The Horrocks-Mumford bundle restricted to planes

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ABSTRACT

We study the behavior of the Horrocks-Mumford bundle F_{HM} when restricted to a plane $\mathbb{P}^2 \subset \mathbb{P}^4$, looking for all possible minimal free resolutions for the restricted bundle. To each of the 6 resolutions (4 stable and 2 unstable) we find, we then associate a subvariety of the Grassmannian $\mathbb{G}(2, 4)$ of planes in $\mathbb{P}^4$. We thus obtain a filtration of the Grassmannian, which we describe in the second part of this work.

1. Introduction

The Horrocks-Mumford bundle is a stable rank-2 complex vector bundle on $\mathbb{P}^4$, and at the present state, it is -up to twist by line bundles and finite pullbacks- the only one of its kind known to be undecomposable (cf. Hartshorne's conjecture, [10]).

It can be defined as the cohomology of the following monad:

$$0 \rightarrow 5\mathcal{O}_{\mathbb{P}^4}(-1) \xrightarrow{B} 2\Omega^2_{\mathbb{P}^4}(2) \xrightarrow{A} 5\mathcal{O}_{\mathbb{P}^4} \rightarrow 0. \tag{1.1}$$

Once we have equipped the 5-dimensional complex vector space V with a basis $\{e_i\}_{i \in \mathbb{Z}_5}$, $A = (a_{ij})$ is a 2×5 matrix of 2-forms (see [4] or [7]):

$$A = \begin{pmatrix} a_{0i} := e_{i+2} \wedge e_{i+3} \\ a_{1i} := e_{i+1} \wedge e_{i+4} \end{pmatrix}$$

and $B = {}^t (A \cdot Q)$, where Q is just the matrix $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The construction of the bundle is due to Horrocks and Mumford, who discovered it in 1972 [13]. Nevertheless via the Hartshorne-Serre correspondence its existence is hand in glove with that of degree 10 abelian varieties in $\mathbb{P}^4$, that had already been proved by Comessatti [6] in 1916.

This work aims at studying the behavior of the HM-bundle F_{HM} when restricted to a plane $\mathbb{P}^2 \subset \mathbb{P}^4$.

Stable and unstable planes (the latter are also called jumping) are already known by [4]. Here we want to study minimal free resolutions for the restricted bundle.

We find out that there are 6 possible minimal resolutions $(\underline{a}, \underline{b})$, of the form:

$$0 \to \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(-a_i) \xrightarrow{\varphi} \oplus_{j=1}^{k+2} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to F_{HM}|_{\mathbb{P}^2} \to 0. \qquad (1.2)$$

We prove that all of them are actually assumed by the HM-bundle on some plane $\mathbb{P}^2 \subset \mathbb{P}^4$.

For each of these 6 resolutions, we then consider points of the Grassmannian $\mathbb{G}(2,4)$ where this particular resolution is assumed, forming the subvarieties that we call $\mathcal{V}_{(\underline{a},\underline{b})}$. Studying resolutions in connection with jumping phenomena, we obtain a detailed description of what these subvarieties look like.

The results are summarized in the:

Main Theorem. Let $\mathcal{V}_{(\underline{a},\underline{b})}$ be the subvarieties of the Grassmannian $\mathbb{G}(2,4)$ defined as follows:

$$\mathcal{V}_{(\underline{a},\underline{b})} := \left\{ \mathbb{P}^2 \subset \mathbb{P}^4 \mid F_{HM}|_{\mathbb{P}^2} \text{ has minimal free resolution } (\underline{a}, \underline{b}) \right\}.$$

Then we have the filtration:
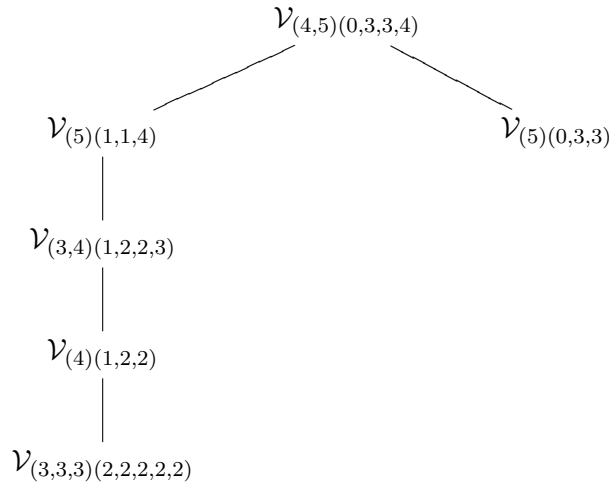
where:

- $\mathcal{V}_{(5)(0,3,3)}$ is a smooth surface of degree 25, formed by jumping planes. It is the image of the well known Shioda's surface under a complete linear system;
- $\mathcal{V}_{(4,5)(0,3,3,4)}$ consists of 25 smooth conics of jumping planes on the surface $\mathcal{V}_{(5)(0,3,3)}$;

- $\mathcal{V}_{(5)(1,1,4)}$ consists of 25 planes, formed by stable planes, each one containing one of the conics above described;
- $\mathcal{V}_{(3,4)(1,2,2,3)}$ is an irreducible 4-fold, formed by stable planes;
- $\mathcal{V}_{(4)(1,2,2)}$ is also formed by stable planes. It has dimension 5 and degree 5;
- $\mathcal{V}_{(3,3,3)(2,2,2,2,2)}$ is an open subset of $\mathbb{G}(2,4)$, associated to the generic stable resolution.

Notation. V is a fixed 5-dimensional vector space over $\mathbb{C}$, and $\mathbb{P}^4 = \mathbb{P}(V)$ is the projective space of hyperplanes in V, so that

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(1)) = V.$$

F_{HM} denotes the normalized Horrocks-Mumford bundle, with Chern classes $c_1 = -1$ and $c_2 = 4$. The stability of F_{HM} (in the Mumford-Takemoto sense) therefore simply means:

$$\mathrm{h}^0(F_{HM}) = \dim(\mathrm{H}^0(F_{HM})) = 0$$

as it is explained in [16], for example.

I would like to thank Professor Hulek for the useful discussion that we had in Ferrara and for the ideas that led us to prove Corollary 4.4.

For the original idea of this paper, however, and for supervision and encouragement, I wish to express my deepest gratitude to Professor Ottaviani.

2. Minimal resolutions

We are looking for all possible minimal free resolutions for $F_{HM}|_{\mathbb{P}^2}$. For this purpose, we use results on moduli spaces contained in [5, 8, 17] and [11].

From Horrocks' Theorem (see [12]) we learn that a vector bundle on $\mathbb{P}^n$ has homological dimension (*i.e.* the length of its minimal free resolution) at most $n-1$. This means that we are actually looking for short exact sequences of the form:

$$0 \to \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(-a_i) \xrightarrow{\varphi} \oplus_{j=1}^{k+2} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to F_{HM}|_{\mathbb{P}^2} \to 0 \qquad (2.1)$$

for some integers a_i, b_j and k. Notice here that $\mathrm{rk}(F_{HM}) = 2$.

We assume that the two sequences of integers a_i and b_j are indexed in nondecreasing order, and we call $(\underline{a}, \underline{b}) = ((a_1, ..., a_k), (b_1, ..., b_{k+2}))$ the *associated pair* to the resolution 2.1.

Our goal is now to determine bounds on a_i and b_j, in order to obtain all possible associated pairs for $F_{HM}|_{\mathbb{P}^2}$. We have:

Proposition 2.1

The integers a_i and b_j must satisfy the following six conditions:

(1) $\sum_{i=1}^{k} a_i - \sum_{j=1}^{k+2} b_j = -1$;

(2) $\sum_{i=1}^{k} a_i^2 - \sum_{j=1}^{k+2} b_j^2 = 7$;

(3) $a_i \geq b_{i+2} + 1, \ \forall\, i = 1...k$;

(4) $b_1 \geq 0$;

(5) $2b_{k+2} + k \leq 11$;

(6) $2a_k + k \leq 12$.

Proof. (1) and (2) come from asking the Chern classes of the bundle to be respectively -1 and 4, as we must have:

$$\sum_{i=1}^{k} a_i - \sum_{j=1}^{k+2} b_j = c_1(F_{HM}) = -1$$

$$\sum_{i=1}^{k} a_i^2 - \sum_{j=1}^{k+2} b_j^2 = 2c_2(F_{HM}) - c_1(F_{HM})^2 = 7.$$

Condition (3) corresponds to the minimality of the resolutions (see [5, Proposition 2]).

Condition (4) comes from a direct computation of the cohomology of the bundle restricted to a plane, which can be obtained from the cohomology of the bundle on $\mathbb{P}^4$, see [13]. More in detail, the Koszul complex splits in 2 short exact sequences:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0 \qquad (2.2)$$

$$\mathcal{I}_{\mathbb{P}^2, \mathbb{P}^4}$$

$$0 \qquad\qquad 0$$

which we can tensorize by F_{HM} and thus obtain that for any $\mathbb{P}^2 \subset \mathbb{P}^4$, $h^0(F_{HM}|_{\mathbb{P}^2}) = 0$. This latter condition implies (4), as proved in [8].

For (5) and (6) we modify an argument from [17] as follows. Using conditions (1), (2) and (3), we have:

$$7 = -b_1^2 - b_2^2 + \sum_{i=1}^{k} (a_i^2 - b_{i+2}^2)$$

$$= -b_1^2 - b_2^2 + \sum_{i=1}^{k} (a_i - b_{i+2})(a_i + b_{i+2} - 2b_2) + 2b_2(b_1 + b_2 - 1)$$

$$\geq -b_1^2 - b_2^2 + 2\sum_{i=1}^{k} (b_{i+2} - b_2) + k + 2b_2(b_1 + b_2 - 1)$$

$$\geq -b_1^2 - b_2^2 + 2b_{k+2} - 2b_2 + k + 2b_2(b_1 + b_2 - 1)$$

$$= -[b_1^2 + b_2^2 - 2b_2(b_1 + b_2 - 2)] + 2b_{k+2} + k.$$

Now looking at the hyperbola in the plane:

$$x^2 - y^2 - 2xy + 4y - 4 = 0,$$

we get that, $\forall\ 0 \leq b_1 \leq b_2 \in \mathbb{Z}$:

$$b_1^2 + b_2^2 - 2b_2(b_1 + b_2 - 2) \leq 4.$$

Thus

$$7 \geq -4 + 2b_{k+2} + k,$$

which proves (5). Condition (6) easily follows with an identical argument as for (5). □

A brute force check after the conditions of Proposition 2.1 proves the following:

Corollary 2.2

The only pairs (thus possible resolutions) satisfying the conditions of Proposition 2.1 are:

$$(5)\ (0,3,3)$$
$$(4,5)\ (0,3,3,4)$$

$$(5)\ (1,1,4)$$
$$(3,4)\ (1,2,2,3)$$
$$(4)\ (1,2,2)$$
$$(3,3,3)\ (2,2,2,2,2)\,.$$

From a first look at the cohomology of bundles associated to these six resolutions, we can immediately see that the first two are not stable, whereas the other four do not have sections, *i.e.* they are stable.

It is also noteworthy that there are two couples of resolutions that have the same cohomology: one is stable $((4)(1,2,2)$ and $(3,4)(1,2,2,3))$, and one is not $((5)(0,3,3)$ and $(4,5)(0,3,3,4))$. The reason why this can happen is that minimality of the resolutions implies that every time we have a constant map:

$$\mathcal{O}(-a) \to \mathcal{O}(-a),$$

it has to be the zero map (again, [5, Proposition 2]).

Anyhow, in case we wanted a cohomological distinction between those resolution forming couples, we could obtain it by tensorizing for an appropriate sheaf of forms.

Notice that all bundles we are dealing with are prioritaries, according to [11].

Remark 2.3 In what follows we will often write $(\underline{a}, \underline{b})$ meaning either the resolution, or the associated pair, or sometimes even a bundle with that resolution (or better, its class of isomorphism in the moduli space). What we are referring to will be clear from the context.

3. Jumping phenomena

It is known that F_{HM} remains stable when it is restricted to any hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ (see [4, 7], or [14]).

We give the following:

DEFINITION 3.1 The subspace $X \subset \mathbb{P}^4$ is a *jumping space* for the HM-bundle iff $\mathrm{H}^0(F_{HM}|_X) \neq 0$.

Hence F_{HM} has no jumping 3-spaces, but it admits jumping spaces of lower dimension, planes and lines.

For lines in particular we have a more subtle definition: jumping lines are those where the bundle doesn't have the generic splitting type entailed by Grauert-Mülich Theorem. The measure of how much a jumping line drifts away from the generic case is given by its jumping order.

In other words:

DEFINITION 3.2 A line $\ell \subset \mathbb{P}^4$ is a *jumping line of order k* for HM (also called a *k-jumping line*) iff $\mathrm{H}^0(F_{HM}|_\ell(-k)) \neq 0$.

HM-bundle's jumping lines and planes have been deeply studied by Barth, Hulek and Moore in [4].

The structure of jumping planes is explained in the following result:

Theorem 3.3

A plane $\mathbb{P}^2 \in \mathbb{G}(2,4) \subset \mathbb{P}^9$, with Plückerian coordinates p_{ij}, is a jumping plane iff $\mathrm{rk}(A \otimes \mathcal{O}_{\mathbb{P}^2}) \leq 1$, where

$$A \otimes \mathcal{O}_{\mathbb{P}^2} = \begin{pmatrix} p_{23} & p_{34} & -p_{04} & p_{01} & p_{12} \\ p_{14} & -p_{02} & -p_{13} & -p_{24} & p_{03} \end{pmatrix}. \tag{3.1}$$

Jumping planes thus form a smooth surface of degree 25 in $\mathbb{P}^9$, called S_{25}, which is birational to Shioda's modular surface $S(5)$. S_{25} may also be seen as the transverse intersection of the Grassmannian $\mathbb{G}(2,4)$ with the (suitably normalized) Segre variety $\mathbb{P}^1 \times \mathbb{P}^4$.

Proof. The result simply derives from translating in Plücker coordinates the condition $\mathrm{h}^0 \neq 0$. Note that the matrix A is the one used to define the monad in (1.1).

Let our $\mathbb{P}^2 \subset \mathbb{P}^4$ be defined as zero locus of the 2 hyperplanes ξ^* and ζ^*. Then $\mathbb{P}^2 = \{\sum_0^4 \xi_i x_i = \sum_0^4 \zeta_i x_i = 0\}$. Restricting A to $\mathbb{P}^2$ means now contracting the 2-forms a_{ij} with the 2-form $\xi^* \wedge \zeta^*$ associated to the plane, i.e. $A \otimes \mathcal{O}_{\mathbb{P}^2} = (\xi^* \wedge \zeta^*(a_{ij}))$, hence Plückerian coordinates $p_{ij} = \xi_i \zeta_j - \xi_j \zeta_i$ appear.

For details, see [7], Proposition 2, and [4], Section 3.4.   □

Notice that we have a natural map $S_{25} \to \mathbb{P}^1$, and that the general fibers are transverse linear sections of $\mathbb{G}(2,4)$, hence they are elliptic normal quintics in $\mathbb{P}^4$.

For the central role of Shioda's modular surface in the understanding of HM-bundle jumping phenomena, we refer the reader to the detailed articles [4, 3], and to [14].

Here we just limit ourselves to mention the result on jumping lines, which we will need later on. Our bundle admits jumping lines of order 1, 2 and 3. More precisely:

Theorem 3.4 ([4], Section 1.2 and 5.5)

Let $J_i(F_{HM}) \subset \mathbb{G}(1,4)$, for $i = 1, 2, 3$, be the subvarieties of jumping lines for the HM-bundle. $J_1(F_{HM})$, is a rational, irreducible 4-fold. It is smooth outside of

$J_2(F_{HM})$. *The variety $J_2 \subset J_1$ has dimension 2, and it is birational to Shioda surface $S(5)$. In fact it is nothing but $S(5)$ with its 25 sections blown down to 25 singular points, forming $J_3(F_{HM})$. J_3 is formed by 25 points, corresponding in $\mathbb{P}^4$ to 25 skew lines, the so called Horrocks-Mumford lines.*

From this first look at jumping spaces, we can already infer that at least two out of the six possible resolutions are actually assumed by $F_{HM}|_{\mathbb{P}^2}$: one of them will be stable, the other not.

The situation so far is then:

$$S_{25} \subset \mathbb{G}(2,4) \subset \mathbb{P}^9 \tag{3.2}$$

where S_{25} "contains" the 2 unstable resolutions, the other 4 being "contained" in $\mathbb{G}(2,4) \setminus S_{25}$.

Keeping in mind Horrocks-Mumford bundle's jumping phenomena, we now want to analyze jumping lines admitted by each of our six resolutions.

The situation is described in the following result:

Proposition 3.5

- *Part A*

  - $(5)(0,3,3)$ *admits jumping lines of order 1 and 2.*

    Take the 4 points of intersection of the 2 conics q_i, zero locus of $q_i : \mathcal{O}_{\mathbb{P}^2}(-5) \to \mathcal{O}_{\mathbb{P}^2}(-3)$. The generic structure then is the following: the 4 pencils of lines through these 4 points are the jumping lines, while the 6 lines connecting pairs among the 4 points are double jumping lines.

    The 4 points are exactly those where the unique section violating stability vanishes.

  - $(4,5)(0,3,3,4)$ *generically admits only 1 jumping line ℓ_1, of order 3. It is the zero locus of $\ell_1 : \mathcal{O}_{\mathbb{P}^2}(-5) \to \mathcal{O}_{\mathbb{P}^2}(-4)$.*

  - $(5)(1,1,4)$ *generically admits only 1 triple jumping line ℓ_2. It is the zero locus of $\ell_2 : \mathcal{O}_{\mathbb{P}^2}(-5) \to \mathcal{O}_{\mathbb{P}^2}(-4)$.*

  - $(3,4)(1,2,2,3)$ *generically admits only 1 double jumping line ℓ_3, zero locus of $\ell_3 : \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-3)$.*

  - $(4)(1,2,2)$ *admits only jumping lines of order 1.*

    Those are, in the generic case, the 6 lines connecting pairs among the 4 points of intersection of the conics zero loci of $q_i : \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-2)$

- *Part B*

  - $(3,3,3)(2,2,2,2,2)$ *admits only 1-jumping lines.*

    They can be either 6 lines with normal crossing, or they can form a conic.

Proof of Part A.

We will not go through all the 6 cases in detail, because we use exactly the same argument for the 5 resolutions of Part A.

Take for example a vector bundle E associated with $(5)(0,3,3)$. We have

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-5) \xrightarrow{\varphi} 2\mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2} \to E \to 0 \qquad (3.3)$$

where

$${}^t\varphi = \boxed{\begin{array}{c|c|c} q_1 & q_2 & r \end{array}}$$

and the q_i's are quadratic polynomials, while r has degree 5.

Once dualized and restricted to a generic line $\ell \subset \mathbb{P}^4$, the sequence (3.3) becomes (remember that $E^* = E(-c_1) = E(1)$):

$$0 \to E|_\ell \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1}(4) \to 0 \,. \qquad (3.4)$$

If we look at its cohomology sequence, we get:

$$0 \to \mathrm{H}^0(E|_\ell(-1)) \to 2V \xrightarrow{\Phi} S^3(V)$$
$$(\mathcal{L}_1, \mathcal{L}_2) \mapsto (\mathcal{L}_1 \cdot q_1|_\ell + \mathcal{L}_2 \cdot q_2|_\ell).$$

Now the two schemes $q_1|_\ell$ and $q_2|_\ell$ both consist of two points, and there is a nonzero $(\mathcal{L}_1, \mathcal{L}_2) \in \ker \Phi$ iff there is a common point between these two schemes, that is iff $\ell \in q_1 \cap q_2$.

In other words, when $\mathcal{L}_1$ and $\mathcal{L}_2$ vary, ℓ describes four pencils of lines through the 4 points of intersections of the two conics.

Moreover these 4 points coincide with the zero locus of the generic section s that breaks the stability.

To see this, first remark that $\mathrm{h}^0(E) = 1$ implies that we have only one section, and that $\mathrm{h}^0(E(-1)) = 0$ and $c_2 = 4$ imply respectively that the zero locus has dimension 0 and degree 4. Now let

$$0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E \to \mathcal{I}_Z(-1) \to 0 \qquad (3.5)$$

be the short exact sequence of defining the subscheme $Z = \{P_1, ..., P_4\}$, of points where s vanishes.

Putting (3.5) together with (3.3), we obtain the diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{O}_{\mathbb{P}^2} & & & & \\
 & & & & \downarrow{\scriptstyle s} & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-5) & \xrightarrow{(q_1,q_2,r)} & 2\mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & E & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & \mathcal{I}_Z(-1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

Now, completing the diagram, we get our thesis:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^2} & = & \mathcal{O}_{\mathbb{P}^2} \\
\downarrow & & \downarrow s \\
0 \to \mathcal{O}_{\mathbb{P}^2}(-5) \xrightarrow{(q_1,q_2,r)} 2\mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2} \to E \to 0 \\
\| \qquad\qquad \downarrow \qquad\qquad \downarrow \\
0 \to \mathcal{O}_{\mathbb{P}^2}(-5) \xrightarrow{(q_1,q_2)} 2\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{I}_Z(-1) \to 0 \\
\downarrow \qquad\qquad \downarrow \\
0 \qquad\qquad 0
\end{array}
$$

To prove the part about the 6 double jumping lines, we repeat the previous argument, except that this time we twist (3.4) by $\mathcal{O}_{\mathbb{P}^1}(-2)$ and the cohomology sequence looks:

$$
0 \to \mathrm{H}^0(E|_\ell(-2)) \to 2\mathbb{C} \xrightarrow{\Phi} S^2(V)
$$
$$
(\alpha, \beta) \mapsto (\alpha \cdot q_1|_\ell + \beta \cdot q_2|_\ell).
$$

It is clear that, generically, to make the map Φ vanish we need:

$$
\alpha \cdot q_1|_\ell = -\beta \cdot q_2|_\ell.
$$

In other words we want the 4 points given on ℓ by the 2 conics to coincide 2 by 2. Generically this is possible for appropriate values of α and β, iff ℓ is exactly one of the 6 lines passing through the 4 intersections of q_1 and q_2, that is what we wanted.

Proof of Part B.

We explain the generic resolution separately, because we treated this last case with different tools, since we deal with a Steiner bundle. We use results from [1] and [18].

From the cohomology table, we know that only order 1 jumping lines are admitted. They can be either 6 lines with normal crossing, or they can form a conic in our plane.

First, let $W(S)$ be the scheme (see [1]) of unstable hyperlanes of a Steiner bundle $S \in S_{n,k}$ on $\mathbb{P}^n$, with dual resolution

$$
0 \to S^* \to W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{f_A} I \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to 0,
$$

where W and I are complex vector spaces of dimension $n+k$ and k respectively. Then we have:

Proposition 3.6 ([1], Section 3.8)

Let p_V be the projection of the Segre variety $\mathbb{P}(V) \times \mathbb{P}(I)$ on the $\mathbb{P}(V)$. Then

$$
W(S)_{\mathrm{red}} = p_V[\mathbb{P}(W) \cap (\mathbb{P}(V) \otimes \mathbb{P}(I))]_{\mathrm{red}}.
$$

In our case where $(n, k) = (2, 3)$, the generic Steiner bundle is logarithmic. In fact, the generic $\mathbb{P}^4$ linearly embedded in $\mathbb{P}^8$ meets the Segre variety $\mathbb{P}^2 \otimes \mathbb{P}^2$ in $\deg(\mathbb{P}^2 \otimes \mathbb{P}^2) = 6 = n + k + 1$ points.

Contrary to what happens for the resolution $(4)(1, 2, 2)$, the six lines don't have a particular configuration, but normal crossing, as it is stated in [1], Theorem 5.6.

In case the jumping lines are not 6, they can only be infinite and thus form a conic (see [1]). In particular, we get that if a generic linear $\mathbb{P}(W)$ meets $\mathbb{P}(V) \otimes \mathbb{P}(I)$ in $n + k + 2$ points, then it meets it in infinitely many points. $\square$

Now we would like to improve (3.2), and to obtain a more detailed filtration of the Grassmannian.

If we look at Theorem 3.3, we see that the characterization of jumping planes has been obtained by translating in Plücker coordinates a cohomological condition.

Can we repeat this argument?

Looking at the cohomology table of the bundles associated to the 6 resolutions:

	$h^0(E(3))$	$h^0(E(2))$	$h^0(E(1))$	$h^0(E)$
(4,5)(0,3,3,4)	12	6	3	1
(5)(0,3,3)	12	6	3	1
(5)(1,1,4)	12	6	2	0
(3,4)(1,2,2,3)	12	5	1	0
(4)(1,2,2)	12	5	1	0
(3,3,3)(2,2,2,2,2)	12	5	0	0

it is clear that the best condition to translate would be $h^0(E(1))$.

Doing this is not so easy as it has been for the previous twist, though. Take the display associated to the monad (1.1):

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow & 5\mathcal{O}_{\mathbb{P}^4}(-1) \longrightarrow & \mathrm{Ker}(A) \longrightarrow & F_{HM} \longrightarrow & 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \longrightarrow & 5\mathcal{O}_{\mathbb{P}^4}(-1) \stackrel{B}{\longrightarrow} & 2\Omega^2_{\mathbb{P}^4}(2) \longrightarrow & \mathrm{Coker}(B) \longrightarrow & 0 \\
& & \downarrow{\scriptstyle A} & & \downarrow & & \\
& & 5\mathcal{O}_{\mathbb{P}^4} =\!=\!= & 5\mathcal{O}_{\mathbb{P}^4} \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}
$$

Looking at it, from the first line we get:

$$h^0(F_{HM}(1)|_{\mathbb{P}^2}) = h^0(\mathrm{Ker}(A(1))|_{\mathbb{P}^2}) - 5$$

and from the first row:

$$h^0(\mathrm{Ker}(A(1))|_{\mathbb{P}^2}) = 20 - \mathrm{rk}(A(1) \otimes \mathcal{O}_{\mathbb{P}^2}).$$

Unlike the previous twist (see [7], Proposition 2 for details), this time we are not able to determine explicitly the map $A(1) \otimes \mathcal{O}_{\mathbb{P}^2}$. Nevertheless, we can compute its rank by embedding everything in a bigger space.

Let's see this in detail. We have an isomorphism:

$$\mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)) \cong \mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)|_{\mathbb{P}^2}). \tag{3.6}$$

This can be verified by tensorizing short exact sequence of definition of an hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ by the sheaf $\Omega^2_{\mathbb{P}^4}(3)$:

$$0 \to \Omega^2_{\mathbb{P}^4}(2) \to \Omega^2_{\mathbb{P}^4}(3) \to \Omega^2_{\mathbb{P}^4}(3)|_{\mathbb{P}^3} \to 0.$$

Looking at the cohomology, since $\Omega^2_{\mathbb{P}^4}(2)$ has vanishing cohomology, we get that:

$$\mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)) \cong \mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)|_{\mathbb{P}^3}). \tag{3.7}$$

Now since $\Omega^2_{\mathbb{P}^4}(2)|_{\mathbb{P}^3}$ has vanishing cohomology as well, repeating the argument for an (hyper)plane $\mathbb{P}^2 \subset \mathbb{P}^3$ we get the desired isomorphism (3.6).

Now we use the Plücker embedding:

$$V \xrightarrow{\psi(\omega)} \bigwedge^3 V$$
$$v \mapsto v \wedge \omega$$

where ψ is given by the 2-form $\omega = x^* \wedge y^*$, once we have defined our plane $\mathbb{P}^2 \subset \mathbb{P}^4$ as the zero locus of the two hyperplanes x^* and y^*:

$$\mathbb{P}^2 = \{x^* = y^* = 0\}.$$

In this case $W = \mathrm{Im}(\psi(\omega))$, where $\mathbb{P}^2 = \mathbb{P}(W)$.

Thus we have:

$$2\mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)|_{\mathbb{P}^2}) \xrightarrow{A(1) \otimes \mathcal{O}_{\mathbb{P}^2}} 5\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(1)) \overset{5i}{\hookrightarrow} 5 \bigwedge^3 \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(1)) \tag{3.8}$$

with $\sim$ on the left vertical, 5π the middle vertical, and $5\psi(\omega)$, lower map:

$$2\,\mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)) \xrightarrow{A(1)} 5\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(1))$$

All in all, we have obtained:

$$\mathrm{rk}(A(1) \otimes \mathcal{O}_{\mathbb{P}^2}) = \mathrm{rk}(5\psi(\omega) \circ A(1)). \tag{3.9}$$

We are now ready to state the first result.

Proposition 3.7

Let π be a plane in $\mathbb{G}(2,4) \subset \mathbb{P}^9$. Let

$$M = A(1) \otimes \mathcal{O}_{\mathbb{P}^2} : 2\,\mathrm{H}^0(\Omega^2_{\mathbb{P}^4}(3)) \to 5 \wedge^3 \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(1))$$

be the 20×50 matrix constructed above. The rank of M determines the minimal resolution of $F_{HM}|_\pi$. More precisely, we have:

- rk $M = 15 \Leftrightarrow F_{HM}|_\pi$ has resolution $(3,3,3)(2,2,2,2,2)$;
- rk $M = 14 \Leftrightarrow F_{HM}|_\pi$ has resolution either $(4)(1,2,2)$ or $(3,4)(1,2,2,3)$;
- rk $M = 13 \Leftrightarrow F_{HM}|_\pi$ has resolution $(5)(1,1,4)$;
- rk $M = 12 \Leftrightarrow F_{HM}|_\pi$ has resolution either $(5)(0,3,3)$ or $(4,5)(0,3,3,4)$.

Remark 3.8 Clearly this classification, being based on a cohomological criterion, doesn't take into account the differences between those couples of resolutions with the same cohomology (see paragraph 2).

4. The M_k subvarieties

Let's define the subvarieties of the Grassmannian $\mathbb{G}(2,4)$:

$$M_k := \{\pi \in \mathbb{G}(2,4) \mid \mathrm{rk}\, M(\pi) \leq k\}.$$

We want to know more about M_{12}, M_{13}, M_{14} and M_{15}.

Notice that even though we do have explicit equations for them, these equations are almost impossible to handle, even with the aid of a computer.

Remark 4.1 All the four M_k's are non empty. In fact we are able to find out points where rk(M) assumes all its four possible values (see next Proposition 4.2 for details).

This already implies that we can confirm the presence of the generic resolution $(3,3,3)(2,2,2,2,2)$ between those really assumed by $F_{HM}|_{\mathbb{P}^2}$.

We can make some other useful observations. First M_{15}, being associated to the generic resolution, hence to the open condition of maximum rank, is an open subset of $\mathbb{G}(2,4)$, so dim $M_{15} = \dim \mathbb{G}(2,4) = 6$. Moreover we know from the previous remark that $M_{15} \neq \emptyset$. From now on, recalling the notation given in the Introduction, we will call this last one $\mathcal{V}_{(3,3,3)(2,2,2,2,2)}$.

Second, the subvariety M_{12} is nothing else than S_{25}, the surface in $\mathbb{P}^9$ that parametrizes jumping planes.

So now the situation described in (3.2) has been improved quite a lot:

$$S_{25} = M_{12} \subset M_{13} \subset M_{14} \subset M_{15} = \mathcal{V}_{(3,3,3)(2,2,2,2,2)} \subset \mathbb{G}(2,4) \subset \mathbb{P}^9. \qquad (4.1)$$

Another useful step in the direction of understanding the M_k's structure is computing their dimension.

Let's take for example M_{13}. The matrix M clearly defines a map ϑ:

$$\mathbb{G}(2,4) \longrightarrow \mathbb{P}(\mathbb{M}(20 \times 50))$$
$$\searrow \qquad \nearrow \vartheta$$
$$\mathbb{P}^9$$

where ϑ is linear in the coordinates p_{ij}.

Now let $R_{13} \subset \mathbb{P}(\mathbb{M}(20 \times 50))$ the subspace consisting of matrices whose rank is 13. We have:
$$M_{13} = \vartheta^{-1}(R_{13}) \cap \mathbb{G}(2,4).$$

For each point (plane) $\pi \in \mathbb{G}(2,4)$ we could then obtain the tangent space $T_\pi(M_{13})$, simply by deriving the equations given by the appropriate minors of M, together with the 5 Plücker quadrics, of course.

Obviously these equations, the 14×14 minors of a 20×50 matrix, are too heavy to handle. But if we are satisfied with local dimension (*i.e.* local tangent space), we can use the point where we know that $\mathrm{rk}(M) = 13$.

In other words, we have a 13×13 minor μ which we know not to be zero. Then we can compute only those minors 14×14 obtained by adding 1 row and 1 column to μ. This diminishes considerably the computational cost of the operation, and can be done without great efforts.

Thanks to these simple observations, and with the aid of Macaulay2 computer system [9], we obtain:

Proposition 4.2

The subvarieties M_k have dimension:

- *dim $M_{12} = 2$, locally around $\pi = (1,0,0,0,0,0,0,0,0,0)$;*
- *dim $M_{13} = 2$, l.a. $\rho = (1,1,1,1,0,0,0,0,0,0)$;*
- *dim $M_{14} = 5$, l.a. $\sigma = (1,1,0,0,1,0,0,0,0,0)$.*

Remark that dim M_{12} is exactly the expected one, since M_{12} is the surface S_{25}.

Going further on with the study of the M_k subvarieties, we use Proposition 3.5. Let's take a jumping line $L_{k,j}$ of order 3 for which we do have explicit equations, from [4]:

$$L_{k,j} = \{z_k = z_{k+2} + e^{j\frac{2\pi i}{5}} z_{k+3} = z_{k+1} + e^{3j\frac{2\pi i}{5}} z_{k+4} = 0\}, \ \ k,j = 0...4.$$

The incidence variety:
$$\{\pi \in \mathbb{G}(2,4) \mid \pi \supset L_{k,j}\}$$

forms a plane $\mathbb{P}^2 \subset \mathbb{P}^9$. This plane cuts the surface S_{25} in a smooth conic.

If we recall that $L_{k,j}$ may be contained only in a plane with resolution either $(5)(1,1,4)$ if it is stable, or $(4,5)(0,3,3,4)$ if it is not, we easily have:

Theorem 4.3

Each of the 25 (skew) jumping lines of order 3 is contained in a smooth conic $C \subset \mathbb{P}^2$ of jumping planes, on which the restriction of F_{HM} takes resolution $(4,5)(0,3,3,4)$.

For each of these lines then there exists a set $\{\mathbb{P}^2 \setminus C\}$ of stable planes containing it, with resolution $(5)(1,1,4)$.

Proof. Now we explicitly use the group of symmetries of the HM bundle, for which we refer the reader to [13, 14] and [15]. In fact the property we are checking is invariant under the action of the symmetry group of F_{HM}, and this group acts transitively on the 25 Horrocks-Mumford lines (see [13]). This means that what we prove for one $L_{k,j}$ is valid for all the 25 triple jumping lines.

Take for example the line:

$$L_{0,0} = \{z_0 = z_2 + z_3 = z_1 + z_4 = 0\}.$$

A plane $\pi = \{\sum_{i=0}^{4} f_i z_i = \sum_{i=0}^{4} g_i z_i = 0\}$ will contain $L_{0,0}$ iff:

$$\text{rk} \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 \\ g_0 & g_1 & g_2 & g_3 & g_4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \leq 3. \tag{4.2}$$

Translating condition (4.2) in Plücker coordinates $p_{ij} = f_i g_j - f_j g_i$, we obtain:

$$\begin{cases} p_{14} = p_{23} = 0 \\ p_{01} = p_{04} = a \\ p_{02} = p_{03} = b \\ p_{13} = p_{12} = -p_{24} = -p_{34} = c\,. \end{cases}$$

Now if we ask for the plane π to be unstable, hence to satisfy Theorem 3.3, we finally get the smooth conic $ab = c^2$. $\qquad\square$

Corollary 4.4

The surface S_{25} contains exactly 25 conics, which are the image of the 25 sections of the Shioda's surface.

Proof. The 25 smooth conics of Theorem 4.3 are exactly the image of the 25 sections of the Shioda's modular surface $S(5)$ under the linear system

$$S(5) \xrightarrow{|I+3F|} S_{25},$$

where I and F are classes of divisors of $S(5)$ (see [3, 4] for details).

This is because if we take any curve C that doesn't correspond neither to a section nor to a singular fibre, then C must have degree ≥ 3, since both $C.I$ and $C.F$ are strictly greater than zero (again, we refer to [3, 4]). Thus the only curves with degree 2 are the 25 sections. $\qquad\square$

Theorem 4.3 confirms the presence of $(4,5)(0,3,3,4)$ and $(5)(1,1,4)$, because we need to have both stable and unstable planes to contain jumping lines of order 3.

If we analyze this result from the M_k's point of view, we can distinguish inside $M_{12} = S_{25}$ two subvarieties:

$$\mathcal{V}_{(4,5)(0,3,3,4)} \leftrightarrow (4,5)(0,3,3,4)$$
$$\mathcal{V}_{(5)(0,3,3)} \leftrightarrow (5)(0,3,3),$$

where of course $M_{12} = \mathcal{V}_{(4,5)(0,3,3,4)} \cup \mathcal{V}_{(5)(0,3,3)} = S_{25}$.
Then Theorem 4.3 takes us to this

Corollary 4.5

- $\mathcal{V}_{(4,5)(0,3,3,4)} = 25$ smooth conics;
- $\mathcal{V}_{(5)(0,3,3)} = S_{25} \setminus \mathcal{V}_{(4,5)(0,3,3,4)}$;
- $M_{13} =: \mathcal{V}_{(5)(1,1,4)} = 25$ planes.

Remark that $\mathcal{V}_{(5)(1,1,4)}$ is reducible.

This result makes us hope we could obtain information on M_{14} repeating a similar argument for a double jumping line.

Just like we have done for M_{12}, we make the distinction:

$$\mathcal{V}_{(3,4)(1,2,2,3)} \leftrightarrow (3,4)(1,2,2,3)$$
$$\mathcal{V}_{(4)(1,2,2)} \leftrightarrow (4)(1,2,2).$$

Now recall that the resolution $(3,4)(1,2,2,3)$ is the only stable one that admits jumping lines of order 2. We have that:

Proposition 4.6

$\mathcal{V}_{(3,4)(1,2,2,3)}$ *is irreducible of dimension 4.*

Proof. Take a 2-jumping line, and recall that 2-jumping lines form a surface $J_2 \subset \mathbb{G}(1,4)$ which is smooth outside 25 points. Since (once again from Proposition 3.5) $(3,4)(1,2,2,3)$ admits only 1 jumping line of order 2, we have the fibration:

$$
\begin{array}{ccc}
\mathcal{V}_{(3,4)(1,2,2,3)} & & \pi \\
\big\downarrow{\scriptstyle g} & & \Big\uparrow \\
J_2 & & \ell
\end{array}
$$

that associates to each plane $\pi \in \mathcal{V}_{(3,4)(1,2,2,3)}$ the only 2-jumping line ℓ contained in it. Each fiber has dimension 2. Moreover, J_2 is irreducible, and for every $\ell \in J_2$, $g^{-1}(\ell) \simeq \mathbb{P}^2$ is irreducible. This concludes the proof. $\qquad\square$

From [4] we learn that generically each double jumping line is contained in exactly 3 jumping planes.

As a first consequence, now we can also confirm the presence of resolutions $(3,4)(1,2,2,3)$ and $(5)(0,3,3)$, for we must have stable and unstable planes containing jumping lines of order 2.

Secondly, with the notation used above, for $\ell \in J_2$ generic:

$$g^{-1}(\ell) \cap S_{25} = 3 \, \text{points}. \tag{4.3}$$

As already remarked, the above described property is generic. In fact when working out some examples explicitly with the aid of a computer, sometimes a degenerate case comes out, where instead of 3 points, the intersection (4.3) consists of a line plus an external point.

The observation of this phenomenon takes us to stress out the fact that there are lines entirely contained in S_{25}.

This we already knew from [4], Section 3.4. In S_{25} we find 12 pentagons, that are the 12 singular fibres of the Shioda's surface, isomorphically mapped in S_{25}. It means we have at least $12 \times 5 = 60$ lines inside our surface.

We can show that:

Corollary 4.7

The surface S_{25} contains exactly 60 lines.

Proof. We use the same argument that entailed Corollary 4.4. If there is a curve in S_{25} with degree 1, then it must be contained in a singular fibre, otherwise it would have degree ≥ 3. $\qquad\square$

As our last step, we want to study the subvariety $\mathcal{V}_{(4)(1,2,2)}$.

Of course we deal with the usual computing problems. Unfortunately in this case jumping phenomena do not help, since we have another stable resolution that admits jumping lines of order 1, the generic $(3,3,3)(2,2,2,2,2)$.

Yet we have explicitly found out a point

$$\widetilde{\pi} = (1,1,1,0,0,0,0,0,0,0) \in \mathbb{G}(2,4)$$

where the resolution $(4)(1,2,2)$ is assumed, so that our subvariety $\mathcal{V}_{(4)(1,2,2)}$ is non-empty.

From Proposition 4.2, we know that (locally) it has dimension 5. We want to compute its degree. The result we obtain is:

Proposition 4.8

The subvariety formed by stable planes $\mathcal{V}_{(4)(1,2,2)}$ is a 5-fold of degree 5.

Proof. $\mathcal{V}_{(4)(1,2,2)} \subset \mathbb{G}(2,4)$ has codimension 1, a hypersurface inside the Grassmannian. Now let's take a line $\mathbb{P}^1 \subset \mathbb{G}(2,4)$, paramatrized by the coordinates (s,t). The intersection $\mathcal{V}_{(4)(1,2,2)} \cap \mathbb{P}^1$ is formed by a number of points equal to the degree we are looking for:

$$\deg(\mathcal{V}_{(4)(1,2,2)} \cap \mathbb{P}^1) = \deg(\mathcal{V}_{(4)(1,2,2)}),$$

but this time we are dealing with a Principal Ideal Domain, which means that $I(\mathcal{V}_{(4)(1,2,2)} \cap \mathbb{P}^1) = (f)$, where $f \in \mathbb{C}[s,t]$ is an homogeneous polynomial of degree $= \deg(\mathcal{V}_{(4)(1,2,2)})$.

The Greatest Common Divisor of two (suitably chosen) 15×15 minors is a degree 5 polynomial in $\mathbb{C}[s,t]$, thus:

$$\deg(\mathcal{V}_{(4)(1,2,2)}) \leq 5.$$

In fact, the degree turns out to be exactly 5.

To show this, we use Invariant Theory. Let's take the standard action of the Heisenberg group H_5 on V (see [13, 15]), and the induced action on $\wedge^2 V$. From the computations contained in the appendix of Manolache's work, it is easy to see that the symmetric powers

$$\mathrm{S}^i(\wedge^2 V)$$

do not contain trivial summands for $1 \leq i \leq 4$, so that the degree of our subvariety must be 5. $\qquad\square$

Remark 4.9 From the proof of Proposition 4.8 we can also infer that the hypersurface we have found inside $\mathbb{G}(2,4)$ is irreducible. Unfortunately we are not yet able to exclude the presence of some components of lower dimension in $\mathcal{V}_{(4)(1,2,2)}$.

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