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Generalized Lions-Peetre methods of constants and means and operator ideals

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Abstract

We establish results on interpolation of Rosenthal operators, Banach-Saks operators, Asplund operators and weakly compact operators by means of generalized Lions-Peetre methods of constants and means. Applications are presented for the $\mathcal{K}$-method space generated by the Calderón-Lozanovskii space parameters.

1. Introduction

It is well known that some properties of operators are stable for the complex as well as for the real method of interpolation. From the point of view of the theory of Banach spaces it is useful to identify the properties of operators or Banach spaces

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which are stable for various interpolation methods. A notable negative result in this context is that none of the following properties is stable for any exponential method of interpolation: the Radon-Nikodým property, the property of not having any subspace isomorphic to c_0, and the property of being a dual space. This follows from a result of Garling and Montgomery-Smith [14], which states that there exists a compatible couple (A_0, A_1) of Banach spaces such that A_0 and A_1 are isometric to ℓ_1 and for every exponential interpolation method $\mathcal{F}$ the interpolation space $\mathcal{F}(A_0, A_1)$ contains a complemented copy of c_0.

The behaviour of weak compactness under interpolation has attracted the attention of many researchers since the time when Davis, Figiel, Johnson and Pełczyński [10] established the well known result on factorization of weakly compact operators. An early contribution was due to Beauzamy [1] who provided a necessary and sufficient condition for the real interpolation space $(A_0, A_1)_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ to be reflexive. Later, Heinrich [15] extended this result to other closed operator ideals. More related results that deal with the classical real method can be found in [1, 9, 18]. The case of general $\mathcal{K}$- and $\mathcal{J}$-methods has been investigated in [6, 7].

In this paper we investigate the interpolation of Rosenthal operators, Banach-Saks operators, Asplund operators and weakly compact operators by means of the generalized Lions-Peetre methods of constants and means. We use similar techniques to those used in [6, 7, 22].

Let us mention that the methods of constants and means were defined in the fundamental paper of Lions and Peetre [19] for the case when the lattices are weighted L_p-spaces with power weights. Their generalizations to the case of arbitrary Banach lattices, considered in this paper, were proposed by Peetre and later developed by Dmitriev in several papers (see [17, 4] for details and relevant references). It is known that the methods of constants $\mathcal{K}_{E_0, E_1}(\cdot)$ and means $\mathcal{J}_{E_0, E_1}(\cdot)$ are equivalent to the $\mathcal{K}$- and $\mathcal{J}$-methods with parameters $\mathcal{K}_{E_0, E_1}(\ell_\infty, \ell_\infty(2^{-m}))$ and $\mathcal{J}_{E_0, E_1}(\ell_1, \ell_1(2^{-m}))$ respectively (see, e.g., [4, Theorems 4.2.11 and 4.2.33]). In general, the description of such space parameters is a subtle problem. Indeed, they have been calculated only for special cases, in particular for the weighted L_p-spaces with power weights or quasi-power weights (see [2, 13]). As one could expect, the results we derive here depend on the Banach lattice parameters E_0 and E_1 which generate the corresponding method of interpolation.

As an application, we give necessary and sufficient conditions for the $\mathcal{K}$-method space generated by the Calderón-Lozanovskii space parameters to be a space not containing ℓ_1, a space with the Banach-Saks property, an Asplund space, or a reflexive space.

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2. Preliminaries and auxiliary results

Let $\omega(\mathbb{Z})$ denote the space of all real-valued sequences modelled on $\mathbb{Z}$, equipped with the topology of pointwise convergence. By a Banach lattice on $\mathbb{Z}$ we shall mean a Banach space E which is a subspace of $\omega(\mathbb{Z})$, such that there exists a sequence in E that is positive on $\mathbb{Z}$ and E satisfies the following condition: if $\xi = (\xi_m) \in E$, $\eta = (\eta_m) \in \omega(\mathbb{Z})$ and $|\eta_m| \leq |\xi_m|$ for every $m \in \mathbb{Z}$, then $\eta \in E$ and $\|\eta\|_E \leq \|\xi\|_E$.

We say that a Banach lattice E on $\mathbb{Z}$ is ***regular*** if for any $(x_n)_{n\in\mathbb{N}} \subseteq E$ with $x_n \downarrow 0$ it follows that $\|x_n\|_E \longrightarrow 0$.

The ***Köthe dual*** of a Banach lattice E on $\mathbb{Z}$ is a Banach lattice E' on $\mathbb{Z}$ which consists of all sequences $(\eta_m) \in \omega(\mathbb{Z})$ for which

$$\|(\eta_m)\|_{E'} = \sup\Big\{\sum_{m=-\infty}^{\infty} |\eta_m \xi_m| \ : \ \|(\xi_m)\|_E \leq 1\Big\} < \infty.$$

Given a positive sequence $(w_m)_{m\in\mathbb{Z}}$, we denote by $E(w_m)$ the space E with the weight $(w_m)_{m\in\mathbb{Z}}$.

If E is a Banach lattice on $\mathbb{Z}$ and X is a Banach space, then by $E(X)$ we denote the Banach space of all sequences $x = (x_m)_{m\in\mathbb{Z}}$ in X equipped with the norm $\|x\|_{E(X)} = \|(\|x_m\|_X)\|_E$.

We shall use standard notation and notions from interpolation theory as presented, e.g., in [2, 4]. We recall that a mapping $\mathcal{F}$ from the category $\overline{\mathcal{B}}$ of compatible couples of Banach spaces into the category $\mathcal{B}$ of Banach spaces is said to be an ***interpolation functor*** (or an ***interpolation method***) if, for any couple $\bar{A} = (A_0, A_1)$, $\mathcal{F}(\bar{A})$ is a Banach space intermediate with respect to $\bar{A}$ (i.e., $A_0 \cap A_1 \hookrightarrow \mathcal{F}(\bar{A}) \hookrightarrow A_0 + A_1$), and $T : \mathcal{F}(\bar{A}) \to \mathcal{F}(\bar{B})$ for all Banach couples $\bar{A}$, $\bar{B}$ and any operator $T : \bar{A} \to \bar{B}$. Here, as usual, we use the notation $T : \bar{A} \to \bar{B}$ to mean that $T : A_0 + A_1 \to B_0 + B_1$ is a linear operator such that the restriction of T to the space A_j is a bounded operator from A_j into B_j for $j = 0, 1$. We denote by $L(\bar{A}, \bar{B})$ the Banach space of all operators $T : \bar{A} \to \bar{B}$ equipped with the norm:

$$\|T\|_{\bar{A}\to\bar{B}} \ = \ \max\{\|T\|_{A_0\to B_0},\, \|T\|_{A_1\to B_1}\}.$$

As a consequence of the closed graph theorem, for any couples $\bar{A}$, $\bar{B}$ there exists a positive constant C such that for all $T \in L(\bar{A}, \bar{B})$ it holds

$$\|T\|_{\mathcal{F}(\bar{A})\to\mathcal{F}(\bar{B})} \leq C\,\|T\|_{\bar{A}\to\bar{B}}.$$

If C can be chosen as equal to 1 for all couples $\bar{A}$, $\bar{B}$, then $\mathcal{F}$ is called ***exact***.

An interpolation functor $\mathcal{F}$ is called ***regular*** if $A_0 \cap A_1$ is dense in $\mathcal{F}(A_0, A_1)$ for any Banach couple (A_0, A_1).

For $t > 0$, let $t\mathbb{R}$ be the space $\mathbb{R}$ with the norm $\|\lambda\|_{t\mathbb{R}} = t|\lambda|$. Let $\mathcal{F}$ be an exact interpolation functor. Following [11] (see also [16]) the ***fundamental function*** $\varphi = \varphi_{\mathcal{F}}$ of $\mathcal{F}$ is defined as follows:

$$\mathcal{F}(\mathbb{R}, (1/t)\mathbb{R}) = (1/\varphi(t))\mathbb{R}.$$

It is known that φ is a quasi-concave function, i.e., $\varphi : (0,\infty) \to (0,\infty)$ and it satisfies $\varphi(s) \leq \max\{1, s/t\}\,\varphi(t)$ for all $s, t > 0$. For a quasi-concave function φ, we define a quasi-concave function φ^* by $\varphi^*(t) = 1/\varphi(1/t)$ for $t > 0$. We denote by $\mathcal{P}_0$ the set of quasi-concave functions φ such that $\varphi(t) \to 0$ as $t \to 0$ and $\varphi(t)/t \to 0$ as $t \to \infty$.

Given a Banach couple (A_0, A_1), for each $t > 0$, we put

$$\begin{aligned} K(t, a) &= K(t, a, A_0, A_1) \\ &= \inf \{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\,;\, a = a_0 + a_1,\, a_i \in A_i\}, \qquad a \in A_0 + A_1, \end{aligned}$$

and

$$J(t, a) = J(t, a, A_0, A_1) = \max\{\|a\|_{A_0}\,,\, t\|a\|_{A_1}\}\,, \qquad a \in A_0 \cap A_1.$$

Let us now recall the definition (in a discrete form) of the generalized Lions-Peetre methods of constants and means. Let E_0 and E_1 be Banach lattices on $\mathbb{Z}$. For any Banach couple $\bar{A} = (A_0, A_1)$, the space $\mathcal{K}_{E_0,E_1}(\bar{A})$ is defined as the set of elements $a \in A_0 + A_1$ for which there exists $a^i = (a^i_m) \in E_i(A_i)$, $i = 0, 1$, such that $a = a^0_m + a^1_m$ for all $m \in \mathbb{Z}$. We set

$$\|a\|_{\mathcal{K}_{E_0,E_1}(\bar{A})} = \inf\Big\{\|a^0\|_{E_0(A_0)} + \|a^1\|_{E_1(A_1)};\, a^i = (a^i_m) \in E_i(A_i), a = a^0_m + a^1_m\Big\}.$$

The space $\mathcal{K}_{E_0,E_1}(\bar{A})$ may contain non-zero elements only when $e \in E_0 + E_1$, where $e = (e_m)$ with $e_m = 1$ for all $m \in \mathbb{Z}$. Furthermore, if $e \in E_0 + E_1$, then $\mathcal{K}_{E_0,E_1}$ is an exact interpolation functor (see [17, 4]). In what follows we shall always assume that $e \in E_0 + E_1$.

Analogously, the Banach space $\mathcal{J}_{E_0,E_1}(\bar{A})$ consists of all $a \in A_0 + A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), where $(u_m) \subset E_0(A_0) \cap E_1(A_1)$. $\mathcal{J}_{E_0,E_1}(\bar{A})$ is equipped with the norm defined by

$$\|a\|_{\mathcal{J}_{E_0,E_1}(\bar{A})} = \inf\Big\{\max\{\,\|(u_m)\|_{E_0(A_0)},\, \|(u_m)\|_{E_1(A_1)}\,\}\,;\, a = \sum_{m=-\infty}^{\infty} u_m\Big\}.$$

In the sequel we always assume that E_0 and E_1 satisfy the condition $E_0 \cap E_1 \hookrightarrow \ell_1$. In that case $\mathcal{J}_{E_0,E_1}$ is an exact interpolation functor (see [17, 4]).

The methods of constants and means coincide with the $\mathcal{K}$- and $\mathcal{J}$-methods with certain parameters (see [4]). Namely, if $\Phi = \mathcal{K}_{E_0,E_1}(\ell_\infty, \ell_\infty(2^{-m}))$ (resp. $\Phi = \mathcal{J}_{E_0,E_1}(\ell_1, \ell_1(2^{-m}))$), it holds that $\mathcal{K}_{E_0,E_1}(\cdot) = \mathcal{K}_\Phi(\cdot)$ (resp. $\mathcal{J}_{E_0,E_1}(\cdot) = \mathcal{J}_\Phi(\cdot)$).

In the special case when $E_j = \ell_{p_j}(2^{(j-\theta)m})$, $1 \le p_j \le \infty$, $0 < \theta < 1$ $(j = 0, 1)$, the generalized Lions-Peetre methods of constants and means reduce to the classical real method:

$$\mathcal{K}_{E_0,E_1}(\bar{A}) = \mathcal{J}_{E_0,E_1}(\bar{A}) = \bar{A}_{\theta,p}\,,$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$ (see [2, Theorem 3.12.1]).

On the other hand the space $\mathcal{K}_E(\bar{A})$ (resp. $\mathcal{J}_E(\bar{A})$) coincides with $\mathcal{K}_{E,E(2^m)}(\bar{A})$ (resp. $\mathcal{J}_{E,E(2^m)}(\bar{A})$) (see [17, Chapter IV, Lemma 2.8 and Lemma 2.9]).

We refer to the books [17, 4] for additional information about the methods of constants and means.

We note that if φ is the fundamental function of an exact interpolation functor $\mathcal{F}$, then

$$\mathcal{F}(X_0, X_1) \hookrightarrow \mathcal{K}_\Psi(X_0, X_1)$$

for any Banach couple (X_0, X_1), with $\Psi = \ell_\infty(1/\varphi(2^m))$ (see, e.g., [16]). When $\varphi \in \mathcal{P}_0$, it implies, in particular, that

$$\mathcal{F}(X_0, X_1) \hookrightarrow X_0^\circ + X_1^\circ,$$

where X_j° denotes the closure of $X_0 \cap X_1$ in X_j for $j = 0, 1$ (see [4, Proposition 2.2.12 and Corollary 3.1.14]).

Moreover, it holds that $K(t, x; X_0, X_1) = K(t, x; X_0^\circ, X_1^\circ)$ for any $x \in X_0^\circ + X_1^\circ$ and $t > 0$. Thus, if $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$, we derive

$$\mathcal{K}_{E_0,E_1}(X_0, X_1) = \mathcal{K}_{E_0,E_1}(X_0^\circ, X_1^\circ), \tag{2.1}$$

from the known formula $\mathcal{K}_{E_0,E_1}(X_0, X_1) = \mathcal{K}_\Phi(X_0, X_1)$, with $\Phi = \mathcal{K}_{E_0,E_1}(\ell_\infty, \ell_\infty(2^{-m}))$ (see [4, Theorem 4.2.11]).

Let us also mention that the equality

$$\mathcal{J}_{E_0,E_1}(X_0, X_1) = \mathcal{J}_{E_0,E_1}(X_0^\circ, X_1^\circ) \tag{2.2}$$

holds for any Banach couple (X_0, X_1).

For each $t > 0$ we put (see [5])

$$\psi_{\mathcal{K}_{E_0,E_1}(\bar{A})}(t) = \sup\{K(t, a);\ \|a\|_{\mathcal{K}_{E_0,E_1}(\bar{A})} = 1\},$$
$$\rho_{\mathcal{J}_{E_0,E_1}(\bar{A})}(t) = \inf\{J(t, a);\ a \in A_0 \cap A_1,\ \|a\|_{\mathcal{J}_{E_0,E_1}(\bar{A})} = 1\}.$$

We set the relationship between functions $\psi_{\mathcal{K}_{E_0,E_1}(\bar{A})}$, $\rho_{\mathcal{J}_{E_0,E_1}(\bar{A})}$ and the fundamental functions $\varphi_{\mathcal{K}_{E_0,E_1}}$, $\varphi_{\mathcal{J}_{E_0,E_1}}$ by means of the lemma (see [6, Lemma 2.1]).

Lemma 2.1

For any Banach couple $\bar{A}$ the following estimates hold:

$$\psi_{\mathcal{K}_{E_0,E_1}(\bar{A})}(t) \le \varphi_{\mathcal{K}_{E_0,E_1}}(t), \qquad t > 0,$$

and

$$\rho^*_{\mathcal{J}_{E_0,E_1}(\bar{A})}(t) \le \varphi^*_{\mathcal{J}_{E_0,E_1}}(t), \qquad t > 0.$$

In our interpolation results we shall assume that $\varphi_{\mathcal{K}_{E_0,E_1}}$ and $\varphi^*_{\mathcal{J}_{E_0,E_1}}$ belong to the class $\mathcal{P}_0$. The following result gives a sufficient condition for these functions to belong to $\mathcal{P}_0$ in terms of the norms of shift operators on E_j $(j = 0, 1)$. For $n \in \mathbb{Z}$, the *shift operator* τ_n is defined by the equality $\tau_n(\xi_m)_{m\in\mathbb{Z}} = (\xi_{m+n})_{m\in\mathbb{Z}}$.

Lemma 2.2

Let $\mathcal{F}$ be either $\mathcal{K}_{E_0,E_1}$ or $\mathcal{J}_{E_0,E_1}$ and let

$$2^{-(1-j)n}\|\tau_n\|_{E_j\to E_j} \longrightarrow 0 \quad \text{and} \quad 2^{-jn}\|\tau_{-n}\|_{E_j\to E_j} \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{2.3}$$

for $j = 0, 1$. Then $\varphi_{\mathcal{F}}$ and $\varphi^*_{\mathcal{F}}$ belong to $\mathcal{P}_0$.

The proof of Lemma 2.2 is based on the following:

Lemma 2.3

Let $\mathcal{F}$ be either $\mathcal{K}_{E_0,E_1}$ or $\mathcal{J}_{E_0,E_1}$. Suppose that $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ are Banach couples and let $T\in L(\bar{A},\bar{B})$. For any $n\in\mathbb{N}$ the following statements hold:

(i) *If $\|T\|_{A_0\to B_0}\le 2^{-n}$ and $\|T\|_{A_1\to B_1}\le 1$, then*

$$\|T\|_{\mathcal{F}(\bar{A})\to\mathcal{F}(\bar{B})}\le \max\{\,2^{-n}\|\tau_n\|_{E_0\to E_0}\,,\,\|\tau_n\|_{E_1\to E_1}\,\},$$

(ii) *If $\|T\|_{A_0\to B_0}\le 1$ and $\|T\|_{A_1\to B_1}\le 2^{-n}$, then*

$$\|T\|_{\mathcal{F}(\bar{A})\to\mathcal{F}(\bar{B})}\le \max\{\,\|\tau_{-n}\|_{E_0\to E_0}\,,\,2^{-n}\|\tau_{-n}\|_{E_1\to E_1}\,\}.$$

Proof of Lemma 2.3 We just show (i) for $\mathcal{F}=\mathcal{J}_{E_0,E_1}$. The rest of the proof is similar. Let $a\in\mathcal{J}_{E_0,E_1}(\bar{A})$ and let us take any representation $a=\sum_{m=-\infty}^{\infty}u_m$ (convergence in A_0+A_1), with $(u_m)\in E_0(A_0)\cap E_1(A_1)$. Since $E_0\cap E_1\hookrightarrow\ell_1$, the series $\sum_{m=-\infty}^{\infty}u_m$ is absolutely summable in A_0+A_1. Therefore,

$$\begin{aligned}\|Ta\|_{\mathcal{J}_{E_0,E_1}(\bar{B})} &\le \max\{\,\|(Tu_{m+n})\|_{E_0(B_0)}\,,\,\|(Tu_{m+n})\|_{E_1(B_1)}\,\}\\ &\le \max\{\,2^{-n}\|\tau_n\|_{E_0\to E_0}\,,\,\|\tau_n\|_{E_1\to E_1}\,\}\,\|(u_m)\|_{E_0(A_0)\cap E_1(A_1)}.\end{aligned}$$

By taking the infimum over all $\mathcal{J}$-representations of a we have that

$$\|Ta\|_{\mathcal{J}_{E_0,E_1}(\bar{B})}\le \max\,\{\,2^{-n}\|\tau_n\|_{E_0\to E_0}\,,\,\|\tau_n\|_{E_1\to E_1}\,\}\,\|a\|_{\mathcal{J}_{E_0,E_1}(\bar{A})}\,. \qquad \square$$

Proof of Lemma 2.2 We prove that $\varphi_{\mathcal{K}_{E_0,E_1}}\in\mathcal{P}_0$. Let us put $C=\|e\|_{E_0+E_1}$, where $e=(e_m)$ with $e_m=1$ for any $m\in\mathbb{Z}$. It is easy to see that $\|\cdot\|_{\mathcal{K}_{E_0,E_1}(\mathbb{R},\mathbb{R})}$ is equal to $C\,|\cdot|$.

Thus,

$$\varphi_{\mathcal{K}_{E_0,E_1}}(t)=C^{-1}\,\|I\|_{\mathcal{K}_{E_0,E_1}(\mathbb{R},(1/t)\mathbb{R})\to\mathcal{K}_{E_0,E_1}(\mathbb{R},\mathbb{R})}\,,\quad t>0,$$

where I is the identity operator. Now using Lemma 2.3(ii), we get that $\varphi_{\mathcal{K}_{E_0,E_1}}(t)\to 0$ as $t\to 0$. Analogously

$$\varphi_{\mathcal{K}_{E_0,E_1}}(t)/t=C^{-1}\,\|(1/t)I\|_{\mathcal{K}_{E_0,E_1}(\mathbb{R},(1/t)\mathbb{R})\to\mathcal{K}_{E_0,E_1}(\mathbb{R},\mathbb{R})}\,,\quad t>0,$$

and from Lemma 2.3(i) we derive that $\varphi_{\mathcal{K}_{E_0,E_1}}(t)/t\to 0$ as $t\to\infty$.

In order to establish that $\varphi^*_{\mathcal{J}_{E_0,E_1}}\in\mathcal{P}_0$ we may reason in an analogous way. In this case,

$$\varphi^*_{\mathcal{J}_{E_0,E_1}}(t^{-1})=D^{-1}\,\|I\|_{\mathcal{J}_{E_0,E_1}(\mathbb{R},\mathbb{R})\to\mathcal{J}_{E_0,E_1}(\mathbb{R},(1/t)\mathbb{R})}\,,\quad t>0,$$

and

$$\varphi^*_{\mathcal{J}_{E_0,E_1}}(t^{-1})/t^{-1} = D^{-1}\,\|tI\|_{\mathcal{J}_{E_0,E_1}(\mathbb{R},\mathbb{R})\to\mathcal{J}_{E_0,E_1}(\mathbb{R},(1/t)\mathbb{R})}\,, \quad t>0,$$

where D is the constant of the embedding $E_0 \cap E_1 \hookrightarrow \ell_1$ and I is the identity operator. Lemma 2.3(ii) and (i) give that $\varphi^*_{\mathcal{J}_{E_0,E_1}}(t) \to 0$ as $t \to 0$ and $\varphi^*_{\mathcal{J}_{E_0,E_1}}(t)/t \to 0$ as $t \to \infty$ respectively.

It can be checked that $\varphi_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$ and $\varphi^*_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$ by repeating the above arguments with minor modifications. □

It is easy to see that the condition (2.3) is satisfied whenever $E_0 = \ell_{p_0}(2^{-\theta m})$ and $E_1 = \ell_{p_1}(2^{(1-\theta)m})$ with $0<\theta<1$ and $1 \le p_j \le \infty$ for $j=0,1$.

3. Interpolation results

We start by studying the interpolation of the Rosenthal operators. Recall that a bounded linear operator $T : X \longrightarrow Y$ between Banach spaces is said to be a ***Rosenthal operator*** if $T(U_X)$ is a weakly precompact subset. Here U_X stands for the closed unit ball of X. In other words, T is a Rosenthal if for every bounded sequence $(x_n) \subset X$, the sequence (Tx_n) admits a weak Cauchy subsequence. Rosenthal operators form an injective and surjective closed operator ideal (we refer to [25] for concepts related to Banach operator ideals).

In what follows we will use some interpolation duality formulas, so let us note that if X and Y are Banach spaces, then as usual $X \cong Y$ means that X and Y are isometrically isomorphic.

Theorem 3.1

Let E_0, E_1 be Banach lattices on $\mathbb{Z}$ that do not contain a copy of ℓ_1. Suppose that $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ are Banach couples and let $T \in L(\bar{A}, \bar{B})$ be such that $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is a Rosenthal operator.

(i) *If $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$, then $T : \mathcal{K}_{E_0,E_1}(A_0,A_1) \longrightarrow \mathcal{K}_{E_0,E_1}(B_0,B_1)$ is a Rosenthal operator.*

(ii) *If $\varphi^*_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$, then $T : \mathcal{J}_{E_0,E_1}(A_0,A_1) \longrightarrow \mathcal{J}_{E_0,E_1}(B_0,B_1)$ is a Rosenthal operator.*

Proof. Let us first show that (ii) holds. Using (2.2), without loss of the generality, we may assume that $A_0 \cap A_1$ is dense in A_i and that $B_0 \cap B_1$ is dense in B_i $(i = 0,1)$. Since $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is a Rosenthal operator, as a consequence of Lemma 2.1 and [8, Corollary 3.5], we conclude that $T : A_0 \cap A_1 \longrightarrow \mathcal{J}_{E_0,E_1}(B_0,B_1)$ is also Rosenthal. Consequently $\widetilde{T}P_k$ has the same property for any $k \in \mathbb{Z}$, where $\widetilde{T} : E_0(A_0) \cap E_1(A_1) \longrightarrow \mathcal{J}_{E_0,E_1}(B_0,B_1)$ is defined by $\widetilde{T}(a_m) = T\Big(\sum_{m=-\infty}^{\infty} a_m\Big)$ and the operator $P_k : A_0 \cap A_1 \longrightarrow E_0(A_0) \cap E_1(A_1)$ is given by $P_k a = (\delta_m^k a)_{m\in\mathbb{Z}}$.

Since $\widetilde{T} = T\pi$, with $\pi : E_0(A_0) \cap E_1(A_1) \longrightarrow \mathcal{J}_{E_0,E_1}(A_0, A_1)$ being the metric surjection $\pi(a_m) = \sum_{m=-\infty}^{\infty} a_m$, the surjectivity of the ideal of Rosenthal operators yields that the proof will be concluded if we prove that $\widetilde{T}$ is a Rosenthal operator.

Let us take any bounded sequence $(a_j)_{j\in\mathbb{N}} \subset E_0(A_0) \cap E_1(A_1)$ and let

$$M = \sup\{\, \|a_j\|_{E_0(A_0)\cap E_1(A_1)}\,;\, j \in \mathbb{N}\}.$$

We show that $(\widetilde{T}a_j)$ admits a weak Cauchy subsequence. Let $Q_k : E_0(A_0)\cap E_1(A_1) \longrightarrow A_0\cap A_1$ be the operator given by $Q_k(a_m) = a_k$. Due to the fact that $\widetilde{T}P_k$ is a Rosenthal operator for every $k \in \mathbb{Z}$, we can choose a subsequence $(\widetilde{a}_j)$ of (a_j) in such a way that $\Big(\sum_{|k|\le N} \widetilde{T}P_kQ_k\widetilde{a}_j\Big)_{j\in\mathbb{N}}$ is a weak Cauchy sequence for any $N \in \mathbb{N}$.

It is known that if E_i does not contain a copy of ℓ_1, then E_i and E_i' are regular (see [28, Theorems 117.3 and 117.2]) and $(E_0(A_0) \cap E_1(A_1))^* \cong E_0'(A_0^*) + E_1'(A_1^*)$ (see [17, Chapter IV, Lemma 2.13]). Thus, for any $f \in \mathcal{J}_{E_0,E_1}(B_0, B_1)^*$ and any $\varepsilon > 0$, by the regularity of E_i', there exists $N \in \mathbb{N}$ such that

$$\Big\|\widetilde{T}^*f - \sum_{|k|\le N} Q_k^*P_k^*\widetilde{T}^*f\Big\|_{E_0'(A_0^*)+E_1'(A_1^*)} \le \varepsilon/4M\,. \tag{3.1}$$

On the other hand, since $\Big(\sum_{|k|\le N} \widetilde{T}P_kQ_k\widetilde{a}_j\Big)_{j\in\mathbb{N}}$ is a weak Cauchy sequence, there exists $j_0 \in \mathbb{N}$ such that

$$\Big|\langle \sum_{|k|\le N} \widetilde{T}P_kQ_k(\widetilde{a}_i - \widetilde{a}_j)\,,\, f\rangle\Big| \le \varepsilon/2 \qquad \text{for all } i, j \ge j_0. \tag{3.2}$$

Finally, by (3.1) and (3.2),

$$\begin{aligned}|\langle\widetilde{T}(\widetilde{a}_i - \widetilde{a}_j),\, f\rangle| &\le \Big|\langle\widetilde{a}_i - \widetilde{a}_j\,, \widetilde{T}^*f - \sum_{|k|\le N} Q_k^*P_k^*\widetilde{T}^*f\rangle\Big| \\ &\quad + \Big|\langle\widetilde{a}_i - \widetilde{a}_j\,,\, \sum_{|k|\le N} Q_k^*P_k^*\widetilde{T}^*f\rangle\Big| \le \varepsilon,\end{aligned}$$

for any $i, j \ge j_0$. This finishes the proof of (ii).

To prove (i) we may assume without loss of generality by (2.1) that $A_0 \cap A_1$ is dense in A_i and that $B_0 \cap B_1$ is dense in $B_i\,(i = 0, 1)$. Then, it is enough to modify slightly the argument in the proof of the statement (ii) by using now the injectivity of the operator ideal and the operators $\widehat{T}$, R_k and S_k defined as follows: $\widehat{T} = jT$, where $j : \mathcal{K}_{E_0,E_1}(B_0, B_1) \longrightarrow E_0(B_0)+E_1(B_1)$ is the metric injection given by $jb = (\cdots, b, b, b, \cdots)$; $R_k : B_0 + B_1 \longrightarrow E_0(B_0) + E_1(B_1)$, with $R_kb = (\delta_m^k b)$, where δ_m^k is the Kronecker delta; and $S_k : E_0(B_0)+E_1(B_1) \longrightarrow B_0+B_1$, defined by $S_kb = b_k^0+b_k^1$ for $b = b^0 + b^1$ with $b^j = (b_m^j) \in E_j(B_j)$, $j = 0, 1$. □

Next, we focus on Banach-Saks operators. A bounded linear operator $T : X \longrightarrow Y$ between Banach spaces is called a ***Banach-Saks operator*** if it maps bounded sequences into sequences possessing Cesaro convergent subsequences. Banach-Saks operators also form an injective and surjective closed operator ideal. A Banach space X is said to have the ***Banach-Saks property*** if the identity operator I_X is a Banach-Saks operator. The Banach-Saks property has attracted considerable attention (see, e.g., [12, 1, 15, 26]).

Theorem 3.2

Assume that E_0, E_1 are Banach lattices on $\mathbb{Z}$ with the Banach-Saks property. Let $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ be Banach couples and let $T \in L(\bar{A}, \bar{B})$ be such that $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is a Banach-Saks operator.

(i) *If $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$, then $T : \mathcal{K}_{E_0,E_1}(A_0, A_1) \longrightarrow \mathcal{K}_{E_0,E_1}(B_0, B_1)$ is a Banach-Saks operator.*

(ii) *If $\varphi^*_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$, then $T : \mathcal{J}_{E_0,E_1}(A_0, A_1) \longrightarrow \mathcal{J}_{E_0,E_1}(B_0, B_1)$ is a Banach-Saks operator.*

Proof. The proof of (i) will be completed if we establish that the operator $\widehat{T} : \mathcal{K}_{E_0,E_1}(A_0, A_1) \longrightarrow E_0(B_0) + E_1(B_1)$, given by $\widehat{T}a = (\cdots, Ta, Ta, Ta, \cdots)$, is a Banach-Saks operator. To show this, we consider the embedding $R_k : B_0 + B_1 \longrightarrow E_0(B_0) + E_1(B_1)$ and the projection $S_k : E_0(B_0) + E_1(B_1) \longrightarrow B_0 + B_1$, defined as in the proof of Theorem 3.1(i). According to Lemma 2.1 and [8, Corollary 3.6] the operator $T : \mathcal{K}_{E_0,E_1}(A_0, A_1) \longrightarrow B_0 + B_1$ is a Banach-Saks operator.

Let $(a_j)_{j\in\mathbb{N}} \subset \mathcal{K}_{E_0,E_1}(A_0, A_1)$ be any bounded sequence. We shall prove that there exists a subsequence $(\widehat{a}_j)$ of (a_j) such that $\Big(\frac{1}{n}\sum_{j=1}^{n} \widehat{T}\,\widehat{a}_j\Big)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $S_k\widehat{T}$ is a Banach-Saks operator for each $k \in \mathbb{Z}$, using a result of Erdös and Magidor [12] we can find for every $N \in \mathbb{N}$ a subsequence $(a_j^N)_{j\in\mathbb{N}}$ of $(a_j)_{j\in\mathbb{N}}$ such that all subsequences of $\Big(\sum_{|k|\le N} R_k S_k \widehat{T}\, a_j^N\Big)_{j\in\mathbb{N}}$ are Cesaro convergent. Hence, using a diagonal argument, we obtain a subsequence $(\widehat{a}_j)$ of (a_j) so that $\Big(\sum_{|k|\le N} R_k S_k \widehat{T}\,\widehat{a}_j\Big)_{j\in\mathbb{N}}$ is Cesaro convergent for all N simultaneously.

For each $j \in \mathbb{N}$ there is $b_j^i \in E_i(B_i)$, $i = 0, 1$, such that $\widehat{T}\,\widehat{a}_j = b_j^0 + b_j^1$ and the sequence $(b_j^i)_{j\in\mathbb{N}}$ is bounded on $E_i(B_i)$, $i = 0, 1$. Let us put $\xi_j^i = (\|S_k b_j^i\|_{B_i})_{k\in\mathbb{Z}} \in E_i$, $j \in \mathbb{N}$, $i = 0, 1$. Taking into account that E_0 and E_1 have the Banach-Saks property, we may assume (by extracting a suitable subsequence by means of Erdös-Magidor's result) that $(\widehat{a}_j)$ has been chosen in such a way that $(\xi_j^i)_{j\in\mathbb{N}}$ is Cesaro convergent in E_i, for $i = 0, 1$. Set

$$\mu^i = (\mu_k^i)_{k\in\mathbb{Z}} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \xi_j^i\,,\ i = 0, 1.$$

Then, given any $\varepsilon > 0$, there is $n_0^i \in \mathbb{N}$ such that if $n > n_0^i$, it holds for all $N \in \mathbb{N}$ that

$$\Big\|\Big(\gamma_k^N\,[\,\frac{1}{n}\sum_{j=1}^{n} \|S_k\, b_j^i\|_{B_i} - \mu_k^i\,]\Big)\Big\|_{E_i} \le \varepsilon/16 \qquad (i = 0, 1),$$

where $\gamma_k^N = 0$ for $|k| \le N$ and $\gamma_k^N = 1$ for $|k| > N$.

On the other hand, E_i is regular because E_i has the Banach-Saks property. Therefore, there exists $q_0^i \in \mathbb{N}$ such that

$$\|(\gamma_k^q \mu_k^i)\|_{E_i} \leq \varepsilon/16 \quad \text{for all} \;\; q \geq q_0^i \qquad (i = 0, 1).$$

Consequently, we can find $N^i \in \mathbb{N}$ such that for every $n > N^i$

$$\Big\|\Big(\gamma_k^{N^i} \frac{1}{n} \sum_{j=1}^{n} \|S_k\, b_j^i\|_{B_i} \Big)\Big\|_{E_i} \leq \varepsilon/8\,,$$

for $i = 0, 1$. Thus, if $n > N = \max\{N^i \,:\, i = 0, 1\}$, then

$$\Big\|\frac{1}{n} \sum_{j=1}^{n} \sum_{|k|>N} R_k S_k \widehat{T}\, \widehat{a}_j \Big\|_{E_0(B_0)+E_1(B_1)} \tag{3.3}$$

$$\leq \Big\|\Big(\gamma_k^N \Big\| S_k\Big(\frac{1}{n} \sum_{j=1}^{n} b_j^0\Big) \Big\|_{B_0} \Big)\Big\|_{E_0} + \Big\|\Big(\gamma_k^N \Big\| S_k\Big(\frac{1}{n} \sum_{j=1}^{n} b_j^1\Big) \Big\|_{B_1} \Big)\Big\|_{E_1} \leq \varepsilon/4\,.$$

Moreover, since $\Big(\sum_{|k|\leq N} R_k S_k \widehat{T}\, \widehat{a}_j \Big)_{j\in\mathbb{N}}$ is Cesaro convergent due to the choice of $(\widehat{a}_j)$, there exists $\overline{N} \in \mathbb{N}$ such that if $n_2, n_1 > \overline{N}$, then

$$\Big\|\frac{1}{n_2} \sum_{j=1}^{n_2} \sum_{|k|\leq N} R_k S_k \widehat{T}\, \widehat{a}_j - \frac{1}{n_1} \sum_{j=1}^{n_1} \sum_{|k|\leq N} R_k S_k \widehat{T}\, \widehat{a}_j \Big\|_{E_0(B_0)+E_1(B_1)} \leq \varepsilon/2. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that $\Big(\frac{1}{n} \sum_{j=1}^{n} \widehat{T}\, \widehat{a}_j\Big)_{n\in\mathbb{N}}$ is a Cauchy sequence.

The proof of (ii) uses similar arguments but deals with the operators $\widetilde{T}, P_k$ and Q_k, defined as in the proof of Theorem 3.1(ii). Namely, given any bounded sequence $(a_j)_{j\in\mathbb{N}} \subset E_0(A_0) \cap E_1(A_1)$, it is possible to choose a suitable subsequence $(\widetilde{a}_j)$ of (a_j) in such a way that $\Big(\sum_{|k|\leq N} \widetilde{T} P_k Q_k\, \widetilde{a}_j \Big)_{j\in\mathbb{N}}$ is Cesaro convergent for all N simultaneously. In this case (3.3) and(3.4) turn into

$$\Big\|\frac{1}{n} \sum_{j=1}^{n} \widetilde{T} \sum_{|k|>N} P_k Q_k\, \widetilde{a}_j \Big\|_{\mathcal{J}_{E_0,E_1}(B_0,B_1)} \leq \varepsilon/4\,,$$

and

$$\Big\|\frac{1}{n_2} \sum_{j=1}^{n_2} \widetilde{T} \sum_{|k|\leq N} P_k Q_k\, \widetilde{a}_j - \frac{1}{n_1} \sum_{j=1}^{n_1} \widetilde{T} \sum_{|k|\leq N} P_k Q_k\, \widetilde{a}_j \Big\|_{\mathcal{J}_{E_0,E_1}(B_0,B_1)} \leq \varepsilon/2\,,$$

respectively. □

We now consider the case of Asplund operators. A bounded linear operator $T : X \longrightarrow Y$ between Banach spaces is said to be an ***Asplund operator*** if T^* is a

Radon-Nikodým operator (recall that $S : U \longrightarrow V$ is called a ***Radon-Nikodým operator*** if for any probability measure μ, S maps each μ-continuous U-valued measure of finite variation into a μ-differentiable V-valued measure; see [25]). A Banach space X is called an ***Asplund space*** if its identity operator I_X is an Asplund operator. Closed subspaces of an Asplund space and reflexive spaces are also Asplund spaces. Full information on Asplund spaces and related questions can be found in [15, 27, 3].

Theorem 3.3

Assume that E_0, E_1 are Asplund Banach lattices on $\mathbb{Z}$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in L(\bar{A}, \bar{B})$ be such that $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is an Asplund operator.

(i) *If $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$, then $T : \mathcal{K}_{E_0,E_1}(A_0, A_1) \longrightarrow \mathcal{K}_{E_0,E_1}(B_0, B_1)$ is Asplund.*

(ii) *If $\varphi^*_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$, then $T : \mathcal{J}_{E_0,E_1}(A_0, A_1) \longrightarrow \mathcal{J}_{E_0,E_1}(B_0, B_1)$ is Asplund.*

Proof. We shall first establish (ii). By (2.2) we may assume without loss of generality that $A_0 \cap A_1$ is dense in A_i and that $B_0 \cap B_1$ is dense in B_i $(i = 0, 1)$. We just need to show that the operator $\widetilde{T} : E_0(A_0) \cap E_1(A_1) \longrightarrow \mathcal{J}_{E_0,E_1}(B_0, B_1)$ is an Asplund operator. Furthermore, E_i is an Asplund space and consequently E_i does not contain a copy of ℓ_1. Therefore E_i and E_i' are regular and $(E_0(A_0) \cap E_1(A_1))^* \cong E_0'(A_0^*) + E_1'(A_1^*)$. For any bounded set H of a Banach space X, we denote by μ_H the seminorm on X^* given by $\mu_H(f) = \sup\{|f(x)| : x \in H\}$, $f \in X^*$.

Let D be a countable subset of the closed unit ball of $E_0(A_0) \cap E_1(A_1)$. Since $\widetilde{T}P_k$ is an Asplund operator, according to [3, Theorem 5.2.11], the space $(\widetilde{T}^*(\mathcal{J}_{E_0,E_1}(B_0, B_1)^*), \mu_{P_kQ_k(D)})$ is separable. Let Δ_k be a countable set dense in $(\widetilde{T}^*(\mathcal{J}_{E_0,E_1}(B_0, B_1)^*), \mu_{P_kQ_k(D)})$. We check that the countable set

$$\Delta = \Big\{ \sum_{|k|\le N} Q_k^* P_k^* g_k \,:\, g_k \in \Delta_k\,,\, N \in \mathbb{N} \Big\}$$

is dense in $(\widetilde{T}^*(\mathcal{J}_{E_0,E_1}(B_0, B_1)^*), \mu_D)$. Choose $f \in \mathcal{J}_{E_0,E_1}(B_0, B_1)^*$ and let $\varepsilon > 0$. Taking into account the regularity of E_i', there is $N \in \mathbb{N}$ such that

$$\Big\| \widetilde{T}^* f - \sum_{|k|\le N} Q_k^* P_k^* \widetilde{T}^* f \Big\|_{E_0'(A_0^*)+E_1'(A_1^*)} \le \varepsilon/2.$$

Moreover, for each $|k| \le N$ we can extract $g_k \in \Delta_k$ verifying that $\mu_{P_kQ_k(D)}(\widetilde{T}^* f - g_k) \le \varepsilon/(4N+2)$. Consequently,

$$\begin{aligned} \mu_D\Big(\widetilde{T}^* f - \sum_{|k|\le N} Q_k^* P_k^* g_k\Big) &\le \mu_D\Big(\widetilde{T}^* f - \sum_{|k|\le N} Q_k^* P_k^* \widetilde{T}^* f\Big) \\ &\quad + \mu_D\Big(\sum_{|k|\le N} Q_k^* P_k^* (\widetilde{T}^* f - g_k)\Big) \\ &\le \varepsilon/2 + \sum_{|k|\le N} \mu_{P_kQ_k(D)}(\widetilde{T}^* f - g_k) \le \varepsilon, \end{aligned}$$

and so (ii) is proved.

(i) A similar reasoning proves that $\widehat{T} : \mathcal{K}_{E_0,E_1}(A_0^\circ, A_1^\circ) \longrightarrow E_0(B_0^\circ) + E_1(B_1^\circ)$ is an Asplund operator. Indeed, if we denote by $S_k : E_0(B_0^\circ) + E_1(B_1^\circ) \longrightarrow B_0^\circ + B_1^\circ$ the projection given by $S_k b = b_k^0 + b_k^1$, for $b = b^0 + b^1$ with $b^j = (b_m^j) \in E_j(B_j^\circ)$, and D is a countable set of the closed unit ball of $\mathcal{K}_{E_0,E_1}(A_0^\circ, A_1^\circ)$, then for every k there exists a countable set Υ_k which is dense in $((B_0^\circ + B_1^\circ)^*, \mu_{S_k \widehat{T}(D)})$ and it can be checked that

$$\Upsilon = \Big\{ \sum_{|k| \le N} S_k^* g_k \,:\, g_k \in \Upsilon_k \,,\, N \in \mathbb{N} \Big\}$$

is a dense set in $(\,(E_0(B_0^\circ) + E_1(B_1^\circ)\,)^*, \mu_{\widehat{T}(D)})$. The proof is complete by (2.1). □

Finally, we research the interpolation of weakly compact operators. Recall that a bounded linear operator $T : X \longrightarrow Y$ between two Banach spaces is said to be *weakly compact* if $T(U_X)$ is a relatively weakly compact subset of Y. By the well-known Gantmacher's theorem, the operator T is weakly compact if and only if $T^{**}(X^{**}) \subset Y$.

We will need the following preliminary technical lemma.

Lemma 3.4

Let E_0 and E_1 be reflexive Banach lattices on $\mathbb{Z}$ and suppose that $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$. Then the following statements are true:

(i) *$\mathcal{K}_{E_0,E_1}$ is a regular interpolation functor.*

(ii) *The closed unit ball of $\mathcal{K}_{E_0,E_1}(X_0, X_1)$ is a closed subset of $X_0 + X_1$ for any Banach couple (X_0, X_1).*

Proof. Let $\Phi = \mathcal{K}_{E_0,E_1}(\ell_\infty, \ell_\infty(2^{-m}))$. To establish (i) we follow the line of reasoning of [4, Lemma 4.6.15.] According to [4, Corollary 3.6.3(b), Theorem 4.2.11], the interpolation functor $\mathcal{K}_{E_0,E_1}$ is regular if and only if $\ell_\infty \cap \ell_\infty(2^{-m})$ is dense in Φ and the lattice Φ is a nondegenerate parameter of the $\mathcal{K}$-method, i.e. $\Phi \setminus \ell_\infty \cup \ell_\infty(2^{-m}) \neq \emptyset$ (see [4, Definition 3.5.4]).

By Theorem 3.1 we get that Φ does not contain a copy of ℓ_1. In particular, the Banach lattice Φ is regular and so it is clear that $\ell_\infty \cap \ell_\infty(2^{-m})$ is dense in Φ.

Let us now see that Φ is a nondegenerate parameter of the $\mathcal{K}$-method. For instance, assume that the embedding $\Phi \hookrightarrow \ell_\infty(2^{-m})$ holds. Then the embedding $i : \ell_\infty \cap \ell_\infty(2^{-m}) \hookrightarrow \ell_\infty(2^{-m})$ is a Rosenthal operator. Let us denote by X the subspace $X = \{x = (x_m) \in \ell_\infty \cap \ell_\infty(2^{-m}) \,:\, x_m = 0 \text{ for all } m > 0\}$. In addition, let $P : \ell_\infty \cap \ell_\infty(2^{-m}) \longrightarrow X$ be the operator of multiplication by the characteristic function of the set $\{m \in \mathbb{Z} \,:\, m \le 0\}$. For every $x = (x_m) \in X$, it follows that

$$\|iPx\|_{\ell_\infty(2^{-m})} = \sup\{2^{-m}|x_m| \,:\, m \le 0\} = \|x\|_{\ell_\infty \cap \ell_\infty(2^{-m})}\,.$$

Thus, the restriction to X of the operator iP coincides with the identity operator on X and hence X does not contain a copy of ℓ_1. However, taking into account the previous equality, the space X is isometrically isomorphic to

$$\ell_\infty^-(2^{-m}) = \{x = (x_m)_{m \le 0} \,:\, \sup\{2^{-m}|x_m| \,:\, m \le 0\} < \infty\}\,.$$

We arrive at a contradiction.

(ii). It is proved in [24] that the Banach lattice Φ has the Fatou property, that is, the closed unit ball of Φ, U_Φ, is closed in $\omega(\mathbb{Z})$. Then, it is easy to derive that the closed unit ball of $K_\Phi(X_0, X_1)$ is closed in $X_0 + X_1$ for any Banach couple (X_0, X_1). Since $K_{E_0,E_1}(X_0, X_1) = K_\Phi(X_0, X_1)$ with equality of norms, the proof of (ii) is complete. □

Theorem 3.5

Assume that E_0 and E_1 are reflexive Banach lattices on $\mathbb{Z}$. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in L(\bar{A}, \bar{B})$ be such that $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is a weakly compact operator. Then the following statements hold:

(i) *If $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$, then the operator $T : \mathcal{K}_{E_0,E_1}(A_0, A_1) \to \mathcal{K}_{E_0,E_1}(B_0, B_1)$ is weakly compact.*

(ii) *If $\varphi^*_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$, then the operator $T : \mathcal{J}_{E_0,E_1}(A_0, A_1) \to \mathcal{J}_{E_0,E_1}(B_0, B_1)$ is weakly compact.*

Proof. (i). According to (2.1), without loss of generality, we may assume that $A_0 \cap A_1$ is dense in A_i and that $B_0 \cap B_1$ is dense in B_i $(i = 0, 1)$.

Applying Lemma 3.4(i) and [17, Chapter IV, Theorem 2.14], it holds that

$$\mathcal{K}_{E_0,E_1}(B_0, B_1)^* \cong \mathcal{J}_{E'_0,E'_1}(B^*_0, B^*_1). \tag{3.5}$$

Now we claim that the inclusion map $\mathcal{K}_{E_0,E_1}(B_0, B_1) \hookrightarrow B_0 + B_1$ is a Tauberian operator (see, e.g., [22] for definition and properties of Tauberian operators). In fact, according to [22, Theorem 5], it is enough to check that $(B_0 + B_1)^*$ is norm-dense in $\mathcal{K}_{E_0,E_1}(B_0, B_1)^*$ and moreover that the closed unit ball of $\mathcal{K}_{E_0,E_1}(B_0, B_1)$ is a closed subset of $B_0 + B_1$. But these facts follow as a straightforward consequence from (3.5) (see also [17, Chapter IV, Lemma 2.14]) and Lemma 3.4(ii) respectively.

We can conclude that $T(U_{\mathcal{K}_{E_0,E_1}(A_0,A_1)})$ is a relatively weakly compact subset of $\mathcal{K}_{E_0,E_1}(B_0, B_1)$ provided that $T : A_0 \cap A_1 \to B_0 + B_1$ is weakly compact (see [22, Theorem 8]).

(ii). Using (2.2), we may suppose without loss of generality that $A_0 \cap A_1$ is dense in A_i and that $B_0 \cap B_1$ is dense in B_i $(i = 0, 1)$. Because of $E_0 \cap E_1 \hookrightarrow \ell_1$, we have $e \in E'_0 + E'_1$, and so $\mathcal{K}_{E'_0,E'_1}$ is a well defined interpolation functor. In addition, it is not hard to derive that $\varphi_{\mathcal{K}_{E'_0,E'_1}} = \varphi^*_{\mathcal{J}_{E_0,E_1}}$ from the definition of fundamental function, and so $\varphi_{\mathcal{K}_{E'_0,E'_1}} \in \mathcal{P}_0$. Indeed, under our assumptions, it holds that (see Lemma 3.4(i) and [17, Chapter IV, Theorem 2.15])

$$\mathcal{J}_{E_0,E_1}(A_0, A_1)^* \cong \mathcal{K}_{E'_0,E'_1}(A^*_0, A^*_1) \quad \text{and} \quad \mathcal{J}_{E_0,E_1}(B_0, B_1)^* \cong \mathcal{K}_{E'_0,E'_1}(B^*_0, B^*_1). \tag{3.6}$$

By Gantmacher's theorem, it follows that $T^* : B^*_0 \cap B^*_1 \to A^*_0 + A^*_1$ is a weakly compact operator. Using (3.6) and the statement (i), we obtain that $T^* : \mathcal{J}_{E_0,E_1}(B_0, B_1)^* \to \mathcal{J}_{E_0,E_1}(A_0, A_1)^*$ is weakly compact, and the proof finishes. □

Since $\mathcal{K}_E(\bar{A})$ (resp. $\mathcal{J}_E(\bar{A})$) coincides with $\mathcal{K}_{E,E(2^m)}(\bar{A})$ (resp. $\mathcal{J}_{E,E(2^m)}(\bar{A})$), Theorems 3.5, 3.1, 3.2 and 3.3 yield [6, Corollaries 4.4, 4.5 and 4.6] as well as [7, Theorems 3.1 and 3.2] respectively (see also [4, Theorem 4.6.8], [21, Theorem 3.3] and [22, Corolary 11]). Moreover, applying Theorems 3.1, 3.2, 3.3 and 3.5 for the special case $E_0 = \ell_{p_0}(2^{-\theta m})$, $E_1 = \ell_{p_1}(2^{(1-\theta)m})$, $1 < p_0, p_1 < \infty$, $0 < \theta < 1$, we derive results due to Heinrich (see [15, Theorem 2.3 and Proposition 2.2]) on interpolation of these operator ideals by the classical real method (see also [1, Propositions II.2.3, II.3.3 and Theorem III.2.1]).

On the other hand, Theorem 3.1 yields:

Corollary 3.6

Let E_0, E_1 be Banach lattices on $\mathbb{Z}$ that do not contain a copy of ℓ_1. Suppose that (A_0, A_1) is a Banach couple such that the inclusion map $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ is a Rosenthal operator.

(i) *If $\varphi_{\mathcal{K}_{E_0,E_1}} \in \mathcal{P}_0$, then $\mathcal{K}_{E_0,E_1}(A_0, A_1)$ does not contain a copy of ℓ_1.*

(ii) *If $\varphi^*_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$, then $\mathcal{J}_{E_0,E_1}(A_0, A_1)$ does not contain a copy of ℓ_1.*

Similar results hold for Banach-Saks operators, Asplund operators and weakly compact operators.

Finally we present some applications. Let $\mathcal{U}$ denotes the set of all concave, positively homogeneous of degree one, nondecreasing continuous in each variable functions $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\psi(0,0) = 0$. If $\bar{E} = (E_0, E_1)$ is a couple of Banach lattices on $\mathbb{Z}$ and $\psi \in \mathcal{U}$, then the ***Calderón-Lozanovskii space*** $\psi(\bar{E}) = \psi(E_0, E_1)$ consists of all $x \in \omega(\mathbb{Z})$ such that $|x| \le \lambda\psi(|x_0|, |x_1|)$ for some $\lambda > 0$, $x_j \in E_j$, with $\|x_j\|_{E_j} \le 1$, $j = 0, 1$. The space $\psi(\bar{E})$ is a Banach lattice equipped with the norm

$$\|x\|_\psi = \inf\{\lambda > 0 : \, |x| \le \lambda\psi(|x_0|, |x_1|), \; \|x_0\|_{E_0} \le 1, \, \|x_1\|_{E_1} \le 1\}$$

(see [20] for details).

A function $\psi \in \mathcal{U}$ is said to be a ***quasi-power*** if the dilatation indices δ_ρ and γ_ρ of the function $\rho(t) = \psi(1, t)$ satisfy $0 < \delta_\rho \le \gamma_\rho < 1$ (see, e.g., [16, 17]).

Theorem 3.7

Let $E_0 = \ell_{p_0}(1/\psi(1, 2^m))$ and $E_1 = \ell_{p_1}(2^m/\psi(1, 2^m))$ be weighted Banach lattices on $\mathbb{Z}$, where $\psi \in \mathcal{U}$ is a quasi-power function and $1 < p_j < \infty$ for $j = 0, 1$. Suppose that $\bar{A} = (A_0, A_1)$ is a Banach couple. Then the following statements are equivalent:

(i) *The inclusion map $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ is a Banach-Saks operator.*

(ii) *$\mathcal{J}_{E_0,E_1}(A_0, A_1)$ has the Banach-Saks property.*

(iii) *The $\mathcal{K}$-method space $\mathcal{K}_{\psi(\ell_{p_0}, \ell_{p_1}(2^{-m}))}(A_0, A_1)$ has the Banach-Saks property.*

Proof. Since $\psi \in \mathcal{U}$ is quasi-power, can easily get that $E_0 \cap E_1 \hookrightarrow \ell_1$. This implies that $\mathcal{J}_{E_0,E_1}$ is a non-trivial functor. Further simple calculations show that

$$2^{-(1-j)n}\|\tau_n\|_{E_j \to E_j} \longrightarrow 0 \quad \text{and} \quad 2^{-jn}\|\tau_{-n}\|_{E_j \to E_j} \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

for $j = 0, 1$. In consequence, it follows by Lemma 2.2 that $\varphi^*_{\mathcal{J}_{E_0,E_1}} \in \mathcal{P}_0$.

It is shown in [23] that under above conditions

$$\mathcal{J}_{E_0,E_1}(A_0, A_1) = \mathcal{K}_{\psi(\ell_{p_0},\ell_{p_1}(2^{-m}))}(A_0, A_1)$$

holds for every Banach couple (A_0, A_1). Since both spaces E_0 and E_1 have the Banach-Saks property, Theorem 3.2 applies. □

Using Theorems 3.1, 3.3 and 3.5, we obtain analogous results concerning the Rosenthal property, the property of being Asplund, and the weak compactness, respectively.

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