ. □

Remark 4 With the conditions in Theorem 2, one similarly shows that τ in (4) is an isomorphism. Moreover; we have maps

$$\mathrm{Def}^A_M \xrightarrow{\alpha} \mathrm{Def}^C_{(M,V_1)} \to \mathrm{Def}^C_{(\Omega^n_C M, V_2)} \to \mathrm{Def}^A_{(\Omega^n_C M \otimes_C A, V_3)} \xrightarrow{\beta} \mathrm{Def}^A_M \tag{6}$$

(where the V_i are the images of $\mathrm{Def}^A_M(k[\varepsilon])$) which all except β exist without the splitting condition or $\mathrm{o}(C/I^2, M) = 0$.

Let M and M' be A-modules and $A = C/I$ any quotient ring. In [20] an obstruction theory for Def^A_M as a subfunctor of Def^C_M is given. Let M_S be a deformation of M as A-module. If the obstruction class o_C for deforming M_S along a small surjection $R \to S$ as C-module is zero, there exists a secondary class o_I which vanish if and only if there is a deformation of M_S as A-module, see [20, Theorem 1]. Moreover, there is a change of rings spectral sequence

$$\mathrm{E}^{pq}_2 = \mathrm{Ext}^p_A(M, \mathrm{Ext}^q_C(A, M')) \Rightarrow \mathrm{Ext}^{p+q}_C(M, M')$$

with d_2-differential $\mathrm{Hom}_A(M, \mathrm{Ext}^1_C(A, M')) \xrightarrow{d_2} \mathrm{Ext}^2_A(M, M')$ induced by cupping with $\mathrm{o}(C/I^2, M) \in \mathrm{Ext}^2_A(M, M \otimes_B I/I^2)$ via the isomorphism

$$\mathrm{Hom}_A(M \otimes_A I/I^2, M') \cong \mathrm{Hom}_A(M, \mathrm{Ext}^1_C(A, M')),$$

see [22, IV 3.1] and [20, Proposition 3]. In [20, Theorem 4] it is shown that o_I is in the image of d_2, hence is zero if $\mathrm{o}(C/I^2, M) = 0$. It follows that α in (6) is formally smooth and tangentially an isomorphism, and if the splitting condition holds, α is an isomorphism.

Modular families

DEFINITION 4 *An S-family of A-modules M_S* is an S-flat, finitely generated $A \otimes_k S$-module with S finitely generated as k-algebra. Then the *Kodaira-Spencer map g_S* :

$\mathrm{Der}_k(S) \to \mathrm{Ext}^1_{A\otimes S}(M_S, M_S)$ is defined by $D \mapsto [D, d^S]$ where d^S is the differential in an $A\otimes_k S$-free resolution of M_S. For each point $t \in \mathrm{Spec}\, S$ there is a *local Kodaira-Spencer map* $g_t : \mathrm{Der}_k(S, k(t)) \to \mathrm{Ext}^1_A(M_t, M_t)$ *at* t induced by g_S where $M_t = M_S \otimes_S k(t)$.

Lemma 4

Let $T_{S,t}$ be the Zariski tangent space of S at the k-point t. The local Kodaira-Spencer map at t equals the natural map $T_{S,t} \to \mathrm{Ext}^1_A(M_t, M_t)$ induced by M_S via the natural isomorphism $\mathrm{Der}_k(S, k(t)) \cong T_{S,t}$.

Proof. Let $k = k(t)$, we may assume S is local. Since

$$\mathrm{Der}_k(S, k) \cong \mathrm{Hom}_k(\Omega_{S/k} \otimes_S k, k) \cong T_{S,0}\,, \tag{7}$$

we may assume $\mathfrak{m}_S^2 = 0$. The differential of an $A\otimes S$-free resolution of M_S can then be written as $d + \sum \xi_i \otimes s_i$ where d is the differential of an A-free resolution F of M, the ξ_i are representatives for a basis of $\mathrm{Ext}^1_A(M, M)$ in the Yoneda complex, and the s_i are elements in $\mathfrak{m}_S$. If $\varphi_D \in T_{S,0}$ is the homomorphism corresponding to $D \in \mathrm{Der}_k(S, k)$ then a calculation shows that $g_0(D)$ is represented by

$$\sum \varphi_D(s_i)\xi_i \in \mathrm{Hom}^1_A(F, F)$$

and hence g_0 is the natural map $T_{S,0} \to \mathrm{Ext}^1_A(M, M)$ induced by the deformation M_S. □

Definition 5 Suppose k is algebraically closed. We say that an S-family of A-modules M_S is *modular* if S is regular and g_t is injective for all k-points $t \in \mathrm{Spec}\, S$. If in addition $\dim S = d$ we say that M_S is a modular family of dimension d.

A local Cohen-Macaulay k-algebra A is *geometrically wild* if for infinitely many d there exists modular families M_S of dimension d such that M_t is indecomposable and maximal Cohen-Macaulay for all k-points $t \in \mathrm{Spec}\, S$.

Remark 5 This definition of geometrically wild differs slightly from the usual one since we allow finite repetition of isomorphism classes in the family, cf. [12].

With notation as in Theorem 2, we have:

Corollary 1

Assume $k = \overline{k}$. If A has a modular family M_S of dimension d (e.g. of maximal Cohen-Macaulay modules), so does C.

Proof. There is an open locus $U_0 \subseteq \mathrm{Spec}\, S$ where M_t is generated by the smallest number of generators μ_0. Localising S (and M_S) sufficiently, we obtain a free cover $(C\otimes S)^{\mu_0} \xrightarrow{\varepsilon} M_S$ which is minimal at all k-points t, i.e. $\ker(\varepsilon)\otimes_S k(t) \cong \Omega_A M_t$. Iterating the argument yields the $n+2$ first terms $F^S_{n+1} \xrightarrow{d_{n+1}} \dots \to F^S_0 \xrightarrow{\varepsilon} M_S$ in a $C\otimes S$-free resolution of M_S inducing minimal presentations $F_{i+1} \xrightarrow{d_{i+1}\otimes_S k(t)} F_i \twoheadrightarrow \Omega^i_C M_t$ for all k-points t and $i = 0, \dots, n$.

One checks that the change of rings map

$$\operatorname{Ext}^1_A(M,M)\to\operatorname{Ext}^1_C(M,M)\ \text{ and }\ \omega^1:\operatorname{Ext}^1_C(M,M)\to\operatorname{Ext}^1_C(\Omega M,\Omega M)$$

commutes with the (local) Kodaira-Spencer map. Since

$$\sigma:\operatorname{Ext}^1_A(M_t,M_t)\to\operatorname{Ext}^1_C(\Omega^n M_t,\Omega^n M_t)$$

is injective for all k-points t by Theorem 2, it follows that d_{n+1} presents a modular family of C-modules of dimension d. If M_S is a family of MCM modules, so is $\operatorname{coker} d_{n+1}$. □

Corollary 2

Assume $k=\overline{k}$. Suppose $C=k[\mathbf{x},\mathbf{v}]^{\mathrm{h}}/(F)$ where F is given as

$$F=f(\mathbf{x})+v_1^{n_1}+\cdots+v_q^{n_q}\quad\text{with } n_i\geqslant 3\ \text{ for all } i,$$

and $A=k[\mathbf{x}]^{\mathrm{h}}/(f)$. If A is geometrically wild, then so is C.

Proof. The conditions in Theorem 2 are satisfied, and by the proof of Corollary 1, we only need the indecomposability of $\Omega^q_C M$ for all indecomposable MCM A-modules M, which is asserted by a result of J. Yoshino [35, Theorem 4.1]. □

A grade condition

The following result gives modules of different depths and dimensions which have isomorphic deformation functors.

Proposition 1

Let M be a finitely generated A-module where A is a local k-algebra. If $\operatorname{Ext}^i_A(M,A)=0$ for all $0<i<g$, and $g\geqslant 3$, then

$$\operatorname{Def}^A_M\xrightarrow{\simeq}\operatorname{Def}^A_{\Omega M}\xrightarrow{\simeq}\ldots\xrightarrow{\simeq}\operatorname{Def}^A_{\Omega^{g-2}M}\,.$$

In particular; if k is the residue field of A, and if A is the A_A-module defined via the multiplication map $A_A\xrightarrow{m}A$, then $(A,\Omega^i_{A_A}A)$ is a mini-versal family for $\operatorname{Def}^A_{\Omega^i k}$ for all $0\leqslant i\leqslant d-2$ where $d=\operatorname{depth}A$.

Proof. Assume $\operatorname{Ext}^1_A(M,A)=\operatorname{Ext}^2_A(M,A)=0$, we show that $\operatorname{Def}^A_M\to\operatorname{Def}^A_{\Omega_A M}$ in Lemma 1 is an isomorphism. For surjectivity, let $(\Omega M)_S\in\operatorname{Def}^A_{\Omega_A M}(S)$ and choose a minimal A_S-free resolution $\ldots\to F_2^S\to F_1^S\twoheadrightarrow(\Omega M)_S$, then a minimal A-free resolution $\ldots\to F_2\to F_1\xrightarrow{d_1}F_0\twoheadrightarrow M$ is obtained by extending $F^S\otimes_S k$. Let

$$b^i(k):\operatorname{Ext}^i_{A_S}((\Omega M)_S,A_S)\otimes_S k\to\operatorname{Ext}^i_A(\Omega M,A)$$

be the natural map which necessarily is surjective for $i=1$. From [1, 1.9] we get that $\operatorname{Ext}^1_{A_S}((\Omega M)_S,A_S)=0$, which in particular is S-flat. From [1, 1.9] we get that $b^0(k)$ is an isomorphism, hence that $\operatorname{Hom}_{A_S}((\Omega M)_S,A_S)\to\operatorname{Hom}_A(\Omega M,A)$ is surjective. We can therefore lift the map $F_0^\vee\to(\Omega M)^\vee$ to a map $\rho_0:(F_0^S)^\vee\to((\Omega M)_S)^\vee$ where $F_0^S=$

$F_0 \otimes_A A_S$. Let σ be the composition of ρ_0 with the natural inclusion $((\Omega M)_S)^\vee \hookrightarrow (F_1^S)^\vee$. Let $d_1^S := \sigma^\vee$ and $M_S := \operatorname{coker} d_1^S$. We have $M_S \otimes_S k \cong M$. Consider the 5-term exact sequence of the spectral sequence $\operatorname{Tor}_p^S(\mathrm{H}_q(F^S), k) \Rightarrow \mathrm{H}_{p+q}(F)$:

$$0 \leftarrow \operatorname{Tor}_1^S(\mathrm{H}_0(F^S), k) \longleftarrow \mathrm{H}_1(F) \longleftarrow \mathrm{H}_1(F^S) \otimes_S k \longleftarrow \operatorname{Tor}_2^S(\mathrm{H}_0(F^S), k) \leftarrow \dots$$

From $\mathrm{H}_1(F) = 0$ and the local criterion of flatness it follows that M_S is S-flat and that $\dots \to F_1^S \xrightarrow{d_1^S} F_0^S \twoheadrightarrow M_S$ is an A_S-free resolution. In particular $\Omega M_S = (\Omega M)_S$.

For the injectivity, let $\psi : \Omega M_S \to \Omega M'_S$ be an isomorphism of deformations. Dualisation of the inclusions in F_0^S gives surjective maps since $\operatorname{Ext}_A^1(M, A) = 0$ by [1, 1.9] again. There is a lifting $\tau : (F_0^S)^\vee \to (F_0^S)^\vee$ of $\psi^\vee$ with $\tau \otimes_S k = \mathrm{id}_{F_0}$. Let $\psi_0 := \tau^\vee$, then ψ_0 induces an isomorphism $M_S \to M'_S$ of deformations since it is compatible with ψ.

For the final statement one checks that A as A_A-module is a mini-versal family for Def_k^A, cf. [20, Example 4]. $\square$

Example 3 If A is Gorenstein and M is an MCM A-module then $\operatorname{Ext}_A^i(M, A) = 0$ for $i > 0$, hence $\operatorname{Def}_M^A \cong \operatorname{Def}_{\Omega^n M}^A$ for all $n > 0$. By a Tate resolution we can define $\Omega_A^n M$ as an MCM A-module for all $n \in \mathbb{Z}$, and the isomorphism of deformation functors is valid for all n.

Remark 6 In general we have that $\operatorname{Ext}_A^i(M, A) = 0$ for $i < \operatorname{depth}_{\operatorname{Ann} M} A$, cf. [14, Proposition 18.4]. Proposition 1 implies that the second map in (6) factors through $n-2$ isomorphisms: $\operatorname{Def}_{(M,V)}^C \cong \operatorname{Def}_{(\Omega_C^1 M, V')}^C \cong \dots \cong \operatorname{Def}_{(\Omega_C^{n-2} M, V'')}^C$.

3. Generalised Knörrer functors

In Theorem 3 we introduce an intermediate ring B and obtain a class of examples not covered by Theorem 2. Corollary 3 and 4 applies the result to the Knörrer functors. Lemma 5 gives conditions implying that the map of deformation functors induced by restriction is formally smooth.

Definition 6 If $I(\rho)$ is the ideal generated by the maximal minors of the $l \times m$-matrix ρ with entries from the maximal ideal of a local ring R, then $I(\rho)$ is *determinental* if $\operatorname{depth} I(\rho) = |l - m| + 1$, the maximal possible value, or if ρ is empty (and then $I(\rho) = (0)$).

Let P be a local k-algebra with residue field k, and let Q and R be the Henselisations of the polynomial rings $P[\mathbf{u}]$ and $P[\mathbf{u}, \mathbf{v}]$ respectively, where $\mathbf{u} = \{u_1, \dots, u_p\}$ and $\mathbf{v} = \{v_1, \dots, v_q\}$ are indeterminants. Let $\mathbf{f}$ and $\mathbf{F}$ be b elements from $\mathfrak{m}_P$ and $\mathfrak{m}_R$ respectively. Set $h_i = F_i - f_i \in R$ for $i = 1, \dots, b$. Moreover, let $\psi = (g_{ij})$ be an $l \times m$-matrix ($l \leqslant m$) with $g_{ij} \in Q$, let $\overline{g}_{ij}$ be the image of g_{ij} under the natural map $Q \to Q \otimes_P k = Q_0 \cong k[\mathbf{u}]^{\mathrm{h}}$ and put $\psi_0 = (\overline{g}_{ij})$.

With this notation we have:

Theorem 3

Assume $(\mathbf{f})$ *is a regular sequence and* $I(\psi_0)$ *is a determinental ideal, and let* $A = P/(\mathbf{f})$, $B = Q/((\mathbf{f}) + I(\psi))$ *and* $C = R/(\mathbf{F})$. *For any finitely generated* A*-module* M, *let* $M' = M \otimes_A B$ *which is a* C*-module via the natural surjective map* $C \to B$.

If $h_i \in (\mathbf{v})(\mathbf{u}, \mathbf{v})R$ *for all* i *and* $g_{ij} \in (\mathbf{u})Q$ *for all* i, j, *and* $n = q + m - l + 1$ ($n = q$ *if* ψ *is empty*), *then there is an isomorphism of deformation functors*

$$\sigma : \mathrm{Def}_M^A \xrightarrow{\simeq} \mathrm{Def}^C_{(\Omega^n_C M', V)}$$

where

$$V = \mathrm{im}\left\{\mathrm{Def}^A_M(k[\varepsilon]) \to \mathrm{Def}^C_{\Omega^n_C M'}(k[\varepsilon])\right\}.$$

Moreover; if M *is a maximal Cohen-Macaulay* A*-module, then* $\Omega^n_C M'$ *is a maximal Cohen-Macaulay* C*-module.*

Remark 7 Fixing M and varying B in Theorem 3 gives many C-modules sharing obstructions. E.g. suppose $M = k$ is the residue field of A, then $M' = k \otimes_A B = Q_0/I(\psi_0)$ and $\mathrm{Def}^C_{(\Omega^n_C M', V)}$ has A as versal deformation ring for all the ψ.

Observe that $\mathrm{o}(C/I^2, M')$ may be non-zero even though $C/I^2 \to A$ has a section, which for instance is the case with Knörrer's H-functor, see Remark 12.

For $p = 0$, Theorem 3 is a special case of Theorem 2.

The proof of Theorem 3 is analogous to the proof of Theorem 2, in particular we need to show that M is a direct summand of $\Omega^n_C M' \otimes_C A$. Since we cannot apply [4, 3.6], we construct a C-free resolution of M' from a P-free resolution L of M, a Koszul resolution $K(\mathbf{v})$ of Q and the Eagon-Northcott resolution $\mathcal{F}(\psi)$ of $Q/I(\psi)$. Both L and $K(\mathbf{v})$ carries "Eisenbud systems" with respect to the regular sequences $(\mathbf{f})$ and $(\mathbf{h})$ respectively, and we define a tensor product of complexes with Eisenbud systems which adds the regular sequences. The construction enables us to observe that many differentials in the resolution vanish with the given conditions, and we obtain the desired splitting.

A tensor product of Eisenbud systems

Definition 7 [D. Eisenbud] Let R be a commutative ring and $J = (f_1, \ldots, f_n)$ a sequence of elements in R. An *Eisenbud system* relative to J on an R-complex $L = (L, d^L)$ is a system of R-linear endomorphisms $\{s_\alpha\}$ of L as graded R-module of degree $2|\alpha| - 1 \geqslant 1$, where α is an n-multi index, satisfying

$$s_\alpha d^L + d^L s_\alpha = -\sum_{\beta_1+\beta_2=\alpha} s_{\beta_1} s_{\beta_2} \tag{8}$$

for $|\alpha| > 1$ and $s_i d + d s_i$ is multiplication by f_i on L, see [13].

If L is an R-free resolution of an $A = R/J$-module M, there exists an Eisenbud system on L. Let $S = R[t_1, \ldots, t_n]$ and let $\mathbb{D} = \mathrm{Hom}_{\text{grad.}\,R\text{-alg.}}(S, R)$ (where $\deg t_i = -2$) be the divided power algebra. It has generators $\tau^{(\alpha)}$ which are dual to the t^α and t_i acts on $\mathbb{D}$ by subtracting the i-th index in α by 1 if possible, or else $t_i \cdot \tau^{(\alpha)} = 0$. If we put $s_0 = d^L$ and $d = \sum_\alpha t^\alpha \otimes s_\alpha$ then $\mathbb{D}\otimes L\otimes A = (\mathbb{D}\otimes_R L\otimes_R A, d\otimes 1)$ is a complex of A-free modules, and if $(f_1, \dots, f_n)$ is a regular sequence then $\mathbb{D}\otimes L\otimes A$ is an A-free resolution of M, for details see [13, 7.2] and [5].

DEFINITION 8 If $\mathcal{E} = (L, \{s_\alpha(\mathbf{f})\})$ and $\mathcal{E}' = (L', \{s_\alpha(\mathbf{g})\})$ are Eisenbud systems relative to the sequences $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ in R, then their *sum* tensor product is the Eisenbud system $\mathcal{E}\otimes\mathcal{E}' = (L\otimes_R L', \{s_\alpha(\mathbf{f})\otimes 1 \pm 1\otimes s_\alpha(\mathbf{g})\})$ relative to the sequence $(f_1 + g_1, \dots, f_n + g_n)$.

Remark 8 This definition generalises a definition by Yoshino of a tensor product of two matrix factorisations (see Definition 9 below) over two power series rings in the following sense. If both L and L' have length 1 and $n = 1$, the second and third "differential";

$$\Psi := d_2 = t\otimes(s_1(f)\otimes 1 + 1\otimes s_1(g)) + 1\otimes(s_0(f)\otimes 1 - 1\otimes s_0(g))$$

and

$$\Phi := d_3 = t\otimes(s_1(f)\otimes 1 - 1\otimes s_1(g)) + 1\otimes(s_0(f)\otimes 1 + 1\otimes s_0(g))$$

in $(\mathbb{D}\otimes_R L\otimes_R L', d)$, give a matrix factorisation (Φ, Ψ) of $f + g$ which is equal to the one in [35, 1.2].

Proof of Theorem 3 We have surjections $C \to B$ and $B \to A$, a splitting $A \to B$ which we claim is flat, and a finitely generated A-module M. Define σ by the composition $\mathrm{Def}^A_M \to \mathrm{Def}^B_{M'} \to \mathrm{Def}^C_{M'} \to \mathrm{Def}^C_\Omega$ (where $\Omega = \Omega^n_C M'$) of maps defined in Lemma 2, by change of rings, and in Lemma 1 respectively. We claim that $n \geqslant \mathrm{pdim}_C B$. Then there is a map $\mathrm{Def}^C_\Omega \to \mathrm{Def}^B_{\overline{\Omega}}$ where $\overline{\Omega} = \Omega^n_C M'\otimes_C B$ by Lemma 2. By change of rings there is a map $\mathrm{Def}^B_{\overline{\Omega}} \to \mathrm{Def}^A_{\overline{\Omega}}$. By the splitting of B as A-module $\Omega^n_C M'\otimes_C A$ becomes a direct summand of $\Omega^n_C M'\otimes_C B$. We claim that M is a direct summand of $\Omega^n_C M'\otimes_C A$. Define

$$\tau : \mathrm{Def}^A_M \to \mathrm{Def}^A_{(\overline{\Omega}, V')}, \quad \text{where} \quad V' = \mathrm{im}\,\mathrm{Def}^A_M(k[\varepsilon]),$$

by $M_S \mapsto \Omega^n_{C_S} M'_S\otimes_{C_S} B_S$ considered as (a possibly not finitely generated) A_S-module. That σ is an isomorphism now follows analogously to the argument in Theorem 1 and 2: Define (id, η^i) for $i > 0$ to be the composition of the natural maps

$$\begin{aligned}\mathrm{Ext}^i_A(M, M) \to \mathrm{Ext}^i_C(M', M') \to \mathrm{Ext}^i_C(\Omega, \Omega) \to \mathrm{Ext}^i_B(\overline{\Omega}, \overline{\Omega})\\ \to \mathrm{Ext}^i_A(\overline{\Omega}, \overline{\Omega}) = \mathrm{Ext}^i_A(M\oplus Y, M\oplus Y).\end{aligned}$$

In particular the (id, η^i) are injective. Considering the obstruction classes as 4-term exact sequences (see the proof of Lemma 7) one can show that

$$\mathrm{o}_C(\pi, \sigma M_S)\otimes_C B \mapsto \mathrm{o}_A(\pi, \tau M_S), \quad \text{so} \quad \mathrm{o}_C(\pi, \sigma M_S) = 0 \Rightarrow \mathrm{o}_A(\pi, M_S) = 0$$

and formal smoothness follows for σ. Given a $L_S \in \mathrm{Def}^C_{(\Omega, V)}(S)$, then there is an $A\hat{\otimes}_k\hat{S}$-module $\hat{M}_{\hat{S}}$ and an isomorphism $\hat{\varphi} : \Omega^n_{\hat{C}_{\hat{S}}} \hat{M}'_{\hat{S}} \to \hat{L}_{\hat{S}}$. Let $L^B = L_S\otimes_{C_S} B_S$ and L^A the A_S-linear direct summand of L^B induced by the splitting off of A in B. We observe that the last claim above also gives that $\hat{M}_{\hat{S}}$ splits off from $\Omega^n_{\hat{C}_{\hat{S}}}\hat{M}'_{\hat{S}}\otimes_{\hat{C}_{\hat{S}}}\hat{A}_{\hat{S}}$ since the conditions on the equations are the same. Define M_S as the image of the from $\hat{\varphi}$ induced map $L^A \to \hat{M}_{\hat{S}}$. We obtain an isomorphism $\sigma M_S \cong L_S$ compatible with $\hat{\varphi} \mod \mathfrak{m}_S^2$ by [28, 7.11]. Hence σM_S is equivalent to the deformation L_S and σ is surjective. For the injectivity of σ, see the proof of Theorem 2.

To show that B is A-flat it is sufficient to show that $Q/I(\psi)$ is P-flat. Since $I(\psi_0)$ is determinental, the Eagon-Northcott complex $\mathcal{F}(\psi_0)$ (cf. [14, A2.6]) gives a Q_0-free resolution of $Q_0/I(\psi_0)$. One can show that the natural map $\mathrm{H}_i(\mathcal{F}(\psi))\otimes_P k \to \mathrm{H}_i(\mathcal{F}(\psi_0))$ is surjective if and only if it is an isomorphism. Hence $\mathcal{F}(\psi)$ is a Q-free resolution of $Q/I(\psi)$ of length $m-l+1$. We have

$$\mathrm{Tor}_i^P(Q/I(\psi),k) \cong \mathrm{Tor}_i^Q(Q/I(\psi),Q_0) = \mathrm{H}_i(\mathcal{F}(\psi)\otimes_Q Q_0) = \mathrm{H}_i(\mathcal{F}(\psi_0)) = 0$$

for $i>0$ by assumption, and we conclude by the local criterion of flatness.

We want to show that $n \geqslant \mathrm{pdim}_C B$. Let $C_0 = C/(\mathbf{v})$ with surjections $C \to C_0 \to B$. As for $\mathcal{F}(\psi)$ the Koszul complex $K(\mathbf{F})$ gives an R-free resolution of C. We have

$$\mathrm{Tor}_i^{k[\mathbf{v}]}(C,k) \cong \mathrm{Tor}_i^R(C,Q) \cong \mathrm{H}_i(K(\mathbf{F})\otimes_R Q) \cong \mathrm{H}_i(K(\mathbf{f}))\otimes_P Q = 0$$

for $i>0$ by assumption, hence $(\mathbf{v})$ is a C-regular sequence and $\mathrm{pdim}_C C_0 = q$. Since $Q/I(\psi)$ is P-flat, $\mathcal{F}(\psi)\otimes_P A$ gives an C_0-free resolution of B and the length of $\mathcal{F}(\psi)$ is $m-l+1$. There is a change of rings spectral sequence

$$\mathrm{E}_2^{ij} = \mathrm{Ext}^i_{C_0}(B,\mathrm{Ext}^j_C(C_0,-)) \Rightarrow \mathrm{Ext}^{i+j}_C(B,-).$$

If $i>m-l+1$ or $j>q$, then $\mathrm{E}_\infty^{ij}=0$, and thus $\mathrm{pdim}_C B \leqslant q+m-l+1$.

If M is a MCM A-module (so A is Cohen-Macaulay), then $M_0 = M\otimes_A C_0$ is a MCM C_0-module. We have that $\mathcal{F}(\overline{\psi}) = \mathcal{F}(\psi)\otimes_Q C_0$ gives a C_0-free resolution of B and

$$\mathrm{H}_i(\mathcal{F}(\overline{\psi})\otimes_{C_0} M_0) \cong \mathrm{Tor}_i^{C_0}(B,M_0) \cong \mathrm{Tor}_i^A(B,M) = 0$$

for $i>0$ since B is A-flat. We get an "M_0"-resolution of M' of length $m-l+1$. Now

$$\operatorname{depth} M' \geqslant \operatorname{depth} M_0 - (m-l+1) = \operatorname{depth} C_0 - \mathrm{pdim}_{C_0} B = \operatorname{depth} B = \dim B$$

since B is Cohen-Macaulay, so M' is a MCM B-module, and $\Omega^n_C M'$ is a MCM C-module since $n \geqslant \mathrm{pdim}_C B$.

For the last claim: Let $\mathcal{E}$ be an Eisenbud system on a minimal P-free resolution L of M relative to the regular sequence $(f_1,\dots,f_b)$ and $\mathcal{E}'$ an Eisenbud system on the R-free Koszul resolution $K(\mathbf{v})$ of Q relative to the sequence $(h_1,\dots,h_b)$. Remark that we may assume $s_\alpha(\mathbf{h})=0$ for $|\alpha|>1$. The tensor product of these complexes with the resolution $\mathcal{F}(\psi)$ gives an R-free complex with $\mathrm{H}_0 \cong M'$ and $\mathrm{H}_i \cong \mathrm{Tor}_i^P(M,Q/I(\psi))=0$ for $i>0$, hence an R-free resolution of M'. The tensor product of the Eisenbud systems yields an Eisenbud system relative to $(F_1,\dots,F_b)$, hence we obtain a C-free resolution

$(\mathcal{L}, d)$ of M'. Assuming $(\mathbb{D}\otimes L\otimes A, d\otimes 1)$ is a minimal A-free resolution of M, we have

$$\begin{aligned}(\Omega^n_C M')\otimes_C A &= \operatorname{coker} d_{n+1}\otimes_C A = \operatorname{coker}(d_{n+1}\otimes_C A)\\ &= \operatorname{coker}\bigoplus_{i=0}^{n}\Big(\sum t^a\otimes s_a(\mathbf{f})\Big)_{i+1}\otimes 1\otimes 1\\ &= \bigoplus_{i=0}^{n}\Omega^i_A(M)\otimes_A G_{n-i}\,,\end{aligned}$$

where $G_{n-i} = \bigoplus_{j=0}^{n-i}(\bigwedge^{n-i-j} A^q)\otimes_A A^{\operatorname{rk}\mathcal{F}(\psi)_j}$, since by assumption $h_i \in (\mathbf{v})(\mathbf{u},\mathbf{v})R$ so we may assume $I_1(s_i(\mathbf{h})) \subseteq (\mathbf{u},\mathbf{v})R$ and thus that the $\mathbb{D}\otimes K(\mathbf{v})\otimes C$- and $\mathcal{F}(\psi)$-differentials vanish when applying $-\otimes_C A$. Non-minimality of $(\mathbb{D}\otimes L\otimes A, d\otimes 1)$ will only give certain extra free addends in $\operatorname{coker}(d_{n+1}\otimes_C A)$, the claim is still valid. □

Remark 9 Any finite functorial complex like $\mathcal{F}$ may be used to obtain results similar to Theorem 3.

The restriction functor

Lemma 5

Let $\pi : C \to A$ be a surjective map of local k-algebras. Assume that $I = \ker\pi$ is generated by a regular sequence. If N is a finitely generated C-module with $\operatorname{Tor}_1^C(N, A) = 0$ and $I{\cdot}\operatorname{Ext}^i_C(N, -) = 0$ for all $i > 0$, then the following holds:

(1) *There is a map of deformation functors*

$$\sigma : \operatorname{Def}^C_N \longrightarrow \operatorname{Def}^A_{(N\otimes_C A, V)},$$

where

$$V = \operatorname{im}\Big\{\operatorname{Def}^C_N(k[\varepsilon]) \to \operatorname{Def}^A_{N\otimes_C A}(k[\varepsilon])\Big\},$$

which is formally smooth and an isomorphism at tangent spaces. This holds in particular if $N = \Omega^n_C M$ with M an A-module and $n \geqslant \operatorname{pdim}_C A$.

(2) *If (S, N_S) is a versal family for Def^C_N, then $(S, N_S\otimes_{C_S} A_S)$ is a versal family for $\operatorname{Def}^A_{(N\otimes_C A, V)}$.*

Remark 10 Lemma 5 should be compared with [31, 1.4].

Proof. The map σ is the one given in Lemma 2. If N is a C-module and the length of the regular sequence is r, one has that $\Omega^r_C(N\otimes_C A) \cong \bigoplus_{j=0}^{r}\bigwedge^{r-j} C^r\otimes_C\Omega^j_C N$ if and only if $I{\cdot}\operatorname{Ext}^i_C(N, -) = 0$ for all $i > 0$, by [30, 2.2]. Define $\tau : \operatorname{Def}^C_N \to \operatorname{Def}^C_{(\Omega^r_C\overline{N}, V')}$, where $\overline{N} = N\otimes_C A$ and $V' = \operatorname{im}\operatorname{Def}^C_N(k[\varepsilon])$, by $N_S \mapsto \Omega^r_{C_S}\overline{N_S}$. Formal smoothness follows as in the proof of Theorem 1. If $N = \Omega^n_C M$ we have $\operatorname{Tor}^C_1(N, A) = \operatorname{Tor}^C_{n+1}(M, A) = 0$ and $\operatorname{Ext}^i_C(N, -) = \operatorname{Ext}^{i+n}_C(M, -)$ (for $i > 0$) which certainly is annihilated by I. The second part follows as in the proof of Theorem 1. □

The Knörrer functors

DEFINITION 9 [Eisenbud] If f is a regular element in a ring P, then a *matrix factorisation* of f is a pair of linear maps (ρ, σ) of free P-modules $L_0 \xrightarrow{\sigma} L_1 \xrightarrow{\rho} L_0$ of finite rank such that $\rho\sigma = f \cdot \mathrm{id}_{L_0}$ and $\sigma\rho = f \cdot \mathrm{id}_{L_1}$.

A matrix factorisation is a special case of an Eisenbud system, see Definition 7. If $A = P/(f)$ one obtains an A-free resolution $\dots \xrightarrow{\overline{\rho}} \overline{L}_0 \xrightarrow{\overline{\sigma}} \overline{L}_1 \xrightarrow{\overline{\rho}} \overline{L}_0 \twoheadrightarrow M$ where $\overline{\rho} = \rho \otimes_P A$ etc. If P is a regular local ring, then M is a MCM A-module, and any MCM A-module is given by a matrix factorisation of f. See [13].

DEFINITION 10 [Knörrer] With notation as in Theorem 3, let $F = f + v^2$ (i.e. $h = v^2$, $p = 0$, $q = 1$), then the G-functor in [26] takes the matrix factorisation (ρ, σ) of f over P to the matrix factorisation of F (in block matrix notation)

$$G(\rho, \sigma) = \left(\begin{bmatrix} \rho & v \cdot \mathrm{id} \\ -v \cdot \mathrm{id} & \sigma \end{bmatrix}, \begin{bmatrix} \sigma & -v \cdot \mathrm{id} \\ v \cdot \mathrm{id} & \rho \end{bmatrix} \right) = (\Sigma, \Sigma')$$

over R.

If $M = \operatorname{coker} \rho$ then M is an $A = P/(f)$-module. Let $G(M) = \operatorname{coker} \Sigma$ which is a $C = R/(F)$-module.

Corollary 3

There is an isomorphism of deformation functors

$$\mathrm{Def}_M^A \xrightarrow{\simeq} \mathrm{Def}_{(G(M),V)}^C$$

where

$$V = \mathrm{im}\left\{ \mathrm{Def}_M^A(k[\varepsilon]) \to \mathrm{Def}_{G(M)}^C(k[\varepsilon]) \right\}.$$

If C defines an isolated hypersurface singularity then $\mathrm{Def}_{G(M)}^C$ has a versal family $(S, G(M)_S)$, and $(S, G(M)_S \otimes_{C_S} A_S)$ is a versal family for $\mathrm{Def}_{(M \oplus \Omega_A M, V')}^A$ where

$$V' = \mathrm{im}\left\{ \mathrm{Def}_{G(M)}^C(k[\varepsilon]) \to \mathrm{Def}_{M \oplus \Omega_A M}^A(k[\varepsilon]) \right\}.$$

Proof. Let $L' = L \otimes_P R \otimes_R C$. One checks that

$$M \twoheadleftarrow L_0' \xleftarrow{[v \cdot \mathrm{id}, \overline{\rho}]} L_0' \oplus L_1' \xleftarrow{\overline{\Sigma}} L_1' \oplus L_0' \xleftarrow{\overline{\Sigma}'} \dots \tag{9}$$

gives a C-free resolution of M. Hence $G(M) = \Omega_C M$ and the first part follows from Theorem 3. For the second part $G(M)$ is an MCM module, hence the existence of a versal family is assured by [33], and the rest thus follows from Lemma 5 since $G(M) \otimes_C A = M \oplus \Omega_A M$. □

EXAMPLE 4 Let P and R be the Henselisations of $k[x]$ and $k[x, v]$ respectively, and $A = A_n = P/(f)$ where $f = x^{n+1}$, so that $C = R/(F)$ where $F = f + v^2$. Let $M = k$, the residue field of A, then $G(k) = \mathfrak{m}_C$ and $G(k) \otimes_C A = k \oplus \mathfrak{m}_A$. Consider the first three maps of deformation functors in (6), but without restricting to the images of $\mathrm{Def}_k^A(k[\varepsilon])$. In fact we have $\mathrm{Def}_k^C \xrightarrow{\simeq} \mathrm{Def}_{\mathfrak{m}_C}^C$. By the general identification $\mathrm{Def}_M^A(k[\varepsilon]) = \mathrm{Ext}_A^1(M,M)$, one calculates $\mathrm{Def}_k^A(k[\varepsilon]) = \langle \xi_{11} \rangle$, $\mathrm{Def}_{G(k)}^C(k[\varepsilon]) = \langle \eta_1, \eta_2 \rangle$, and $\mathrm{Def}_{k\oplus\mathfrak{m}_A}^A(k[\varepsilon]) = \langle \xi_{ij} \rangle_{1\leqslant i,j\leqslant 2}$ as k-vector spaces. Recalling ω^1 in (3), the maps in (6) give $\xi_{11} \mapsto \omega_C^1(\xi_{11}) = \eta_1 \mapsto \xi_{11}+\xi_{22}$ where $\xi_{22} = \omega_A^1(\xi_{11})$, and $\eta_2 \mapsto \xi_{12}+\xi_{21}$. Let the images of variables t_{ij} and s_i in the cotangent spaces correspond to the k-duals of the ξ_{ij} and the η_i. Then we know by Proposition 1 that $S_1 = k[t_{11}]^{\mathrm{h}}/(t_{11}^{n+1})$ and $S_2 = k[s_1,s_2]^{\mathrm{h}}/(s_1^{n+1}+s_2^2)$ are the versal deformation rings of Def_k^A and $\mathrm{Def}_{G(k)}^C$ respectively. The obstruction calculus, involving cup and Massey products (see [27, 32, 20]), gives the obstruction ideal. It terminates after $n+1$ steps, and we obtain the versal deformation ring S_3 for $\mathrm{Def}_{k\oplus\mathfrak{m}_A}^A$ as

$$S_3 = k[t_{11}, t_{12}, t_{21}, t_{22}]^{\mathrm{h}} \Big/ \begin{pmatrix} t_{11}^{n+1}+t_{12}t_{21}\,, & t_{11}t_{12}-t_{12}t_{22}\,, \\ t_{21}t_{11}-t_{22}t_{21}\,, & t_{22}^{n+1}+t_{21}t_{12} \end{pmatrix} \tag{10}$$

where the equations are valid even without assuming that the t_{ij} commute. The choice of liftings of the dual maps of the Zariski tangent spaces of the functors to the deformation rings given by $t_{11}, t_{22} \mapsto s_1 \mapsto t_{11}$, and $t_{12}, t_{21} \mapsto s_2 \mapsto 0$ is respected by the equations. However, there is no map $S_1 \to S_3$ such that the composition $S_1 \to S_3 \to S_2 \to S_1$ is the identity! Hence there cannot be any "projection" map $\mathrm{Def}_{k\oplus\mathfrak{m}_A}^A \to \mathrm{Def}_k^A$ for which the natural $\mathrm{Def}_k^A \to \mathrm{Def}_{k\oplus\mathfrak{m}_A}^A$ is a section.

The above example together with Corollary 4 shows that for the A_n-singularities with n odd in odd dimension, there are (indecomposable) MCM modules which have versal deformation spaces with two components. One suspects that the versal deformation space of any MCM module on a simple singularity of even dimension is irreducible.

Definition 11 [Knörrer] With notation as in Theorem 3, let $F = f + uv$ (i.e. $h = uv$, $p = q = 1$), then the H-functor in [26] takes the matrix factorisation (ρ, σ) of f over P to the matrix factorisation of F (in block matrix notation)

$$H(\rho,\sigma) = \left(\begin{bmatrix} \rho & u\cdot \mathrm{id} \\ -v\cdot \mathrm{id} & \sigma \end{bmatrix}, \begin{bmatrix} \sigma & -u\cdot \mathrm{id} \\ v\cdot \mathrm{id} & \rho \end{bmatrix} \right) = (\Phi, \Phi')$$

over R.

Let $A = P/(f)$ and $C = R/(F)$, then $M = \operatorname{coker}\rho$ is an A-module and $H(M) = \operatorname{coker}\Phi$ is a C-module. Knörrer's main result is that H induces an equivalence between the stable category of MCM A-modules and the stable category of MCM C-modules in the case P is complete and regular, see [26, 3.1].

With this notation we have:

Corollary 4

If (ρ,σ) is a matrix factorisation of f, $M = \operatorname{coker}\rho$ and H is the Knörrer functor, then

$$\mathrm{Def}_M^A \cong \mathrm{Def}_{H(M)}^C\,.$$

Proof. Let $L' = L \otimes_P R \otimes_R C$. One checks that

$$M' \twoheadleftarrow L'_0 \xleftarrow{[v \cdot \mathrm{id}, \overline{\rho}]} L'_0 \oplus L'_1 \xleftarrow{\overline{\Phi}} L'_1 \oplus L'_0 \xleftarrow{\overline{\Phi}'} \dots \tag{11}$$

gives a C-free resolution of $M' = M \otimes_A B$ (with C-module structure induced from the natural surjection $C \to B$). Hence $H(M) = \Omega_C M'$ and the conclusion follows by Theorem 3 if we can prove the tangential isomorphism $\mathrm{Def}^A_M(k[\varepsilon]) \xrightarrow{\simeq} \mathrm{Def}^C_{H(M)}(k[\varepsilon])$. Since $(\sigma^\vee, \rho^\vee)$ is a matrix factorisation of f, we have $\mathrm{Ext}^1_A(M, A) = 0$, which implies that

$$\begin{aligned} \mathrm{Ext}^1_A(M, M) &\cong \underline{\mathrm{Hom}}_A(\Omega_A M, M) \\ &\cong \underline{\mathrm{Hom}}_C(H(\Omega_A M), H(M)) \quad \text{by [26, 3.1]} \\ &\cong \underline{\mathrm{Hom}}_C(\Omega_C H(M), H(M)) \quad \text{by [26, 3.5]} \\ &\cong \mathrm{Ext}^1_C(H(M), H(M)) \end{aligned}$$

where $\underline{\mathrm{Hom}}$ is the quotient of stable maps. □

Remark 11 Notice that H is also well defined for matrix factorisations over non-local rings. At least if we restrict the functors to Artinian local k-algebras, the conclusion in Corollary 4 is still valid. The argument in Theorem 3 can be followed for the syzygy *defined* as $H(M)$ only using the obstruction theory. For the tangential result one explicitly constructs a chain homotopy from $H(\xi_A)$ to a given cocycle ξ_C with $[\xi_C] \in \mathrm{Ext}^1_C(H(M), H(M))$ where $\xi_A = \xi_C \otimes_C A$, proving surjectivity, as was done in [19, 7.4.18]. This result was proved by the author in his Master's thesis, see [18, 2.5.4]. A proof of Corollary 4 for P regular, i.e. for MCM modules, was published by Pfister and Popescu in 1996 [31, 3.16]. The obvious generalisation of H is obtained if we in Theorem 3 assume that ψ is empty. Indeed the initial motivation for this work was to get a better understanding of Corollary 4 and thereby possibly obtain generalisations of it.

Remark 12 Observing that $(\Omega_A M)' = \Omega_B(M')$, there is a (non-split) short exact sequence

$$0 \to M' \longrightarrow H(M) \otimes_C B \longrightarrow (\Omega_A M)' \to 0 \tag{12}$$

which represents $\mathrm{o}(C/(v^2), M')$ via the connecting homomorphism. The exact sequences arising from applying $\mathrm{Hom}_B(\overline{H(M)}, -)$ and $\mathrm{Hom}_B(-, \overline{H(M)})$ splits into short exact sequences since the connecting maps may be shown to be zero. E.g.

$$0 \to \mathrm{Ext}^1_A(\Omega_A M, M) \to \mathrm{Ext}^1_B(\overline{H(M)}, \overline{H(M)}) \to \mathrm{Ext}^1_A(M, M) \to 0 \tag{13}$$

which in particular shows that we cannot expect surjectivity in Lemma 5 without restricting to deformations above the image of the tangent space. Cf. [31, 1.17-18].

4. Splitting criteria

Various criteria for the splitting of the map $C/I^2 \to A$ in Theorem 2 are given together with some applications. In particular we consider cones over smooth projective varieties. When we use geometric language, all schemes are supposed to be above a fixed field k. In this section Ω_X will denote the Kähler differentials on a scheme X relative to k.

Let X and Y be reduced and connected k-schemes such that Y is a locally complete intersection closed subscheme of X defined by the ideal sheaf $\mathcal{I}$. Let $Y^{(2)}$ be the double structure on Y defined by $\mathcal{O}_{Y^{(2)}} = \mathcal{O}_X/\mathcal{I}^2$. The cotangent sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X|Y} \to \Omega_Y \to 0 \tag{14}$$

of $\mathcal{O}_Y$-modules defines an element $\eta(\mathcal{I}) \in \mathrm{Ext}^1_Y(\Omega_Y, \mathcal{I}/\mathcal{I}^2)$.

Lemma 6

With the above assumptions, the map $\mathcal{O}_{Y^{(2)}} \to \mathcal{O}_Y$ splits if and only if $\eta(\mathcal{I}) = 0$. If Y is smooth over k, then $\eta(\mathcal{I}) = \delta(\mathrm{id}_{\mathcal{I}/\mathcal{I}^2})$ where δ is the connecting map

$$\delta : \mathrm{H}^0(\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2)) \to \mathrm{H}^1(T_Y \otimes \mathcal{I}/\mathcal{I}^2),$$

for the short exact sequence obtained by applying $\mathcal{H}om_{\mathcal{O}_Y}(-, \mathcal{I}/\mathcal{I}^2)$ to (14).

Proof. In general $\Omega_{X|Y} = \Omega_{Y^{(2)}|Y}$. The first part is a consequence of the obvious generalisation of D. Bayer and Eisenbuds classification theorem of "ribbons" (double structures) in [6, 1.2]. A splitting of $\rho : \mathcal{O}_{Y^{(2)}} \to \mathcal{O}_Y$ induces a splitting of $\varphi : \Omega_{Y^{(2)}|Y} \to \Omega_Y$. For the converse, if φ splits by a map $\psi : \Omega_Y \to \Omega_{Y^{(2)}|Y}$ then the pullback of ψ along the universal derivation $d : \mathcal{O}_Y \to \Omega_Y$ gives a splitting of ρ since $\mathcal{O}_{Y^{(2)}}$ is the pullback of φ along d as shown in the proof of [6, 1.2].

The second part follows by definition of $\eta(\mathcal{I})$ and the natural isomorphism $\mathrm{Ext}^*_Y(\mathcal{F}, \mathcal{I}/\mathcal{I}^2) \cong \mathrm{H}^*(Y, \mathcal{F}^\vee \otimes \mathcal{I}/\mathcal{I}^2)$ for locally free coherent sheaves $\mathcal{F}$. □

We are interested in the local situation, i.e. the vertex of the affine cone over a smooth and projectively normal variety X embedded in some $\mathbb{P}^m$ and not contained in a hyperplane. The splitting condition forces Y to be cut out in X by hyperplanes:

Lemma 7

Suppose $\pi : C \to A$ is a surjective map of local rings and assume k is the residue field of A. If $I = \ker \pi$ is generated by a regular sequence of length n, then

$$\mathrm{o}(C/I^2, k) = 0 \quad \Longleftrightarrow \quad \mathrm{edim}\, C = \mathrm{edim}\, A + n.$$

Proof. ($\Leftarrow$): Let $L \twoheadrightarrow k$ be a minimal C-free resolution of $k = C/(x_1, \ldots, x_e)$, $e = \mathrm{edim}\, C$. Choose an Eisenbud system $\{s_\alpha\}$ relative to $I = (h_1, \ldots, h_n)$ on L, and let $F = (\mathbb{D} \otimes_C L \otimes_C A, d)$, as given after Definition 7. Then $\mathrm{o}(C/I^2, k) \in \mathrm{Ext}^2_A(k, k) \otimes_k I/\mathfrak{m}I$ is represented in the complex $\mathrm{Hom}_A(F, k) \otimes_k I/\mathfrak{m}I$ by the cocycle η given as $F_2 = \overline{L}_0^n[2] \oplus \overline{L}_2 \xrightarrow{(\mathrm{id}[2],0)} \overline{L}_0^n = A^n$ composed with $A^n \twoheadrightarrow I/\mathfrak{m}I$, see [20, Proposition 3]. We have $s_i : L_0 \to L_1$ with $(x_1, \ldots, x_e)s_i = h_i$. We may assume that $h_i = x_i + g_i$, $g_i \in \mathfrak{m}^2$

for $i = 1, \ldots, n$. Hence $s_i = e_i + \delta_i$ where $I(\delta_i) \subseteq \mathfrak{m}$, and η is the coboundary induced from $[\mathrm{id}\,|\,0] : L_1 \to L_0^n$ composed with $L_0^n \twoheadrightarrow I/\mathfrak{m}I$.

($\Rightarrow$): Applying $-\otimes_C A$ to the short exact sequence $0 \to \mathfrak{m}_C \to C \to k \to 0$ gives a 4-term exact sequence $0 \to \mathrm{Tor}_1^C(k, A) \to \mathfrak{m}_C \otimes_C A \to A \to k \to 0$. It represents $\mathrm{o}(C/I^2, k)$, cf. [4, 3.5]. The connecting $\mathrm{Ext}_A^1(\mathfrak{m}_A, k) \to \mathrm{Ext}_A^2(k, k)$ is an isomorphism, so $\mathrm{o}(C/I^2, k) = 0$ implies that

$$0 \to \mathrm{Tor}_1^C(k, A) \longrightarrow \mathfrak{m}_C \otimes_C A \longrightarrow \mathfrak{m}_A \longrightarrow 0 \tag{15}$$

splits. Since $\mathrm{Tor}_1^C(k, A) \cong I/\mathfrak{m}_C I$, we have, after applying $-\otimes_A k$ to (15), a splitting $\mathfrak{m}_C/\mathfrak{m}_C^2 = \mathfrak{m}_A/\mathfrak{m}_A^2 \oplus I/\mathfrak{m}_C I$. □

Remark 13 By Example 1, liftability of an MCM module is not sufficient to imply changing embedding dimension.

Suppose k is algebraically closed. By Lemma 7 we henceforth restrict the attention to smooth (and positive dimensional) hyperplane sections $Y = X \cap H$ for which $\eta(\mathcal{I}) \in \mathrm{H}^1(T_Y(-1))$. E.g. if $\dim Y = 1$, then this group vanish if and only if Y is a rational curve of degree less than 4.

Corollary 5

Let C be the local k-algebra defined by the vertex of the cone of the 2-dimensional rational normal scroll $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ of degree 3 in $\mathbb{P}^4$. Then there are maximal Cohen-Macaulay C-modules N_1 and N_2 of rank 8 and 9 respectively, such that the following holds:

(1) *There is a 10-dimensional k-vector subspace $V \subseteq \mathrm{Def}_{N_1}^C(k[\varepsilon])$ such that the reduced versal deformation space R_{red} of $\mathrm{Def}_{(N_1,V)}^C$ is an isolated singularity of dimension 4 with a (small) resolution $T \to R_{\mathrm{red}}$ given by contraction of the zero section in the vector bundle*

$$T = \mathbf{Spec}\,\mathrm{Sym}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)).$$

(2) *There is a 16-dimensional k-vector subspace $V \subseteq \mathrm{Def}_{N_2}^C(k[\varepsilon])$ such that the reduced versal deformation space R_{red} of $\mathrm{Def}_{(N_2,V)}^C$ is given as the cone over the image of the Segre embedding $\mathbb{P}^5 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^{17}$ intersected with two (special) hyperplanes.*

Proof. Let A be the local k-algebra defined by the cone of any smooth hyperplane section Y of X. Since Y is a rational normal curve of degree 3, $\mathrm{H}^1(T_Y(-1)) = 0$, so $C \to A$ satisfies the splitting condition of Theorem 2. We have $A \cong k[u^3, u^2v, uv^2, v^3]^{\mathrm{h}}$ and rank one MCM A-modules $L_1 = (u, v)$ and $L_2 = (u^2, uv, v^2)$. Let $M_1 = L_1 \oplus L_2^{\oplus 2}$ and $M_2 = L_2^{\oplus 3}$ and let N_i be the first C-syzygy of M_i. The rank of N_i equals the minimal number of generators of M_i. The descriptions of the reduced versal deformation spaces of the $\mathrm{Def}_{M_i}^A$ given in Example 1 and 2 in [17] together with Theorem 2 gives the result. □

In [17] we more generally give explicit descriptions of certain canonical resolutions of strata in the reduced versal deformation space of any MCM module on the cone over the rational normal curve of degree m in $\mathbb{P}^m$. Hence (1) and (2) in Corollary 5 are only two out of an infinite class of examples on C.

EXAMPLE 5 If $\mathrm{H}^i(T_X(-1-i)) = 0$ for $i = 0, 1$, the short exact sequence $T_X(-2) \to T_X(-1) \to T_X|_Y(-1)$ implies that $\mathrm{H}^0(T_X|_Y(-1)) = 0$, hence by Lemma 6 $\eta(\mathcal{I}) \neq 0$. E.g. if either X is embedded by a sufficiently high multiple of an ample line bundle or X is an abelian variety, then for all smooth hyperplane sections $\eta(\mathcal{I}) \neq 0$.

EXAMPLE 6 If $\dim X = 2$, $\mathrm{H}^0(T_Y(-1)) = 0$ if $\deg Y > 2 - 2g_Y$. Hence

$$\eta(\mathcal{I}) = 0 \Longleftrightarrow \mathrm{H}^0(T_X|_Y(-1)) \neq 0 \qquad \text{if } g_Y > 0 \text{ or } \deg Y > 2\,. \tag{16}$$

EXAMPLE 7 Suppose G is a cyclic group acting linearly on $\mathbb{P}^m$ inducing an action on $X \subseteq \mathbb{P}^m$. If G fixes a hyperplane H such that $Y := X^G = X \cap H$ is smooth, then $T_Y \to T_X|_Y$ splits by the map $T_X|_Y(U) \ni v \mapsto |G|^{-1} \sum_{h \in G} hv \in T_Y(U)$ and hence $\eta(\mathcal{I}) = 0$. An explicit example is the hypersurface given by $F = f(\mathbf{x}) + u^n g(\mathbf{x})$, $n > 1$, with $G = \langle \xi \rangle$, where ξ is a primitive n^{th} root of 1 which acts by multiplication in the u-coordinate. This in particular gives examples where $\mathrm{H}^0(T_X|_Y(-1)) \neq 0$.

Consider the following geometric condition:

(C) There is a point p, not contained in the hyperplane H, such that the tangent space T_yX of X at y contains p for all $y \in Y$.

If (C) is satisfied, then the cotangent sequence splits. This was observed by A. Beauville and J.-Y. Mérindol in [7][1]. They also proved the following strong converse:

Proposition 3 (Beauville and Mérindol)

If either $\dim X \geqslant 3$ *and* $\mathrm{H}^1(\mathcal{O}_Y(-1)) = 0$ (*e.g. if* $k = \mathbb{C}$), *or* $\dim X = 2$ *and* $\mathrm{H}^0(\omega_Y) \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^m}(1)) \to \mathrm{H}^0(\omega_Y(1))$ *is surjective, then the splitting of the cotangent sequence implies condition* (C).

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