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On some density theorems in regular vector lattices of continuous functions

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Abstract

In this paper, we establish some density theorems in the setting of particular locally convex vector lattices of continuous functions defined on a locally compact Hausdorff space, which we introduced and studied in [3, 4] and which we named regular vector lattices. In this framework, by using properties of the subspace of the so-called generalized affine functions, we give a simple description of the closed vector sublattice, the closed Stone vector sublattice and the closed subalgebra generated by a subset of a regular vector lattice.

As a consequence, we obtain some density results. Finally, a connection with the Korovkin type approximation theory is also shown.

1. Introduction

In [3], we introduced and studied a particular class of locally convex vector lattices of continuous functions defined on a locally compact Hausdorff space X, which we named regular vector lattices. These spaces are endowed with the natural (pointwise) order and with a locally convex topology which is finer than the topology of pointwise convergence and it is generated by a saturated family of M-seminorms. Moreover, they contains the space $C_c(X, \mathbb{R})$ of all continuous functions with compact support as a dense subspace.

Keywords: Vector lattice of continuous function, generalized affine function, Stone-Weierstrass theorem.

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This rather general class of locally convex vector lattices includes, for example, every weighted function space as introduced by Nachbin (see [13]) and every vector sublattice of continuous functions containing $C_c(X,\mathbb{R})$ endowed with the topology of uniform convergence on compact subsets or the topology of pointwise convergence (see [3, Examples 2.3]).

In the framework of regular vector lattices, among other things, we introduced the space of generalized affine functions and, by means of them, we established several results related to the Korovkin type approximation theory (see [3, 4]).

In this paper we continue the study of generalized affine functions and we use them to describe those subspaces which are intersection of a closed convex inf-stable cone and its opposite. Moreover, we give a simple description of the closed vector sublattice and the closed Stone vector sublattice generated by a subset of a regular vector lattice. All these results immediately lead to some density results, i.e. Stone-Weierstrass type theorems.

Furthermore, we shall determine sufficient conditions in order that the closure of a subalgebra of a regular vector lattice is a Stone vector lattice; these results generalize the ones obtained, for example, in [2, Lemma 4.4.3]. As a consequence, we shall characterize the closed subalgebra generated by a subset of a regular vector lattice.

Finally, a connection with the Korovkin type approximation theory is shown, revealing a deep relationship between the Stone-Weierstrass type theorems established here and some Korovkin type theorems established in [4]. A similar relationship was already discovered in the setting of the space of all continuous functions vanishing at infinity on a locally compact Hausdorff space (which is a regular vector lattice) (see [2, Section 4.4]).

In fact, all the main results of this paper extend those obtained in [2, Section 4.4].

2. Regular vector lattices and generalized affine functions

Let X be a locally compact Hausdorff space. We shall denote by $\mathbb{R}^X$ and $C(X,\mathbb{R})$ the space of all real valued functions on X and the space of all continuous real valued functions on X. We shall also denote by $C_b(X,\mathbb{R})$ (resp., $C_c(X,\mathbb{R})$) the subspace of $C(X,\mathbb{R})$ of all continuous real valued bounded functions on X (resp., the subspace of all continuous real valued functions on X whose support is compact). Moreover, $C_0(X,\mathbb{R})$ stands for the subspace of $C(X,\mathbb{R})$ consisting of all continuous real valued functions on X which vanish at infinity. Finally, we shall denote by $\mathbf{1}$ the function of constant value 1 on X.

Let B_X be the σ-algebra of all Borel subsets of X and denote by $M^+(X)$ the cone of all regular Borel measures on X. In particular, for every $x \in X$ we shall denote by ϵ_x the Dirac measure concentrated at x.

For every $\mu \in M^+(X)$ and $p \in [1,+\infty[$, we shall denote by $\mathcal{L}^p(X,\mu)$ the space consisting of all Borel-measurable functions $f \in \mathbb{R}^X$ which are p-fold μ-integrable.

If $\mu \in M^+(X)$, we shall denote by $\operatorname{supp}(\mu)$ the support of μ, i.e. the complement of the largest open subset of X on which μ is zero.

We recall that if $f \in \mathcal{L}^1(X,\mu) \cap C(X,\mathbb{R})$ and $f = 0$ on $\operatorname{supp}(\mu)$, then $\int f\, d\mu = 0$. Moreover, the following result holds.

Lemma 2.1

Let X be a locally compact Hausdorff space, F a subset of X and $\mu \in M^+(X)$. Assume that for every $y \notin F$ there exists $f \in \mathcal{L}^1(X, \mu) \cap C(X, \mathbb{R})$, $f \geq 0$, such that $f(y) > 0$ and $\int f \, d\mu = 0$. Then $\operatorname{supp}(\mu) \subset F$.

Proof. Assume that there exists $y \in \operatorname{supp}(\mu)$ such that $y \notin F$; hence there exists $f \in \mathcal{L}^1(X, \mu) \cap C(X, \mathbb{R})$, $f \geq 0$, such that $f(y) > 0$ and $\int f \, d\mu = 0$. Let K be a compact neighborhood of y such that $f(x) > 0$ for every $x \in K$ and fix a function $\varphi \in C_c(X, \mathbb{R})$, $\operatorname{supp}(\varphi) \subset K$, such that $\int \varphi \, d\mu \neq 0$. Without loss of generality, we may also assume that $0 \leq \varphi$.

Set $m := \min_{x \in K} f(x)$; hence $0 \leq m\varphi \leq f \, || \, \varphi \, ||_\infty$. Accordingly,

$$0 \leq m \int \varphi \, d\mu \leq || \, \varphi \, ||_\infty \int f \, d\mu = 0$$

and this leads to a contradiction. □

If E is a vector space, then the subspace generated by a subset H of E will be denoted by $\mathcal{L}(H)$.

Moreover, a subset P of a vector space E is said to be a ***convex cone*** (resp., ***absolutely convex***) if $f + g \in P$ and $\lambda f \in P$ for every $f, g \in P$ and for every $\lambda \geq 0$ (resp., if $\lambda f + \mu g \in P$ for every $f, g \in P$ and for every $\lambda, \mu \in \mathbb{R}$, $|\lambda| + |\mu| \leq 1$).

Consider now a vector lattice E. A subset P of E is said to be ***inf-stable*** if $\inf\{f, g\} \in P$ for every $f, g \in P$.

Further, a subset P of E is said to be a ***solid*** set (resp., a ***sublattice***) if for every $f \in E$ and $g \in P$ such that $|f| \leq |g|$, it follows that $f \in P$ (resp., if $\sup\{f, g\} \in P$ for every $f, g \in P$).

A solid linear sublattice P of a vector lattice E is also said to be an ***ideal*** of E. Finally, if H is a subset of a vector lattice E, we shall set

$$H_+ := \{h \in H \mid h \geq 0\}.$$

Let (E, τ) be a topological vector space. We shall denote by $(E, \tau)'$ the space of all τ-continuous linear forms on E. If in addition (E, τ) is a topological vector lattice, then we will denote by $(E, \tau)'_+$ the cone of all τ-continuous positive linear forms on E.

For more details about topological vector lattices see, for example, [1, 17].

We now proceed to recall the definition of a particular class of locally convex vector lattices of continuous functions, in the framework of which we shall develop all the results of this paper (see also [3]).

Let X be a locally compact Hausdorff space and E a subspace of $C(X, \mathbb{R})$, endowed with the natural (pointwise) order $\leq$ and with a locally convex topology τ. We shall assume that the locally convex space (E, τ) satisfies the following properties:

- (R_1) $(E, \tau, \leq)$ is a locally convex vector lattice possessing a neighborhood base $\mathcal{U}_\tau$ of the origin consisting of absolutely convex solid sublattices;
- (R_2) The topology τ is finer than the topology τ_s of pointwise convergence on X;
- (R_3) $C_c(X, \mathbb{R})$ is dense in (E, τ).

In the sequel, we shall refer to this class of locally convex vector lattices as *regular* locally convex vector lattices, in short regular vector lattices.

We point out that Property (R_1) is equivalent to require that the topology τ is generated by a saturated family $(p_\alpha)_{\alpha\in A}$ of seminorms on E such that for every $\alpha \in A$

(i) $p_\alpha(f) \leq p_\alpha(g)$ for every $f, g \in E$, $|f| \leq |g|$;
(ii) $p_\alpha(\sup\{f,g\}) = \sup\{p_\alpha(f), p_\alpha(g)\}$ for every $f, g \in E_+$.

Such seminorms are also called M-seminorms.

Below we present some examples of regular vector lattices, (see [3, Examples 2.3] for more details).

Example 2.2 **1**. Every vector sublattice E of $C(X,\mathbb{R})$ containing $C_c(X,\mathbb{R})$ and endowed with the topology τ_c of uniform convergence on compact subsets of X or the topology τ_s of pointwise convergence on X is a regular vector lattice.

In particular, let H be an adapted subspace of $C(X,\mathbb{R})$ (see [7, Section 2.2], [10, Chapter 8]) and set

$$E_H := \{f \in C(X,\mathbb{R}) \mid \ \textit{there exists}\ \ h \in H_+ \ \ \textit{such that} \ \ | f | \leq h\}.$$

Then E_H is a vector sublattice of $C(X,\mathbb{R})$ containing $C_c(X,\mathbb{R})$. Accordingly, the subspace E_H endowed with the topologies τ_s and τ_c, respectively, is a regular vector lattice.

2. (*Weighted function spaces*.) Let W be a family of positive upper semicontinuous functions on X such that, if $w_1, w_2 \in W$, there exist $w \in W$ and $\alpha > 0$ such that $w_1 \leq \alpha w$ and $w_2 \leq \alpha w$. Further, assume that for every $x \in X$ there exists $w \in W$ such that $w(x) > 0$.

Consider the *weighted function space* $C_W(X,\mathbb{R})$ of all functions $f \in C(X,\mathbb{R})$ such that wf vanishes at infinity for all $w \in W$.

The space $C_W(X,\mathbb{R})$ is endowed with the locally convex topology τ_W generated by the family of seminorms $(p_w)_{w\in W}$ defined by

$$p_w(f) := \sup_{x\in X} w(x)|f(x)| \qquad (f \in E).$$

Then $(C_W(X,\mathbb{R}), \tau_W)$ is a regular vector lattice on X.

Some examples of weighted function spaces can be found in [3, 13, 14].

Among the properties of regular vector lattices, we recall the following results, which have been proved in [3, Proposition 2.5 and Theorem 2.6].

Theorem 2.3

Every regular vector lattice (E,τ) on a locally compact Hausdorff space X satisfies Dini's property, i.e. if $(f_i)^{\leq}_{i\in I}$ is a filtering decreasing net in E such that $\inf_{i\in I} f_i = 0$, then $\lim_{i\in I}^{\leq} f_i = 0$ in (E,τ).

Moreover, for for every $\rho \in (E,\tau)'_+$, there exists a (unique) $\mu \in M^+(X)$ such that $E \subset \mathcal{L}^1(X,\mu)$ and

$$\rho(f) = \int f\, d\mu \quad \textit{for every} \ \ f \in E.$$

From now on, we shall fix a locally compact Hausdorff space X and a regular vector lattice (E, τ) on X.

For every $\mu \in M^+(X)$ such that $E \subset \mathcal{L}^1(X, \mu)$, consider the positive linear form $I_\mu : E \to \mathbb{R}$ defined by

$$I_\mu(f) := \int f \, d\mu \qquad (f \in E)$$

and set

$$M^+_{\tau,E}(X) := \left\{ \mu \in M^+(X) \mid E \subset \mathcal{L}^1(X, \mu) \ \text{ and } \ I_\mu \in (E, \tau)'_+ \right\}.$$

We point out (see [3, Section 3]) that, if $(p_\alpha)_{\alpha \in A}$ is a saturated family of M-seminorms which generates τ, then a Borel measure $\mu \in M^+(X)$ belongs to $M^+_{\tau,E}(X)$ if and only if there exist $c \geq 0$ and $\alpha \in A$ such that for every $\varphi \in C_c(X, \mathbb{R})$

$$\left| \int \varphi \, d\mu \right| \leq c \, p_\alpha(\varphi).$$

Furthermore, given a linear subspace H of E, for every $x \in X$ we set

$$M_{\tau,x}(H) := \left\{ \mu \in M^+_{\tau,E}(X) \mid \int h \, d\mu = h(x) \text{ for every } h \in H \right\}.$$

Every $\mu \in M_{\tau,x}(H)$ is said to be an ***H-representing*** measure for x with respect to the topology τ.

Moreover, for every $f \in E$ and $x \in X$ we shall consider the following envelopes of f:

$$\hat{f}_\tau(x) := \sup_{U \in \mathcal{U}_\tau} \left(\inf_{\substack{k \in H \\ (f-k)^+ \in U}} k(x) \right) \in \mathbb{R} \cup \{+\infty\}$$

and

$$\check{f}_\tau(x) := \inf_{U \in \mathcal{U}_\tau} \left(\sup_{\substack{h \in H \\ (h-f)^+ \in U}} h(x) \right) \in \mathbb{R} \cup \{-\infty\},$$

where $\mathcal{U}_\tau$ is a neighborhood base as in (R_1). Here the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$ have to be observed.

Clearly,

$$\check{f}_\tau \leq f \leq \hat{f}_\tau \quad \text{for every} \quad f \in E$$

and

$$\check{h}_\tau = h = \hat{h}_\tau \quad \text{for every} \quad h \in H.$$

A function $f \in E$ such that

$$\check{f}_\tau = f = \hat{f}_\tau$$

is said to be a ***generalized*** H***-affine*** function (with respect to the topology τ).

The subspace of all generalized H-affine functions will be denoted by $\widehat{H}_\tau$. This subspace has been introduced and studied in [3, Section 3 and 4] as a particular case of a more general space of generalized H-affine functions associated to a positive linear operator. There, we showed that $\widehat{H}_\tau$ is closed (see [3, Corollary 4.4]).

Moreover, we also proved the following characterization (see [3, Corollary 3.6, Theorem 3.7 and Theorem 4.3]).

Theorem 2.4

Let (E,τ) a regular vector lattice on a locally compact Hausdorff space X, H a subspace of E and $f \in E$. Then, the following statements are equivalent:

(a) *f is a generalized H-affine function with respect to τ;*

(b) *For every $V \in \mathcal{U}_\tau$ there exist $n, m \in \mathbb{N}$, $k_1, ..., k_n, k'_1, ..., k'_m \in H$ such that*

- *(i) $(f - k_i)^+ \in V$ for every $i = 1, ..., n$ and $(k'_j - f)^+ \in V$ for every $j = 1, ..., m$;*
- *(ii) $\inf\limits_{1\le i\le n} k_i - \sup\limits_{1\le j\le m} k'_j \in V$;*

(c) *$\int f \, d\mu = f(x)$ for every $x \in X$ and $\mu \in M_{\tau,x}(H)$.*

We finally point out that, if (E,τ) is a regular vector lattice, then $\widehat{H}_\tau$ coincides with the so-called ***Korovkin closure*** $\mathrm{Kor}(H)_\tau$ of H in E for the topology τ which is defined as the subspace of all $f \in E$ such that $\lim\limits_{i\in I}^{\le} L_i(f) = f$ in (E,τ) for every τ-equicontinuous net $(L_i)^{\le}_{i\in I}$ of positive linear operators from E into E satisfying $\lim\limits_{i\in I}^{\le} L_i(h) = h$ in (E,τ) for every $h \in H$ (see [4, Theorem 2.2, Theorem 4.2] for more details and some applications).

3. Generalized affine functions and Stone-Weierstrass type theorems

In this section, by using the properties of generalized affine functions, we shall present a characterization of the closed vector sublattice and the closed Stone vector sublattice generated by subsets of regular vector lattices. As a consequence, we shall determine some density results, i.e. Stone-Weierstrass type theorems.

Our results generalize, in particular, the ones obtained in [2, Section 4.4] and would be compared with several other results scattered in the literature (see, e.g., [5, 6, 8, 11, 13, 14, 15, 16, 19]).

Throughout this section we fix a regular vector lattice (E,τ) on a locally compact Hausdorff space X.

We first proceed to characterize those subspaces of E with coincide with the corresponding space of generalized affine functions.

A first result in this direction has been already obtained in [3, Corollary 3.8 and Corollary 4.4].

Proposition 3.1

Let (E,τ) be a regular vector lattice on a locally compact Hausdorff space X and H a closed vector sublattice of E. Then $H = \widehat{H}_\tau$.

More generally, we have the following result (see also [2, Theorem 4.1.17]).

Theorem 3.2

Let (E,τ) be a regular vector lattice on a locally compact Hausdorff space X and H a subspace of E. Then the following statements are equivalent:

(a) $\widehat{H}_\tau = H$;

(b) *There exists a closed convex inf-stable cone* P *of* (E,τ) *such that* $H = P \cap (-P)$.

Proof. (a)⇒(b). Set

$$P := \left\{ f \in E \mid \int f \; d\mu \le f(x) \text{ for every } x \in X \text{ and } \mu \in M_{\tau,x}(H) \right\}.$$

Then P is obviously a convex cone of E and, taking Theorem 2.4 into account, we get that $H = P \cap (-P)$.

P is also inf-stable, since, if $f, g \in P$, denoting by $w := \inf\{f, g\}$, for every $x \in X$ and $\mu \in M_{\tau,x}(H)$, we get

$$\int w \; d\mu \le \int f \; d\mu \le f(x)$$

and

$$\int w \; d\mu \le \int g \; d\mu \le g(x).$$

Hence

$$\int w \; d\mu \le \inf\{f(x), g(x)\} = w(x).$$

Finally, we have to show that P is closed in (E,τ). To this end, fix $f \in \overline{P}$, $x \in X$ and $\mu \in M_{\tau,x}(H)$.

Since the positive linear form

$$g \in E \mapsto \int g \; d\mu$$

is τ-continuous, given $\epsilon > 0$, there exists $V \in \mathcal{U}_\tau$ such that for every $g \in E$ with $f - g \in V$ we get

$$\left| \int (f-g) \; d\mu \right| \le \epsilon. \tag{3.1}$$

Moreover, the positive linear form $\delta_x : E \to \mathbb{R}$ defined by $\delta_x(h) := h(x)$, $(h \in E)$, by Property (R_2) is τ-continuous and hence there exists $U \in \mathcal{U}_\tau$ such that for every $g \in E$, $f - g \in U$

$$\mid f(x) - g(x) \mid \le \epsilon. \tag{3.2}$$

Set $W := V \cap U$; then there exists $g \in P$ such that $f - g \in W$. Accordingly, from (3.1) and (3.2) it follows that

$$\int f \; d\mu = \int (f-g) \; d\mu + \int g \; d\mu \le \left| \int (f-g) \; d\mu \right| + g(x) \le f(x) + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary chosen, we obtain $\int f \; d\mu \le f(x)$.

(b)⇒(a). We have only to show that $\widehat{H}_\tau \subset H$ and, to this end, it is enough to show that $\widehat{H}_\tau \subset P$. Hence, fix $f \in \widehat{H}_\tau$, $V \in \mathcal{U}_\tau$ and $W \in \mathcal{U}_\tau$ such that $W + W + W \subset V$.

Taking Theorem 2.4 into account, there exist $k_1, ..., k_n, k'_1, ..., k'_m \in H \subset P$ such that $(f - k_i)^+ \in W$ for every $i = 1, ..., n$, $(k'_j - f)^+ \in W$ for every $j = 1, ..., m$ and $\inf\limits_{1 \le i \le n} k_i - \sup\limits_{1 \le j \le m} k'_j \in W$.

Moreover, for every $i = 1, ..., n$

$$f - k_i \leq (f - k_i)^+ \leq \sup_{1 \leq i \leq n} (f - k_i)^+ =: u \in W$$

and for every $j = 1, ..., m$

$$k'_j - f \leq (k'_j - f)^+ \leq \sup_{1 \leq j \leq m} (k'_j - f)^+ =: v \in W.$$

Accordingly, we get

$$f - \inf_{1 \leq i \leq n} k_i \leq u \qquad \text{and} \qquad \sup_{1 \leq j \leq m} k'_j - f \leq v.$$

Hence

$$\inf_{1 \leq i \leq n} k_i - f \leq \inf_{1 \leq i \leq n} k_i - \sup_{1 \leq j \leq m} k'_j + \sup_{1 \leq j \leq m} k'_j - f \leq \inf_{1 \leq i \leq n} k_i - \sup_{1 \leq j \leq m} k'_j + v,$$

so that

$$\left| \inf_{1 \leq i \leq n} k_i - f \right| \leq \inf_{1 \leq i \leq n} k_i - \sup_{1 \leq j \leq m} k'_j + v + u \in W + W + W \subset V.$$

Then $f \in \overline{P}$ and, since by assumption P is closed, this completes the proof. $\square$

In order to present the next results we need to introduce some additional notation.

Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X and let M be a subset of E. Then, for every $x \in X$ and $\alpha \geq 0$ we set

$$M(x, \alpha) := \{y \in X \mid h(y) = \alpha h(x) \;\text{ for every }\; h \in M\} \tag{3.3}$$

and

$$A(x) := \{\alpha \geq 0 \mid M(x, \alpha) \neq \emptyset\}.$$

Clearly, $1 \in A(x)$, since $x \in M(x, 1)$.

We shall also consider the sets

$$Z(M) := \{x \in X \mid h(x) = 0 \;\text{ for every }\; h \in M\}$$

and

$$I(M) := \begin{cases} \{f \in E \mid f(x) = 0 \;\text{ for every }\; x \in Z(M)\} & \text{if } Z(M) \neq \emptyset; \\ E & \text{if } Z(M) = \emptyset. \end{cases}$$

We are now in a position to state the following result.

Theorem 3.3

Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X and let H be a vector sublattice of E. Then

$$\begin{aligned} \overline{H} = \widehat{H}_\tau = I(H) \cap \{f \in E \mid f(y) = \alpha f(x) \\ \textit{for every } \; x \in X, \alpha \in A(x) \;\textit{ and }\; y \in H(x, \alpha)\}. \end{aligned} \tag{3.4}$$

Proof. The first equality follows from Proposition 3.1. We shall prove the second one.

Consider $f \in \widehat{H}_\tau$; taking Theorem 2.4 into account, it is obvious that $f \in I(H)$.

Moreover, fix $x \in X$, $\alpha \in A(x)$ and $y \in H(x,\alpha)$. Setting $\mu = \alpha\epsilon_x$, by (3.3), we get that $\mu \in M_{\tau,y}(H)$ and, from this, it follows that $f(y) = \int f \ d\mu = \alpha f(x)$.

Fix now a function $f \in I(H)$ such that $f(y) = \alpha f(x)$ for every $x \in X$, $\alpha \in A(x)$ and $y \in H(x,\alpha)$; further, consider $x \in X$ and $\mu \in M_{\tau,x}(H)$. Because of Theorem 2.4, it will be enough to prove that $\int f \ d\mu = f(x)$.

To this end, assume first $x \in Z(H)$; accordingly, since H is a vector sublattice of E, for every $h \in H$

$$0 = |h(x)| = \int |h| \ d\mu.$$

Hence, $\mathrm{supp}(\mu) \subset \{x \in X \mid h(x) = 0 \text{ for every } h \in H\} = Z(H)$; then, since $f \in I(H)$, we get

$$\int f \ d\mu = 0 = f(x).$$

Assume now that there exists $h \in H$ such that $h(x) \neq 0$. Without loss of generality, since H is a vector sublattice of E, we may assume that $h \geq 0$ and $h(x) = 1$.

We now prove that $\mathrm{supp}(\mu) \subset \bigcup_{\alpha\in A(x)} H(x,\alpha)$.

To this end, consider $y \in X$ such that $y \notin \bigcup_{\alpha\in A(x)} H(x,\alpha)$. If $h(y) \notin A(x)$, then $H(x,h(y)) = \emptyset$, while, if $h(y) \in A(x)$, then $y \notin H(x,h(y))$; hence, in any case, there exists $k \in H$ such that $k(y) \neq h(y)k(x)$.

Set $k_0 :=| k - k(x)h | \in H$; then

$$k_0 \geq 0, \qquad k_0(x) = 0 \qquad \text{and} \qquad k_0(y) > 0.$$

Consequently, from Lemma 2.1 it follows that $\mathrm{supp}(\mu) \subset \bigcup_{\alpha\in A(x)} H(x,\alpha)$. Accordingly, if $f(x) = 0$, then $f = 0$ on $H(x,\alpha)$ for each $\alpha \in A(x)$; thus

$$\int f \ d\mu = 0 = f(x).$$

On the other hand, if $f(x) \neq 0$, then, choosing again $h \in H$ such that $h(x) \neq 0$, for every $\alpha \in A(x)$ and $y \in H(x,\alpha)$ we get

$$h(y) = \alpha h(x) = \frac{f(y)}{f(x)} h(x).$$

Hence, $h = \frac{h(x)}{f(x)} f$ on $\mathrm{supp}(\mu)$, so that

$$h(x) = \int h \ d\mu = \frac{h(x)}{f(x)} \int f \ d\mu$$

and $\int f \ d\mu = f(x)$. $\square$

We now recall that a subset H of $C(X,\mathbb{R})$ *separates linearly* the points of X if for every $x, y \in X$, $x \neq y$, there exist $h, k \in H$ such that $h(x)k(y) \neq h(y)k(x)$.

It is easy to show that H separates linearly the points of X if and only if for every $x, y \in X$, $x \neq y$, and for every $\alpha \in \mathbb{R}$ there exists $h \in H$ such that $h(x) \neq \alpha h(y)$.

In particular, if $H = H_+ - H_+$ (and, hence, in particular if H is a vector sublattice of $C(X,\mathbb{R})$) then H separates linearly the points of X if and only if for every $x, y \in X$, $x \neq y$, and for every $\alpha \geq 0$ there exists $h \in H$ (that can be chosen to be positive) such that $h(x) \neq \alpha h(y)$.

Finally, if H separates linearly the points of X, then for every $x \in X$ there exists $h \in H$ such that $h(x) \neq 0$.

From Theorem 3.3 the following density result, which improves [3, Corollary 4.5], immediately follows.

Corollary 3.4

Let (E,τ) be a regular vector lattice on a locally compact Hausdorff space X and let H be a vector sublattice of E. Then H is dense in (E,τ) if and only if H separates linearly the points of X.

Proof. Indeed, if H separates linearly the points of X, taking the previous theorem into account, it is obvious that H is dense in (E,τ).

Conversely, assume that H is dense in (E,τ); this implies that $I(H) = E$ and

$$\{f \in E \mid f(y) = \alpha f(x) \;\text{ for every }\; x \in X, \alpha \in A(x) \;\text{ and }\; y \in H(x,\alpha)\} = E,$$

since τ is finer than τ_s.

Assume that H doesn't separate linearly the points of X. Consequently, there exists $x, y \in X$, $x \neq y$ and $\alpha \geq 0$ such that $h(y) = \alpha h(x)$ for every $h \in H$; hence $\alpha \in A(x)$ and $y \in H(x,\alpha)$. Accordingly, $f(y) = \alpha f(x)$ for every $f \in E$, but this is not possible, since $C_c(X,\mathbb{R}) \subset E$. The proof is now complete. ☐

Let M be a subset of a regular vector lattice (E,τ). We shall denote by $R(M)$ the closed vector sublattice generated by M, i.e. the smallest closed vector sublattice of (E,τ) containing M. With the next result, we shall characterize $R(M)$.

Corollary 3.5

Let (E,τ) be a regular vector lattice on a locally compact Hausdorff space X and let M be a subset of E. Then

$$\begin{aligned} R(M) = I(M) \cap \{f \in E \mid f(y) = \alpha f(x) \\ \textit{for every }\; x \in X, \alpha \in A(x) \;\textit{and}\; y \in M(x,\alpha)\}. \end{aligned} \tag{3.5}$$

Therefore $R(M) = E$ if and only if M separates linearly the points of X.

Proof. Because of Theorem 3.3, it is enough to show that $Z(M) = Z(R(M))$ and that $M(x,\alpha) = R(M)(x,\alpha)$ for every $x \in X$ and $\alpha \geq 0$.

Obviously, we get that $Z(R(M)) \subset R(M)$; consider now $x \in X$ such that $h(x) = 0$ for every $h \in M$ and set

$$F := \{f \in E \mid f(x) = 0\}.$$

Since the functional $\delta_x(h) := h(x)$, $(h \in E)$, is τ-continuous, F is a closed vector sublattice of E containing M. Accordingly, $R(M) \subset F$ and thus $x \in Z(R(M))$.

Since $M \subset R(M)$ it is also clear that $R(M)(x, \alpha) \subset M(x, \alpha)$ for every $x \in X$ and $\alpha \geq 0$. To show the opposite inclusion, fix $x \in X$, $\alpha \geq 0$ and $y \in M(x, \alpha)$ and set

$$F := \{f \in E \mid f(x) = \alpha f(y)\}.$$

Since (E, τ) is a regular vector lattice, F is a closed vector sublattice of E containing M, so that $R(M) \subset F$; hence, $y \in R(M)(x, \alpha)$ and the proof is complete. $\square$

With some further assumptions, it is possible to present another generalization of the classical Stone-Weierstrass Theorem.

We recall that a vector sublattice H of $C(X, \mathbb{R})$ is said to be a *Stone vector sublattice* if $\inf\{f, \mathbf{1}\} \in H$ for every $f \in H$.

Moreover, we recall that a subset H of $C(X, \mathbb{R})$ *separates strongly* the points of X if

(i) For every $x \in X$ there exists $h \in H$ such that $h(x) \neq 0$;
(ii) For every $x, y \in X$, $x \neq y$, there exists $h \in H$ such that $h(x) \neq h(y)$.

Finally, let (E, τ) be a regular vector lattice and let M be a subset of E. We shall set

$$M(x) := \{y \in X \mid h(y) = h(x) \text{ for every } h \in M\}.$$

We are now in a position to state the following result.

Theorem 3.6

Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X and let H be a Stone vector sublattice of E. Then

$$\overline{H} = \widehat{H}_\tau = I(H) \cap \{f \in E \mid f \text{ is constant on } H(x) \text{ for every } x \in X\}. \tag{3.6}$$

Consequently, H is dense in (E, τ) if and only if H separates strongly the points of X.

Proof. The first equality clearly follows from Proposition 3.1; we shall prove the second one.

If $f \in \widehat{H}_\tau$, then by Theorem 2.4 it easily follows that $f \in I(H)$. Consider now $x \in X$, $y \in H(x)$ and $\mu \in M_{\tau,x}(H)$, so that $\int f \, d\mu = f(x)$. Then $\mu \in M_{\tau,y}(H)$; hence, again from Theorem 2.4, we get $f(y) = \int f \, d\mu = f(x)$ and this implies that f is constant on $H(x)$.

Conversely, fix a function $f \in I(H)$ such that f is constant on $H(x)$ for every $x \in X$. Moreover, consider $x \in X$ and $\mu \in M_{\tau,x}(H)$. We shall now prove that $\int f \, d\mu = f(x)$.

If $h(x) = 0$ for each $h \in H$, then $f(x) = 0$ and, arguing as in the proof of Theorem 3.3, we get $\int f \, d\mu = 0 = f(x)$.

On the other hand, if there exists $h \in H$ such that $h(x) \neq 0$, clearly $H(x) \cap Z(H) = \emptyset$. Moreover, if $y \notin H(x) \cup Z(H)$, there exists $k \in H$ such that $k(y) \neq k(x)$. Furthermore, since $x, y \notin Z(H)$ and H is a vector sublattice of E, there exist $h_1, h_2 \in H$, $h_1, h_2 \geq 0$, such that $h_1(x) = h_2(y) = 1$.

Put

$$h_0 := \inf\{\mathbf{1}, \sup\{h_1, h_2\}\} \in H$$

and

$$h := |k - k(x)h_0| \in H.$$

Then

$$h \geq 0, \qquad h(x) = 0 \qquad \text{and} \qquad h(y) > 0.$$

By applying Lemma 2.1, we get $\operatorname{supp}(\mu) \subset H(x) \cup Z(H)$.

Consequently, if $f(x) = 0$, then $f = 0$ on $H(x) \cup Z(H)$ and hence $\int f \, d\mu = 0 = f(x)$.

On the other hand, if $f(x) \neq 0$, after choosing $h \in H$ such that $h(x) \neq 0$, we get that for every $y \in H(x) \cup Z(H)$

$$h(y) = \frac{f(y)}{f(x)} h(x).$$

Hence, $h = \frac{h(x)}{f(x)} f$ on $\operatorname{supp}(\mu)$, so that

$$h(x) = \int h \, d\mu = \frac{h(x)}{f(x)} \int f \, d\mu$$

and this completes the proof.

The final statement is a consequence of the first part. □

Assume now that the whole space (E, τ) is a Stone vector sublattice of $C(X, \mathbb{R})$.

It is easy to show that the regular vector lattices presented in Examples 2.2 are Stone vector sublattices of $C(X, \mathbb{R})$.

If M is a subset of E, we shall denote by $S(M)$ the closed Stone vector sublattice generated by M (i.e. the smallest closed Stone vector sublattice of (E, τ) containing M).

With the next result, we shall present a characterization of $S(M)$.

Corollary 3.7

Let (E, τ) be a Stone regular vector lattice on a locally compact Hausdorff space X and let M be a subset of E. Then

$$S(M) = I(M) \cap \{f \in E \mid f \ \text{ is constant on } \ M(x) \ \text{ for every } \ x \in X\}. \tag{3.7}$$

Therefore $S(M) = E$ if and only if M separates strongly the points of X.

Proof. We shall use Theorem 3.6 and so we show that $Z(M) = Z(S(M))$ and $M(x) = S(M)(x)$ for every $x \in X$. In fact, we have only to prove the second equality.

Since $M \subset S(M)$ it is clear that $S(M)(x) \subset M(x)$ for every $x \in X$. Fix now $x \in X$ and $y \in M(x)$ and set

$$F := \{f \in E \mid f(x) = f(y)\}.$$

Since τ is finer than τ_s, F is a closed Stone vector sublattice of (E, τ) containing M; accordingly, $S(M) \subset F$. Therefore $y \in S(M)(x)$ and the proof is complete. $\square$

4. Subalgebras and Stone vector lattices

In what follows, under suitable assumption, we shall present sufficient conditions in order that the closure of a subalgebra of a regular vector lattice is a Stone vector lattice. This will lead us to a characterization of the closed subalgebra generated by a subset of a regular vector lattice (see also [2, Theorem 4.4.4]). Finally, a relationship with Korovkin closures will be also shown.

We recall that a linear subspace P of a vector lattice E is said to be an ***ideal*** of E if P is a solid vector sublattice of E.

Every ideal of E is a Stone vector sublattice of $C(X,\mathbb{R})$ because $|\inf\{f, \mathbf{1}\}| \leq |f|$ for every $f \in C(X,\mathbb{R})$.

For example, every weighted function space on a locally compact Hausdorff space X (see Example 2.2 (2)) is an ideal in $C(X,\mathbb{R})$.

Further, an ideal P of a vector lattice E is said to be a ***band*** in E if, for every $D \subset P$ for which there exist $f = \sup D \in E$, we have that $f \in P$.

For more details about bands and ideals in vector lattices, see, for example, [20, Section 7].

In the sequel, for a subalgebra of $C(X,\mathbb{R})$ we mean a linear subspace A of $C(X,\mathbb{R})$ such that $f \cdot g \in A$ for every $f, g \in A$.

We state a preliminary lemma, which holds true in the context of more general vector lattices.

Lemma 4.1

Let X be a locally compact Hausdorff space and (E, τ) a locally convex vector sublattice of $C(X,\mathbb{R})$ which is a subalgebra and a band in $C(X,\mathbb{R})$ and satisfies Dini's property (see Theorem 2.3). Let A be a subalgebra of E. For every $f \in A$, $f \geq 0$, and $m \in \mathbb{N}$, consider the function $\varphi_{f,m} : X \to \mathbb{R}$ defined by

$$\varphi_{f,m}(x) := f^m(x)e^{-f(x)} \qquad (x \in X).$$

Then $\varphi_{f,m} \in \overline{A}$.

Proof. Fix a function $f \in A$, $f \geq 0$. Then, $e^{-f} \in E$, since $0 \leq e^{-f} \leq \inf\{f, \mathbf{1}\}$ and E is a Stone vector lattice. Hence, $\varphi_{f,m} \in E$ for every $m \in \mathbb{N}$.

Furthermore, for a given $m \in \mathbb{N}$ and $x \in X$,

$$\varphi_{f,m}(x) = f^m(x)e^{-f(x)} = f^m(x)\sum_{k=0}^{\infty}\frac{(-f(x))^k}{k!}$$
$$= \sum_{k=0}^{\infty}\frac{f(x)^{2k+m}}{(2k)!} - \sum_{k=0}^{\infty}\frac{f(x)^{2k+m+1}}{(2k+1)!}.$$

$\left(\sum\limits_{k=0}^{p} \frac{f^{2k+m}}{(2k)!}\right)_{p\geq 0}$ is an increasing sequence in A and, clearly, for every $x \in X$

$$\sum_{k=0}^{\infty} \frac{f(x)^{2k+m}}{(2k)!} = \lim_{p\to+\infty} \sum_{k=0}^{p} \frac{f(x)^{2k+m}}{(2k)!},$$

and $\sum\limits_{k=0}^{\infty} \frac{f^{2k+m}}{(2k)!} \in E$ since E is a band in $C(X,\mathbb{R})$.

Accordingly, since (E,τ) satisfies Dini's property, it follows that

$$\sum_{k=0}^{\infty} \frac{f^{2k+m}}{(2k)!} = \lim_{p\to+\infty} \sum_{k=0}^{p} \frac{f^{2k+m}}{(2k)!} \quad \text{in } (E,\tau);$$

thus, $\sum\limits_{k=0}^{\infty} \frac{f^{2k+m}}{(2k)!} \in \overline{A}$.

Analogously, $\sum\limits_{k=0}^{\infty} \frac{f^{2k+1+m}}{(2k+1)!} \in \overline{A}$; consequently, $\varphi_{f,m} \in \overline{A}$. □

The following two lemmas improve some results proved in [2, Lemma 4.4.3] and in [13, Lemma 2, p. 45].

Lemma 4.2

Let X be a locally compact Hausdorff space and (E,τ) a regular vector lattice on X which is a subalgebra and a band in $C(X,\mathbb{R})$. Then the closure of every subalgebra of (E,τ) is a Stone vector sublattice of E.

Proof. Let A be a subalgebra of (E,τ); we shall prove that $\overline{A}$ is a sublattice of E and $\inf\{f,\mathbf{1}\} \in \overline{A}$ for every $f \in \overline{A}$.

Firts of all we shall show that $|f| \in \overline{A}$ for every $f \in A$. To this end, consider the function $\varphi : [0,+\infty[\to \mathbb{R}$ defined by $\varphi(t) := t^{1/2}$, $(t \geq 0)$ and, for every $n \geq 1$, let s_n be the function obtained by applying to φ the n-th Szász-Mirakjan operator, i.e.

$$s_n(t) := \sum_{k=0}^{\infty} e^{-nt} \frac{(nt)^k}{k!} \left(\frac{k}{n}\right)^{1/2} \qquad (t \geq 0).$$

Since φ is concave, we have that $s_n \leq s_{n+1}$ and $s_n(t) \to t^{1/2}$ for $n \to +\infty$ for every $t \geq 0$ (see [9, Theorem at p. 246]).

Now fix $f \in A$; for every $x \in X$

$$|f(x)| = \left(f^2(x)\right)^{1/2} = \lim_{n\to+\infty} s_n\left(f^2(x)\right) = \lim_{n\to+\infty} f_n(x),$$

where

$$f_n(x) := s_n\left(f^2(x)\right) = \sum_{k=0}^{\infty} e^{-nf^2(x)} \frac{(nf^2(x))^k}{k!} \left(\frac{k}{n}\right)^{1/2}.$$

We have that $f_n \in \overline{A}$ for any $n \geq 1$.

Indeed, for a given $n \geq 1$, $\left(\sum\limits_{k=0}^{p} e^{-nf^2} \frac{(nf^2)^k}{k!} \left(\frac{k}{n}\right)^{1/2}\right)_{p \geq 0}$ is an increasing sequence in $\overline{A}$ (see Lemma 4.1) and for any $x \in X$

$$\lim_{p \to +\infty} \sum_{k=0}^{p} e^{-nf^2(x)} \frac{(nf^2(x))^k}{k!} \left(\frac{k}{n}\right)^{1/2} = \sum_{k=0}^{\infty} e^{-nf^2(x)} \frac{(nf^2(x))^k}{k!} \left(\frac{k}{n}\right)^{1/2};$$

since E is a band in $C(X, \mathbb{R})$, $\sum\limits_{k=0}^{\infty} e^{-nf^2} \frac{(nf^2)^k}{k!} \left(\frac{k}{n}\right)^{1/2} \in E$.

On the other hand, (E, τ) satisfies Dini's property (see Theorem 2.3), so that $f_n \in \overline{A}$.

Further, $f_n \leq f_{n+1}$ and $f_n \to |f|$ pointwise on X; since (E, τ) satisfies Dini's property, $f_n \to |f|$ in (E, τ) and hence $|f| \in \overline{A}$.

By Property (R_1), the mapping $f \longmapsto |f|$ from E into E is τ-continuous and hence $|g| \in \overline{A}$ for every $g \in \overline{A}$, i.e. $\overline{A}$ is a sublattice of E.

Arguing in the same way, one can show that $|f|^{1/2} \in \overline{A}$ for every $f \in A$ and hence, by induction, $|f|^{1/2^n} \in \overline{A}$ for every $n \geq 1$.

After these preliminaries, we pass to prove that $\inf\{f, \mathbf{1}\} \in \overline{A}$ for every $f \in A$.

Indeed, for a given $f \in A$, $x \in X$ and $n \geq 1$, we have that, if $f(x) \leq 1$,

$$\inf\{|f|^{1/2^n}, f\}(x) = f(x) = \inf\{f, \mathbf{1}\}(x)$$

while, if $f(x) \geq 1$,

$$\inf\{|f|^{1/2^n}, f\}(x) = |f|^{1/2^n}(x) \to 1 \qquad (n \to +\infty).$$

Consequently, $\inf\{f, \mathbf{1}\} = \lim\limits_{n \to +\infty} \inf\{|f|^{1/2^n}, f\}$ pointwise on X and the sequence $(\inf\{|f|^{1/2^n}, f\})_{n \geq 1}$ is decreasing. Taking again Dini's property into account, we get

$$\inf\{f, \mathbf{1}\} = \lim_{n \to +\infty} \inf\{|f|^{1/2^n}, f\}$$

in (E, τ), and hence $\inf\{f, \mathbf{1}\} \in \overline{A}$.

Moreover, note that for every $f, g \in E$

$$|\inf\{f, \mathbf{1}\} - \inf\{g, \mathbf{1}\}| \leq |f - g|,$$

and, hence, by Property (R_1), the mapping $f \longmapsto \inf\{f, \mathbf{1}\}$ from E into E is τ-continuous. Therefore, $\inf\{g, \mathbf{1}\} \in \overline{A}$ for every $g \in \overline{A}$ and this completes the proof. $\square$

The spirit of the proof of the next result is very similar to the one of Lemma 4.2.

Lemma 4.3

Let X be a locally compact Hausdorff space, (E, τ) a Stone regular vector lattice on X. Then the closure of every subalgebra of $E \cap C_b(X, \mathbb{R})$ is a Stone vector sublattice of E.

Proof. Let A be a subalgebra of $E \cap C_b(X, \mathbb{R})$; first of all we shall prove that $|f| \in \overline{A}$ for every $f \in A$.

To this end, consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by $\varphi(t) := t^{1/2}$, $(t \geq 0)$ and, for every $n \geq 1$, the corresponding Bernstein polynomial:

$$b_n(t) := \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \left(\frac{k}{n}\right)^{1/2} \qquad (0 \leq t \leq 1).$$

Being φ concave, we have that $b_n \leq b_{n+1}$ and $b_n(t) \to t^{1/2}$ for $n \to +\infty$ uniformly on $[0,1]$ (see [2, Example 1 to Theorem 4.2.7 and Theorem 6.3.8]).

Fix, now a function $f \in A$; for every $x \in X$

$$\begin{aligned} |f(x)| &= \|f\|_\infty \left(\frac{f^2(x)}{\|f\|_\infty^2}\right)^{1/2} \\ &= \|f\|_\infty \lim_{n\to+\infty} b_n \left(\frac{f^2(x)}{\|f\|_\infty^2}\right) = \|f\|_\infty \lim_{n\to+\infty} f_n(x), \end{aligned}$$

where

$$f_n(x) := b_n \left(\frac{f^2(x)}{\|f\|_\infty^2}\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^{1/2} \left(\frac{f^2(x)}{\|f\|_\infty^2}\right)^k \left(1 - \frac{f^2(x)}{\|f\|_\infty^2}\right)^{n-k}.$$

Note that $b_n(0) = 0$, so that $f_n \in A$; moreover, $f_n \leq f_{n+1}$ and $\|f\|_\infty f_n \to |f|$ uniformly on X.

Since (E, τ) satisfies Dini's property (see Theorem 2.3), $|f| = \lim_{n\to+\infty} \|f\|_\infty f_n$ in (E, τ); hence, $|f| \in \overline{A}$.

By Property (R_1), the mapping $f \longmapsto |f|$ from E into E is τ-continuous and hence $|g| \in \overline{A}$ for every $g \in \overline{A}$, i.e. $\overline{A}$ is a sublattice of E.

Reasoning as in the proof of Lemma 4.2, one can also prove that $\inf\{f, \mathbf{1}\} \in \overline{A}$ for every $f \in \overline{A}$ and this completes the proof. $\square$

As a first consequence of the lemmas above we obtain the following two results, which would be compared with [13, Lemma 1, p. 94] and [12, Proposition 3.7].

Assuming that E is a subalgebra of $C(X, \mathbb{R})$, for a given subset M of E, we denote by $B(M)$ the subalgebra generated by M, i.e. the smallest subalgebra of E containing M.

Proposition 4.4

Let X be a locally compact Hausdorff space and (E, τ) a regular vector lattice on X which is a subalgebra and a band in $C(X, \mathbb{R})$. Let M be a subset of E. Then $S(M) \subset \overline{B(M)}$.

Therefore, if M separates strongly the points of X, then $B(M)$ is dense in (E, τ).

Proof. By Lemma 4.2, $\overline{B(M)}$ is a Stone vector sublattice of E and hence $S(M) \subset \overline{B(M)}$.

The last part of the statement follows from Corollary 3.7. $\square$

By applying a similar argument, one can show the following result.

Proposition 4.5

Let X be a locally compact Hausdorff space, (E, τ) a Stone regular vector lattice on X and M a subset of E. Assume that there exists $B(M)$ and it is contained in $C_b(X, \mathbb{R})$. Then $S(M) \subset \overline{B(M)}$.

Therefore, if M separates strongly the points of X, then $B(M)$ is dense in (E, τ).

By combining Corollary 3.7 and Lemma 4.2, we also get a description of the closed subalgebra $A(M)$ of (E, τ) generated by a subset M of E, i.e. the smallest closed subalgebra of (E, τ) containing M. Clearly, $\overline{B(M)} \subset A(M)$.

Theorem 4.6

Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X and assume that E is a subalgebra and a band of $C(X, \mathbb{R})$. Let M be a subset of E. Then

$$\begin{aligned} \overline{B(M)} &= A(M) = S(M) \\ &= I(M) \cap \{f \in E \mid f \ \text{ is constant on } \ M(x) \ \text{ for every } \ x \in X\}. \end{aligned} \tag{4.1}$$

Therefore, $A(M) = E$ if and only if M separates strongly the points of X.

Proof. From Proposition 4.4, we get $S(M) \subset \overline{B(M)} \subset A(M)$.

Moreover, $I(M) \cap \{f \in E \mid f$ is constant on $M(x)$ for every $x \in X\}$ is a closed subalgebra of (E, τ) containing M, so that

$$A(M) \subset I(M) \cap \{f \in E \mid f \text{ is constant on } M(x) \text{ for every } x \in X\}.$$

Then, by applying Corollary 3.7, our statement follows. □

Similarly, by using Lemma 4.3, we get the next result.

Theorem 4.7

Let (E, τ) be a Stone regular vector lattice on a locally compact Hausdorff space X and assume that E is a subalgebra of $C(X, \mathbb{R})$. Let M be a subset of E such that $A(M) \subset C_b(X, \mathbb{R})$. Then

$$\begin{aligned} \overline{B(M)} &= A(M) = S(M) \\ &= I(M) \cap \{f \in E \mid f \ \text{ is constant on } M(x) \ \text{ for every } \ x \in X\}. \end{aligned} \tag{4.2}$$

Therefore, $A(M) = E$ if and only if M separates strongly the points of X.

Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X; if $f_0 \in E$ and M is a subset of E, we shall set

$$f_0 M^n := \{f_0 \cdot g^n \mid g \in M\} \qquad (n \geq 1). \tag{4.3}$$

With the next last result, we shall characterize the space of the generalized affine functions related to some subspaces generated by subsets of the form (4.3) and we show their connection with the closed Stone sublattice as well as with the subalgebra generated by M.

Theorem 4.8

Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X. Let M be a subset of E and assume that there exists a strictly positive function $f_0 \in \mathcal{L}(M)$ such that $f_0 M, f_0 M^2 \subset E$. Then, denoting by H the subspace generated by $\{f_0\} \cup f_0 M \cup f_0 M^2$ and by $\mathrm{Kor}(H)_\tau$ its Korovkin closure (see the final part of Section 2),

$$\mathrm{Kor}(H)_\tau = \widehat{H}_\tau = \{f \in E \mid f \text{ is constant on } M(x) \text{ for every } x \in X\}. \tag{4.4}$$

In particular, if E is a Stone vector lattice, then

$$S(M) = \widehat{H}_\tau = \mathrm{Kor}(H)_\tau \tag{4.5}$$

and, if E is a subalgebra and a band in $C(X, \mathbb{R})$ (resp., a Stone vector lattice and a subalgebra such that $A(M) \subset C_b(X, \mathbb{R})$), then

$$A(M) = \widehat{H}_\tau = \mathrm{Kor}(H)_\tau. \tag{4.6}$$

Moreover, $\widehat{H}_\tau = E$ if and only if M separates the points of X.

Proof. Consider $f \in \widehat{H}_\tau$ and $x \in X$. If $y \in M(x)$, then $h(x) = h(y)$ for every $h \in H$, so that $f(x) = f(y)$ by Theorem 2.4.

Conversely, fix a function $f \in E$ such that f is constant on $M(x)$ for every $x \in X$ and consider $x \in X$ and $\mu \in M_{\tau,x}(H)$. If $y \notin M(x)$, there exists $h \in M$ such that $h(x) \neq h(y)$; then the function

$$k := f_0(h(x) - h)^2 \in H$$

satisfies the following properties:

$$k \geq 0, \qquad k(x) = 0 \qquad \text{and} \qquad k(y) > 0.$$

Then, by Lemma 2.1, $\mathrm{supp}(\mu) \subset M(x)$. Consequently, since $f = \frac{f(x)}{f_0(x)} f_0$ on $\mathrm{supp}(\mu)$,

$$\int f \, d\mu = \frac{f(x)}{f_0(x)} \int f_0 \, d\mu = f(x)$$

and this completes the proof by virtue of Theorem 2.4.

The last part of the theorem follows from Corollary 3.7, Theorem 4.6 and Theorem 4.7. □

Remark 4.9 Let (E, τ) be a regular vector lattice on a locally compact Hausdorff space X and assume that there exist a subset M of E and a strictly positive function $f_0 \in \mathcal{L}(M)$ such that $f_0 M, f_0 M^2 \subset E$, where $f_0 M^i$ is defined by (4.3) for $i = 1, 2$. Then, denoting by H the subspace generated by $\{f_0\} \cup f_0 M \cup f_0 M^2$, on account of the final remark of Section 2, we get that H is a *Korovkin subspace* in E for the topology τ (i.e. the Korovkin closure of H in E for τ coincides with E) if and only if M separates the points of X (compare this result with [4, Proposition 4.7, Proposition 4.8]).

Some concrete examples of Korovkin subspaces of this form in regular vector lattices are also presented in [4, Examples 4.9]; here we limit ourselves to notice that if (E, τ) is a regular vector lattice on a locally compact Hausdorff space X and there exists a strictly positive function $f_0 \in E$ such that $f_0^2, f_0^3 \in E$, then the subspace generated by $\{f_0, f_0^2, f_0^3\}$ is a Korovkin subspace in E for τ if and only if f_0 is one-to-one.

We finally point out that, as in the space $C_0(X, \mathbb{R})$, also in the general setting of regular vector lattices, Stone-Weierstrass type results such as Corollary 3.7, Theorem 4.6 and Theorem 4.7 are equivalent to a Korovkin-type result such as [4, Proposition 4.7].

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