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Collect. Math. **58**, 2 (2007), 193–198
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Existence of coherent systems of rank two and dimension four

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Received July 17, 2006. Revised November 22, 2006

Abstract

We show that the moduli space of coherent systems of rank two and dimension four on a generic curve of genus at least two is non-empty for any value of the parameter when the Brill-Noether number is at least one and the degree is odd or when the Brill-Noether number is at least five and the degree is even. In all these cases there is one component of the moduli space of coherent systems of the expected dimension. The case of rank two and dimension four is particularly relevant as it is the first case that cannot be treated by reduction to smaller rank or dimension.

1. Introduction

The purpose of this note is to show existence of stable coherent systems of rank two and dimension four on a generic curve of genus at least two.

One can define a coherent system on a curve C as a pair (E, V) where E is a vector bundle on C and $V \subset H^0(E)$ is a subspace of sections of E. A coherent subsystem is a pair (E', V') where E' is a subbundle of E and $V' \subset V \cap H^0(E')$. Given a real number α, a coherent system is said to be α-(semi)-stable if for every coherent subsystem

$$\frac{\deg E' + \alpha \dim V'}{\mathrm{rk}\ E'} < (\leq) \frac{\deg E + \alpha \dim V}{\mathrm{rk}\ E}.$$

For general results about coherent systems, the reader is refered to [**?**]. Moduli space of α-stable coherent systems of rank r, degree d and dimension k will be denoted

Keywords: Vector bundle, coherent system, algebraic curve.
MSC2000: 14H60.

by $G^k_{r,d}(\alpha)$ (see [?]). The expected dimension of $G^k_{r,d}(\alpha)$ is given by the Brill-Noether number

$$\rho^k_{r,d} = r^2(g-1) + 1 - k(k - d + r(g-1)).$$

Any component of the moduli space has at least this dimension ρ. It is expected that when the Brill-Noether number is positive, the moduli space is non-empty, but this is not always true.

Here, we deal with the case in which $r = 2$, $k = h^0(E) = 4$.

Theorem 1.1

Let C be a generic curve of genus at least two. For any positive value of α, the moduli space of coherent systems of rank two and dimension four on C is non-empty for $\rho \geq 1$ and d odd or for $\rho \geq 5$ and d even. In all these cases there is one component of the moduli space of coherent systems of the expected dimension.

This result is sharper than our result in [?] that applies to a more general setting. The reason why we focus on this particular case is that it seems to be the first example that is not treatable with commonly used methods. Coherent systems of dimension $k < r$ can be constructed as extensions of a trivial coherent system by a vector bundle (without sections). On the other hand, for $k > r$ and under suitable additional conditions there is a correspondence between coherent systems with invariants r, d, k and coherent systems with invariants $k - r, d, k$ assigning E, V to E', V' where $V' = V^*$ and E' is defined by the exact sequence

$$0 \to E^{'*} \to V \otimes \mathcal{O}_C \to E \to 0.$$

This allows for example to treat the case of dimension $k = r + 1$. Nevertheless, the method is useless in our situation as the numerical data are self dual.

This paper was written in response to a question by P. Newstead. The author would like to thank him for organizing a workshop around this subject in August 2005.

P. Newstead and U. Bhosle dealt with similar questions using different methods (work in progress).

2. A sketch of proof and preliminary results

The main point of the proof is the following fact (that was already used in previous papers [?, ?]): the dimension of $G^k_{2,d}$ at any point is at least ρ. One can also consider the case of a family of curves

$$\mathcal{C} \to T,$$

and define

$$\mathcal{G}^k_{r,d} = \{(t, E, V) | t \in T,\ V \subset H^0(C_t, E)\}.$$

Here E denotes a vector bundle on C_t of rank r and degree d. Then,

$$\dim \mathcal{G}^k_{2,d} \geq \rho^k_{2,d} + \dim T,$$

at every point. If one can find a particular curve C_0 such that the dimension of $G^k_{2,d}(C_0)$ is ρ, then the dimension of the generic fiber of the map $\mathcal{G}^k_{2,d} \to T$ attains its minimum in a neighborhood of the curve. Hence, for a generic curve C in an open neighborhood of C_0, the dimension of $G^k_{2,d}(C)$ is ρ (and the locus is non-empty). We only need to explicitly exhibit a curve C_0 and the corresponding family of vector bundles in $G^k_{2,d}(C_0)$. Our C_0 is a reducible curve that we define as follows:

DEFINITION 2.1 Let $C_1 ... C_g$ be elliptic curves. Let P_i, Q_i be generic points in C_i. Then C_0 is the chain obtained by gluing the elliptic curves when identifying the point Q_i in C_i to the point P_{i+1} in C_{i+1}, $i = 1...g-1$.

We shall be using the following well-known fact:

Lemma 2.2

Let C be an elliptic curve and L a line bundle of degree d on C. One can define a subspace of dimension k of sections of L by specifying the k distinct (minimum) desired vanishing of a basis of the subspace at two different points P, Q so that the sum of the corresponding vanishings at P and Q is $d-1$. In the case when $L = \mathcal{O}(aP + (d-a)Q)$ two of the vanishings could be chosen to be a, $d-a$ adding to d rather than $d-1$. These are the only two vanishings that can add up to d if P, Q are generic.

Similarly, let E be a vector bundle obtained as the sum of two line bundles of degree d. Then one can find a space of sections of E with desired vanishing at two points P, Q if the sum of the vanishings at P, Q is at most $d-1$ and each vanishing appears at most twice.

The proof is left to the reader (or see [**?**]).

When dealing with reducible curves, the notion of a line bundle and a space of its sections needs to be replaced by the analogous concept of limit linear series as introduced by Eisenbud and Harris. A similar definition can be given for vector bundles (cf. [**?**]). For the convenience of the reader, we reproduce this definition here:

DEFINITION 2.3 A limit linear series of rank r, degree d and dimension k on a chain of M (not necessarily elliptic) curves consists of data I,II below for which data III, IV exist satisfying conditions a)-c)

I) For every component C_i, a vector bundle E_i of rank r and degree d_i and a k-dimensional space V_i of sections of E_i

II) For every node obtained by gluing Q_i and P_{i+1}, an isomorphism of the projectivisation of the fibers $(E_i)_{Q_i}$ and $(E_{i+1})_{P_{i+1}}$

III) A positive integer b

IV) For every node obtained by gluing Q_i and P_{i+1} basis $s^t_{Q_i}, s^t_{P_{i+1}}$, $t = 1...k$ of the vector spaces V_i, V_{i+1} of I.

Subject to the conditions

a)$\sum_{i=1}^{M} d_i - r(M-1)b = d$

b) The orders of vanishing at Q_i, P_{i+1} of the sections of the chosen basis satisfy $ord_{Q_i} s^t_{Q_i} + ord_{P_{i+1}} s^t_{P_{i+1}} \geq b$

c) Sections of the vector bundles $E_i(-bP_i), E_i(-bQ_i)$ are completely determined by their value at the nodes.

Finally notice that in order to prove the stability of a coherent system (E, V) for every positive value of α, it suffices to prove the following two facts:

1) The vector bundle E is stable.
2) For every proper subbundle E',

$$\frac{\dim(H^0(E') \cap V)}{\text{rk } E'} \leq \frac{\dim V}{\text{rk } E}.$$

This is what we shall do in the next section for our particular case.

3. Proof of the result

We now give the proof of the Theorem 1.1.

We can assume $d < 2g+2$ as otherwise every vector bundle of rank two and degree d has at least four sections.

We construct the limit linear series on the chain of elliptic curves described above first in the case of $d = 2a$ even. The integer b that appears in III of the definition of limit linear series above will be taken to be a.

For $i = 1, ..., g-a+1$ take the vector bundle on C_{2i-1} to be

$$[\mathcal{O}((i-1)P_i + (a-i+1)Q_i)]^{\oplus 2}.$$

Take the four dimensional space of sections generated by the two sections that vanish with order $i-1$ at P_i and $a-i+1$ at Q_i and the two sections that vanish with order i at P_i and $a-i-1$ at Q_i.

For $i = 1, ..., g-a+1$ take on C_{2i} the vector bundle

$$[\mathcal{O}((i+1)P_i + (a-i-1)Q_i)]^{\oplus 2}.$$

Take the four dimensional space of sections generated by the two sections that vanish with order $i+1$ at P_i and $a-i-1$ at Q_i and the two sections that vanish with order $i-1$ at P_i and $a-i$ at Q_i.

On C_j, $j = 2(g-a)+3, ..., g$, take the vector bundle to be the direct sum of two generic line bundles of degree a. Take the space of sections generated by the two sections that vanish at P_j with multiplicity $a-g+j-2$ and at Q_j with multiplicity $g-j+1$ and the two sections that vanish at P_j with multiplicity $a-g+j-1$ and at Q_j with multiplicity $g-j$.

One can check that this gives rise to a limit linear series. The vector bundle obtained in this way is stable because all the restriction to the elliptic components are semistable and the assumption $\rho \geq 5$ is equivalent to the fact that there is at least one curve where the line bundles are generic. Then the destabilising line bundles for each component do not glue with each other (see [**?**]).

The number of moduli on which the family depends can be computed as follows: the first $2(g-a+1)$ line bundles are completely determined and have a four dimensional family of automorphisms while the remaining ones move in a two dimensional family and have a two dimensional family of endomorphisms. The gluings are arbitrary and therefore move in four dimensional families. The resulting vector bundle being stable has a one dimensional family of endomorphisms. Therefore, the dimension of the family is

$$2(g-2(g-a+1))+4(g-1)-4(2(g-a+1))-2(g-2(g-a+1))+1=\rho.$$

It is easy to check that the family is not contained in a higher dimensional family of limit linear series: In order to obtain a larger family, one needs to make either the restriction of the vector bundles to the various components more general by replacing an $\mathcal{O}(\alpha P+(a-\alpha)Q)$ by a generic line bundle, or make the gluing more general (or both). But either change would decrease the vanishing at Q_i and therefore requires to increase the vanishing at P_{i+1}. This implies that the vanishing at Q_{i+1} must decrease as well and so on. Then the vanishing at Q_g is negative, which is impossible.

It then follows from the general argument at the beginning of section two that such a limit linear series deforms to the generic curve.

Consider now the case of odd $d=2a+1$.

On C_1 take the vector bundle to be $\mathcal{O}(aQ_1)^{\oplus 2}$. Take the four dimension space of sections that vanish with order at least $a-2$ at Q_1.

On C_2 take the vector bundle to be $\mathcal{O}(aQ_2)\oplus\mathcal{O}(2P_2+(a-2)Q_2)$. Take the four dimensional space of sections generated by the section s_1 that vanishes with order a at Q_2, the section s_2 that vanishes with order $a-1$ at Q_2, the section s_3 that vanishes with order 2 at P_2 and $a-2$ at Q_2 and the section s_4 that vanishes with order 2 at P_2 and $a-3$ at Q_2.

On C_3 take the vector bundle to be indecomposable of degree d. Take the section that vanishes with order 1 at P_3 and $a-1$ at Q_3 and glue its direction at P_3 with the direction of s_2, take the section that vanishes with order 3 at P_3 and $a-3$ at Q_3 and glue its direction at P_3 with the direction of s_4. Take then the section that vanishes with order $a-1$ at Q_3 that glues with s_1 and the section that vanishes with order two at P_3 and $a-3$ at Q_3 and glues with s_3.

On C_4 take the vector bundle to be $[\mathcal{O}(3P_4+(a-3)Q_4)]^{\oplus 2}$. Take the four dimension space of sections generated by the two sections that vanish with order 1 at P_4 and $a-2$ at Q_4 and the two sections that vanish with order 3 at P_4 and $a-3$ at Q_4.

On $C_i,\ i=5...g$, take the limit linear series exactly as in the case of even degree.

The resulting vector bundle is stable because the restriction to each component is semistable and one of them (the restriction to C_3) is stable.

The number of moduli on which the family depends can be computed as follows: the first two and the fourth to $2(g-a+1)^{th}$ vector bundles are completely determined and have a four dimensional family of automorphisms except for the second which has a two dimensional family. The third vector bundle moves in a one dimensional family and has a one dimensional family of endomorphisms. The remaining vector bundles move in a two dimensional family and have a two dimensional family of endomorphisms. The gluings are arbitrary and therefore move in four dimensional families except for the gluing between the second and third curve that moves in a two dimensional family. The resulting vector bundle being stable has a one dimensional family of endomorphisms. Therefore, the dimension of the family is

$$2(g-2(g-a+1))+1+4(g-2)+2-4(2(g-a+1)-2)-2-1-2(g-2(g-a+1))+1=\rho.$$

As in the even degree case, it is easy to check that the family is not contained in a higher dimensional family of limit linear series, therefore such a limit linear series deforms to the generic curve.

The statement about the coherent system follows from the fact that on most curves, the limit linear series has only two different values for the vanishing at each node. Therefore, any sublinebundle of E has at most two limit sections. As E is stable, for any positive value of α, the stability condition is satisfied. $\square$

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