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Remarks on the Hilbert transform and on some families of multiplier operators related to it

Enrico Laeng

Dipartimento di Matematica, Politecnico di Milano, Piazza L. da Vinci 32
20133 Milano (Italy)

E-mail: enrlae@mate.polimi.it

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Abstract

We give an overview of the behavior of the classical Hilbert Transform H seen as an operator on $L^p(\mathbb{R})$ and on weak-$L^p(\mathbb{R})$, then we consider other operators related to H. In particular, we discuss various versions of Discrete Hilbert Transform and Fourier multipliers periodized in frequency, giving some partial results and stating some conjectures about their sharp bounds $L^p(\mathbb{R}) \to L^p(\mathbb{R})$, for $1 < p < \infty$.

1. Introduction

In this paper we discuss the Hilbert Transform H and other multiplier operators related to it. In particular, we discuss operators that share with H (or we conjecture that they share) the same sharp bounds as operators mapping $L^p(\mathbb{R}) \to L^p(\mathbb{R})$, for $1 < p < \infty$.

Section 2 contains an overview of what is known and what is not known about the norms of H seen as an operator in three different ways: (i) from L^p into L^p; (ii) from weak-L^p into weak-L^p; (iii) from L^p into weak-L^p. The main new result is the observation (Theorems 2.1 and 2.2) that, although some truncations of the weak-L^p functions $|x|^{-1/p}$ and $\operatorname{sgn}(x)|x|^{-1/p}$, when plugged into H make it achieve "naturally" its norm for the case (i), yet these same weak-L^p functions are not extremals in (ii), and the norms can be strictly bigger in this case.

In Section 3 we discuss some multipliers "of Hilbert Transform type" like the ***segment multiplier***, the ***gap Hilbert transform***, general Fourier multipliers associated with

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monotonic functions, *the Riesz projection*, the *Truncated Hilbert Transform*. Among these, in Theorem 3.1, we show that a compactly supported multiplier corresponding to one period of the sawtooth function has the same norms of H, a result that will be used in Section 5.

In Section 4 we introduce various versions of *Discrete Hilbert Transform* and discuss the 80-year-old problem of showing that two different types of such a transform share the same norms of $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. We give a short history of this remarkable problem and some preliminary facts, then we present a result communicated to us by I.E. Verbitsky a few years ago: the norms of $D : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$, where D is a Discrete Hilbert Transform, coincide with the norms of $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ when p is a power of 2 or the dual exponent of such a p.

In Theorem 4.6 we give a different kind of result: the norms of D and the norms of the truncated Hilbert transform $H^{(1/2)}$, differ at most by an absolute constant, independent of p.

In Section 5 we introduce the family of those multiplier operators whose multiplier function is periodic in the frequency domain. Because of Theorem 4.2, this is a natural setting in which to discuss continuous and discrete versions of operators like H. We prove some simple, but general, facts (Propositions 5.1 and 5.2) and then show that the main problem of the previous section can be seen as a special case of more general problems on these multipliers (see Theorem 5.3 and the following remarks and conjectures). We also prove (Theorem 5.6) a lower estimate for the norms of some "asymmetric" discrete Hilbert transforms (4.5), conjecturing that this lower bound is actually the norm itself.

2. Behavior of the classical Hilbert Transform on L^p and weak-L^p

The *Hilbert transform*

$$(Hf)(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-t)}{t}\,dt$$

is well-defined for $f \in C_c^\infty(\mathbb{R})$ and it can be extended to a bounded linear operator that maps $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ and satisfies the inequality of M. Riesz

$$\|Hf\|_p \le n_p \|f\|_p \qquad 1 < p < \infty. \tag{2.1}$$

The operator norm $||H||_{p,p}$ is the best constant n_p in this inequality and it is given by

$$n_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } \ 1 < p \le 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } \ 2 \le p < \infty. \end{cases} \tag{2.2}$$

Several essentially different proofs of this fundamental inequality are known, but only one proof technique so far leads to (2.1) with the best constant (2.2). Such a technique involves the use of subharmonic (or superharmonic) functions together with the fact that $f + iHf$ is the boundary value of a holomorphic function in the upper half plane when f is real-valued. Actually, the upper half plane can be replaced by the unit disk, and it turns out that n_p is also the norm of the *conjugate function operator*, the analogue of H on the unit circle. This is what Pichorides originally proved, in [19], and from that he derived the same result on the line. We refer the reader to the recent work of Grafakos [11] for a shorter proof that deals directly with the case of H. We should note that earlier than Pichorides (and Cole, unpublished) the constant (2.2) had been shown to be optimal by Gohberg and Krupnik for a discrete family of exponents $p = 2^n$, with $n = 1, 2, \dots$ (see [9] and also paragraph 4 here). Their technique does not use subharmonic functions, but a direct extension of their result to all values of p would require the knowledge of some strong "a priori" property of the norm.

In fact, a related general problem is the following: *what assumptions on a linear operator mapping L^p into L^p are sufficient to guarantee that its norm is an analytic function of p in some range $p_1 < p < p_2$*? A positive answer to this question appears to depend on some kind of "uniqueness" or, at least, "good separation" of the extremals that make our operator achieve its norm. Further complications arise when the norm is a non-attained best constant.

Since H is an L^2-isometry, the equal sign holds in (2.1) when $p = 2$ and every $f \in L^2$ is an extremal (note that the analyticity of n_p breaks down around $p = 2$). When $p \neq 2$ no L^p extremals are known, but plugging into H certain truncations of the weak-L^p functions $|x|^{1/p}$ and $\operatorname{sgn}(x)|x|^{1/p}$ we can approach both the supremum and the infimum of the ratio $||Hf||_p/||f||_p$. More precisely, in [5] we have proven the following:

Theorem 2.1

Let ϕ_δ be an $L^p(\mathbb{R})$-function, defined for each $1 < p < \infty$ and $\delta \in (0,1)$ by

$$\phi_\delta(x) = (-4\log\delta)^{-1/p}\chi_{\{x:\delta<|x|<1/\delta\}}(x)|x|^{-1/p},$$

and let ψ_δ be the odd version of the above even function ϕ_δ, namely

$$\psi_\delta(x) = \phi_\delta(x)\operatorname{sgn}(x).$$

Then we have $||\phi_\delta||_p = ||\psi_\delta||_p = 1$. The limit $\lim_{\delta\to0^+} ||H\phi_\delta||_p$ coincides with the norm n_p of H for $1 < p \le 2$ and with the subnorm $1/n_p$ of H for $2 \le p < \infty$ (we denote by subnorm the quantity $\inf ||Hf||_p/||f||_p$).

Vice-versa, the limit $\lim_{\delta\to0^+} ||H\psi_\delta||_p$ coincides with the norm n_p of H for $2 \le p < \infty$ and with the subnorm $1/n_p$ of H for $1 < p \le 2$.

Sketch of proof. The first step is to show that

$$H(|x|^{-1/p}) = \tan\left(\frac{\pi}{2p}\right)\operatorname{sgn}(x)|x|^{-1/p}$$

and that

$$H(\operatorname{sgn}(x)|x|^{-1/p}) = -\cot\left(\frac{\pi}{2p}\right)|x|^{-1/p} \quad \text{for all} \quad 1 < p < \infty.$$

This can be done, after a change of variable, using residues. We choose a contour of integration in the complex plane "adapted" to the truncation given by the characteristic function $\chi_{\{x:\delta<|x|<1/\delta\}}(x)$. The two above equalities are obtained as $\delta \to 0^+$ and we show that the remainder term remains bounded in L^p. Denoting the identity operator with I, the result then follows from (2.1), (2.2) and the fact that $H^2 = -I$ both as an operator on L^p and on weak-L^p. See [5] for details. □

It is useful both for the present discussion and for introducing some of the following topics, to also take a look at the "positive" Hilbert operator defined by

$$H_+ f(x) = \frac{1}{\pi} \int_0^\infty \frac{f(t)}{x+t}\, dt, \tag{2.3}$$

where $f \in L^p([0, +\infty))$ and $1 < p < \infty$.

This operator is clearly related to H, but it has a simpler structure because no principal values (or cancellations of any kind) are involved in the convergence of the integral. In fact, we can give a straightforward proof that H_+ maps boundedly L^p of the half-line into itself, while making quite explicit the special role of the functions $x^{-1/p}$ in determining the best constant for each p. To do that, we consider the bilinear form

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\, dx\, dy, \tag{2.4}$$

where $f \in L^p([0, +\infty))$, $g \in L^q([0, +\infty))$ and $1/p + 1/q = 1$. Note that in what follows we can consider $f \geq 0$ and $g \geq 0$ without loss of generality. The change of variables $y = ux$ in (2.4) and Fubini's theorem yield

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\, dx\, dy &= \int_0^\infty f(x) dx \int_0^\infty \frac{g(ux)}{x+ux}\, x\, du \\ &= \int_0^\infty \frac{du}{1+u} \int_0^\infty f(x)\, g(ux) dx. \end{aligned}$$

Observing that

$$\left(\int_0^\infty g(ux)^q dx \right)^{1/q} = u^{-1/q} ||g||_q,$$

we apply Hölder's inequality to the inner integral, showing that (2.4) is bounded by

$$\int_0^\infty \frac{u^{-1/q}}{1+u}\, du\, ||f||_p\, ||g||_q\, .$$

Using duality and remembering definition (2.3), we obtain that

$$\|H_+ f\|_p \leq c_p \|f\|_p \qquad 1 < p < \infty\,, \tag{2.5}$$

with

$$c_p = \frac{1}{\pi} \int_0^\infty \frac{u^{-1/q}}{1+u}\, du = \frac{1}{\sin(\frac{\pi}{p})}, \tag{2.6}$$

where the given "closed" formula for c_p can be obtained, e.g., with contour integration. Now it is not difficult to show that (2.6) is the best possible constant in (2.5) and that, although it is never attained, it can be approached, as $\delta \to 0^+$, applying H_+ to the truncated functions $\chi_{\{x:\delta<x<1/\delta\}}(x) x^{-1/p}$ (see [14, Chapter IX]). Note that H_+ is not an isometry when $p = 2$, that there are no extremals, and that the analyticity of c_p

does not break down around $p = 2$. On the other hand, both $c_p = c_q$ and $n_p = n_q$, if $1/p + 1/q = 1$, a fact dictated by duality.

If we apply the above proof strategy in the case of H we can show that

$$\text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{x-y}\, dx\, dy = \text{p.v.} \int_{-\infty}^{\infty} \frac{du}{1-u} \int_{-\infty}^{\infty} f(x)\, g(ux) dx, \tag{2.7}$$

but then we run into an obstacle, because now it is no longer correct to assume that f and g are non-negative. We are forced to consider changes of sign in both integrals of the last expression in (2.7) and, in fact, the outer principal value integral would not even converge without cancellations. Hölder's inequality tells us that, for each given $u \in \mathbb{R}$, we have

$$\left| \int_{-\infty}^{\infty} f(x)\, g(ux) dx \right| \leq ||f||_p\, ||g(u\ \cdot)||_q,$$

which is equivalent to the following equality

$$\int_{-\infty}^{\infty} f(x)\, g(ux) dx = \eta(u)\ |u|^{-1/q} ||f||_p\, ||g||_q, \tag{2.8}$$

with $\eta(u) = \eta(f, g, u)$ a suitable bounded function satisfying $-1 \leq \eta(u) \leq 1$. These considerations imply that the norm n_p in (2.2) must coincide with the supremum of

$$\left| \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta(u)\ |u|^{-1/q}\ du}{1-u} \right|$$

taken over the admissible space of bounded functions η. Assuming Theorem 2.1 we can say that the maximizers we are looking for (and also the corresponding minimizers) are given by $\eta(u) \equiv 1$ or $\eta(u) = \text{sgn}\,(u)$, depending on p being bigger or smaller than 2. A direct solution of this variational problem would be very interesting both for clarifying our understanding of H and for potential applications to other Fourier multipliers, including those discussed in the next sections.

It is well known that H maps weak-L^p into weak-L^p boundedly. Also, it has been proven by Colzani, in [2], that any sublinear translation-invariant operator bounded on weak-L^p is also bounded on L^p, with a norm on L^p which is smaller than or equal to its norm on weak-L^p. Explicit examples are known where the two norms are the same, and it might seem natural to guess that H also belongs among these kind of cases, with norm n_p attained on the weak-L^p extremals $|x|^{-1/p}$ or $\text{sgn}\,(x)|x|^{-1/p}$ as suggested by Theorem 2.1. Perhaps surprisingly, it turns out that this is not the case.

Theorem 2.2

The norm of H as an operator mapping weak-$L^p(\mathbb{R})$ into weak-$L^p(\mathbb{R})$ can be strictly bigger than n_p, its norm on $L^p(\mathbb{R})$. In particular, this certainly happens for $p = 2$ and for an open interval of exponents p containing $p = 2$.

Proof. A property of H, proven by Stein and Weiss in [20], is that the distribution function of the Hilbert transform of a characteristic function of a set E only depends on the Lebesgue measure $|E|$ of such a set. In fact we have the equality

$$|\{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\}| = \frac{2|E|}{\sinh(\pi\lambda)},$$

which is equivalent, in terms of the decreasing-rearrangement of $H\chi_E$, to

$$(H\chi_E)^*(t) = \frac{1}{\pi} \operatorname{arcsinh}\left(\frac{2|E|}{t}\right) \qquad (t > 0).$$

The weak-L^p norm of $H\chi_E$ is given by

$$\begin{aligned} ||H\chi_E||_{L^{p,\infty}} &= \sup_{\lambda>0} \lambda \,|\, \{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\} \,|^{1/p} \\ &= \sup_{\lambda>0} \lambda \left(\frac{2|E|}{\sinh(\pi\lambda)}\right)^{1/p} = \alpha(p)|E|^{1/p}, \end{aligned}$$

where

$$\alpha(p) = \sup_{\lambda>0} \lambda \left(\frac{2}{\sinh(\pi\lambda)}\right)^{1/p} = \sup_{t>0} \frac{t}{\pi} \left(\frac{2}{\sinh t}\right)^{1/p}. \tag{2.9}$$

Since $H^2 = -I$ on weak-L^p, its norm on that space is greater than or equal to $\max\{\alpha(p), \alpha(p)^{-1}\}$. For example, when $p = 2$, we have

$$\alpha(2) = \sup_{t>0} \frac{t}{\pi} \left(\frac{2}{\sinh t}\right)^{1/2} = 0.473127337424\ldots$$

and $\max\{\alpha(2), \alpha(2)^{-1}\} = 2.113595898819\cdots > 1 = n_2$. When $p \neq 2$ we observe that the function of t that appears in (2.9) is unimodal, with limit 0 both as $t \to 0^+$ and as $t \to +\infty$, increasing for $t \in (0, t_0]$ and decreasing for $t \in [t_0, +\infty)$, having denoted by $t_0 = t_0(p)$ the only point where it attains its maximum $\alpha(p)$. The continuity of $\alpha(p)$ and n_p as functions of p implies that there is an open interval of values p, centered around $p = 2$, such that $\max\{\alpha(p), \alpha(p)^{-1}\} > n_p$. It is possible to make this interval larger, by plugging suitable *simple* functions into H instead of characteristic functions of sets, but given the negative nature of this result we choose to state it and prove it in its simplest form. □

To conclude this section, we note that H can also be studied as an operator mapping $L^p(\mathbb{R})$ into weak-$L^p(\mathbb{R})$. The corresponding norm is the best constant in the weak-type (p, p) inequality. The case $p = 1$ can now be considered, and the corresponding norm was shown by Davis to be

$$\frac{1 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots}{1 - 1/3^2 + 1/5^2 - 1/7^2 + \ldots},$$

using Brownian motion and probabilistic methods. A different analytic proof was given later by A. Baernstein II. Curiously, while these best constants are known for $1 \leq p \leq 2$, they are only “almost” known for $p > 2$, since there is still an open alternative: either they are obtained by a suitable extension of Baernstein’s method, or they are the solution of a problem on variational inequalities of Alt-Caffarelli type (some references for this problem, in chronological order, are [3, 4, 1, 22, 8]).

3. Some Fourier multipliers of Hilbert Transform type

The Hilbert transform can be written as a Fourier multiplier operator

$$(Hf)(x) = \int_{\mathbb{R}} -i \ \mathrm{sgn}\,(\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi. \tag{3.1}$$

In this section we want to discuss some other remarkable multiplier operators

$$T_m f(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi, \tag{3.2}$$

whose norm from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ coincides with, or is related to, the norm of (3.1), i.e., the quantity n_p defined in (2.2). If we choose $m(\xi) = \chi_{[-r,r]}(\xi)$ in (3.2), with $r > 0$, we get the ***segment multiplier*** (or ***partial Fourier inversion multiplier***)

$$S_r(x) = \int_{-r}^{r} \hat{f}(\xi)e^{2\pi i x\xi}d\xi, \tag{3.3}$$

whose norm turns out to be equal to n_p, for $1 < p < \infty$ and any $r > 0$. This is the main result in [5] where we also show that the ***gap Hilbert Transform***, i.e., the operator

$$H_r = \int_{\mathbb{R}} -i \ \mathrm{sgn}\,(\xi)\chi_{(-\infty,-r]\cup[r,+\infty)}(\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi, \tag{3.4}$$

has norm n_p for $1 < p < \infty$ and any $r > 0$.

We remind the reader that multiplier operator (p,p)-norms are invariant by translation and dilation of the multiplier function $m(\xi)$. In particular the above results do not change for non-centered segment multipliers or for non-centered gap Hilbert Transforms.

Note that the (p,p) norms of S_r, H_r and H are originally defined by taking suprema over ***real-valued*** $L^p(\mathbb{R})$ functions. It turns out that all these norms are unchanged if we consider ***complex-valued*** $L^p(\mathbb{R})$ functions. This non-trivial fact is the consequence of a general theorem of J. Marcinkiewicz and A. Zygmund about vector-valued linear operators, (see [18], or [12, pages 311–315]). This theorem implies that a linear operator that maps boundedly a real-valued L^p space into itself also maps the complex-valued version of the same space into itself ***with the same norm***. This is not true for multipliers in general: an important counterexample is the ***Riesz projection*** (half-line multiplier) P, defined by

$$(Pf)(x) = \frac{f(x) + i(Hf)(x)}{2} = \int_0^{+\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi. \tag{3.5}$$

The (p,p) norm of P does depend on the choice of the domain and it is bigger, for $p \neq 2$, if we choose the complex-valued $L^p(\mathbb{R})$ space instead of the real-valued one. We just observed that we can replace the multiplier associated with $[a,b]$ with the multiplier associated with $[-r,r]$ for any fixed $r > 0$ without affecting the corresponding (p,p)-norm. This "centered segment" multiplier clearly maps real-valued functions into real-valued functions. The same trick does not work for the multiplier P associated to the half-line $[0,\infty)$. The norm of P, in the real case, has been found by I.E. Verbitsky, and later independently by M. Essén (see [7, 23]). We have

$$||P||^{\mathbb{R}}_{p,p} = \frac{1}{2}\sqrt{1 + n_p^2}. \tag{3.6}$$

The norm in the complex case has been determined recently by B. Hollenbeck and I.E. Verbitsky (see [16]) and is given by

$$||P||^{\mathbb{C}}_{p,p} = \frac{1}{\sin(\frac{\pi}{p})}, \tag{3.7}$$

an expression that coincides with (2.6), the norm c_p of H_+ over $L^p([0,+\infty))$.

Note that the operator P is injective on real-valued L^p functions, while there is a whole Hardy space of complex-valued L^p functions which are mapped into 0.

In [6] we have shown that the multiplier function $\operatorname{sgn}(\xi)$ in (3.1) can be replaced by any monotonic function increasing from -1 to 1 without affecting the operator norm n_p. More generally, the norm of any operator (3.2) associated to a monotonic function $m(\xi)$ increasing (or decreasing) between two different finite values, has the same (p,p) norm of the operator (3.2) associated with two half-lines at the height of those two values.

Let us denote by $|||m|||_p$ the (p,p)-norm of the Fourier multiplier operator (3.2) associated to m. The following result will be applied in the next section

Theorem 3.1

The Fourier multiplier operator (3.2) associated to the compactly-supported multiplier function μ defined by

$$\mu(\xi) = \begin{cases} 1-2\xi & if \;\; \xi \in [0,1/2] \\ -1-2\xi & if \;\; \xi \in [-1/2,0) \\ 0 & if \;\; \xi \notin [-1/2,1/2], \end{cases} \tag{3.8}$$

for $1 < p < \infty$, maps boundedly $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ with norm $|||\mu|||_p = |||\operatorname{sgn}(\cdot)|||_p = n_p$ equal to the norm of the Hilbert Transform H.

Proof. Note that if we change the sign of the values of $\mu(\xi)$ in left interval $[-1/2,0])$ we obtain a continuous "triangular" function $\phi(\xi)$ which corresponds to Fejér's kernel. This is a positive convolution kernel and we have $|||\phi|||_p = 1$ for all p's. Remembering that H corresponds to $-i \operatorname{sgn}(\xi)$, we obtain $|||\mu|||_p \le |||\operatorname{sgn}(\cdot)|||_p \; |||\phi|||_p = n_p$ (the constant factor $-i$ has absolute value 1 and is immaterial here). Now, as we already observed, operator norms are invariant by (finite) dilations of the function μ. This fact, together with Fatou's lemma, leads to the reverse inequality $n_p = |||\operatorname{sgn}(\cdot)|||_p \le |||\mu|||_p$. □

In many other cases, although there might be good reasons to think that the norms of a multiplier operator are n_p, the problem is still open. Two such examples are given by $m(\xi) = 1-\chi_{[0,1]}(\xi)$ and by the *Haar multiplier* $m(\xi) = \chi_{[0,1]}(\xi) - \chi_{[-1,0]}(\xi)$. A third example, which also will play a role in the next section, is the *Truncated Hilbert Transform*, defined for any fixed $\epsilon > 0$ by

$$H^{(\epsilon)}f(x) = \frac{1}{\pi}\int_{|t|\ge\epsilon} \frac{f(x-t)}{t} dt. \tag{3.9}$$

It is known that the norms $||H^{(\epsilon)}||_{p,p}$ do not depend on the size 2ϵ of the truncation around $t = 0$. Since $\lim_{\epsilon \to 0} H^{(\epsilon)}(x) = H(x)$, Fatou's lemma implies that $||H^{(\epsilon)}||_{p,p} \geq n_p$.

4. An 80-year-old tantalizing problem: evaluating the (p, p) norms of the discrete Hilbert transform.

D. Hilbert, followed by many others, has studied the discrete bilinear form

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n}. \tag{4.1}$$

Let us assume that the sequence $a_m \not\equiv 0$ is non-negative and in $\ell^p(\mathbb{N})$, that the sequence $b_n \not\equiv 0$ is non-negative and in $\ell^q(\mathbb{N})$, with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, among other things, (4.1) satisfies the following inequality, with best constant

$$\frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{1}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}, \tag{4.2}$$

which (see [14, Chapter IX]) is remarkably easy to obtain from the corresponding one for the continuous bilinear form (2.4). In other words, the operator (2.3) has a discrete analogue with the same norm c_p as in (2.6). It is natural to ask if the Hilbert Transform H also has a discrete analogue with norm n_p as in (2.2).

Historically, the first attempt to answer this question seems to date back to 1926, to an article by E.C. Titchmarsh (see [21]) where he introduces the following discrete operator

$$(D_{1/2}b)_n = \text{p.v.} \frac{1}{\pi} \sum_k \frac{b_{n-k}}{k+1/2}. \tag{4.3}$$

which maps boundedly the space of bilateral sequences $\{b_n\} \in \ell^p(\mathbb{Z})$ into itself and does behave in many respects like H. The index k in (4.3) runs over all the integers in $\mathbb{Z}$, and the sum is taken in the "principal value" sense, i.e., as the limit of balanced partial sums from $-N$ to N, with cancellations playing a role in convergence. Curiously, Titchmarsh states as a theorem that the (p, p) norms of H and (4.3) coincide, but his proof turns out to be incorrect. He later acknowledges this mistake in another issue of the same Journal, Mathematische Zeitschrift, without any comment on the putative truth of the original statement. Note that the exact value n_p of the norms of H was not known to Titchmarsh at the time. In fact he did not even know their exact order of growth as $p \to 1$ or $p \to \infty$, although he did know that these norms blow up at least with polynomial order one, and not much faster than that.

There are other good candidates for a discrete Hilbert transform. Another one is defined by

$$(Db)_n = \text{p.v.} \frac{1}{\pi} \sum_{k \neq 0} \frac{b_{n-k}}{k}, \tag{4.4}$$

where k runs over all the non-zero integers in $\mathbb{Z}$, with the bilateral sum taken, again, in the principal value sense. More generally, we can consider the operators D_α defined for each fixed $\alpha \in (0,1)$ by

$$(D_\alpha b)_n = \text{p.v.} \frac{\sin \pi\alpha}{\pi} \sum_k \frac{b_{n-k}}{k+\alpha}, \tag{4.5}$$

where k runs over all the integers in $\mathbb{Z}$, and where the normalization factor $\frac{\sin \pi\alpha}{\pi}$ compensates for one term in the sum that can be very big when its denominator $k+\alpha$ is close to zero. It is known that D_α, normalized as in (4.5), maps $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with norm 1 (this includes the special case of $D_{1/2}$). Furthermore, also D with its normalization constant as in (4.4) maps $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with norm 1. For a recent and elementary proof of this fact see [10].

In the rest of this section we want to say something about the following, which is to the best of our knowledge still an open problem.

Conjecture 4.1 The (p,p) norm of the discrete operators $D_{1/2}$ and D, defined in (4.3) and (4.4), coincide with the norm n_p of the classical Hilbert transform H, for all $1 < p < \infty$.

In the next section we will say something about the "asymmetric" case of D_α for $\alpha \neq 1/2$, whose (p,p)-norms behave in a slightly different way.

The next elementary, but general, result relates discrete operators like D or D_α to operators acting on functions on the real line.

Theorem 4.2

Let's associate with the discrete operator

$$(Tb)_n = \text{p.v.} \sum_k a_k \; b_{n-k},$$

mapping bilateral sequences into bilateral sequences another operator

$$(Mf)(x) = \text{p.v.} \sum_k a_k \; f(x-k),$$

which maps functions of $x \in \mathbb{R}$ into functions of $x \in \mathbb{R}$. In both cases $\{a_k\}$ is the same bilateral sequence of coefficients in $\ell^\infty(\mathbb{Z})$. Then T maps $\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ if and only if M maps $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ and the two norms coincide, namely $||T||_{p,p} = ||M||_{p,p}$ for $1 < p < \infty$.

Proof. If we apply M only to the subspace of $L^p(\mathbb{R})$ consisting of those step functions which are constant on the intervals $[n-1/2, n+1/2)$ for $n \in \mathbb{Z}$ we obtain a function in the same subspace. This implies that $||T||_{p,p} \leq ||M||_{p,p}$. In order to obtain the reverse inequality we observe that

$$\int_{\mathbb{R}} \left| \sum_k a_k \; f(x-k) \right|^p dx = \sum_n \int_0^1 \left| \sum_k a_k \; f(t+n-k) \right|^p dt$$
$$\leq ||T||_{p,p}^p \sum_n \int_0^1 |f(t+n)|^p \, dt = ||T||_{p,p}^p \int_{\mathbb{R}} |f(x)|^p dx. \qquad \square$$

We are ready now to prove:

Theorem 4.3

Let $D: \ \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ *and* $D_{1/2}: \ \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ *be the discrete linear operators defined in* (4.4) *and* (4.3) *and let* $n_p = ||H||_{p,p}$ *be the expression defined in* (2.2)*. Then* $||D||_{p,p} \geq n_p$ *and* $||D_{1/2}||_{p,p} \geq n_p$ *for all* $1 < p < \infty$*.*

Proof. Because of Theorem 4.2 we can work on the real line and replace D with

$$(Gf)(x) = \text{p.v.}\frac{1}{\pi}\sum_{k\neq 0}\frac{f(x-k)}{k}. \tag{4.6}$$

Now, in order to show that $||G||_{p,p} \geq ||H||_{p,p}$ let us consider the dilation operators T_ϵ defined for any fixed $\epsilon > 0$ and $1 < p < \infty$ by $(T_\epsilon f)(x) = \epsilon^{1/p} f(\epsilon x)$. It is easy to check that $||T_\epsilon||_{p,p} = 1$ for all $\epsilon > 0$. Note that

$$(T_{1/\epsilon} G T_\epsilon f)(x) = \text{p.v.}\frac{1}{\pi}\sum_{k\neq 0}\frac{f(x-\epsilon k)}{\epsilon k}\epsilon$$

is a Riemann sum for $\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(x-t)}{t}\,dt$. When $\epsilon \to 0$ this sum tends to the principal value of the corresponding integral. We have

$$||H||_{p,p} \leq \sup_{\epsilon} ||T_{1/\epsilon} G T_\epsilon||_{p,p} \leq ||G||_{p,p}.$$

The proof in the case of $D_{1/2}$ is very similar, so we skip the details (it will also be a special case of our discussion of D_α in the next section). □

In [9], using the non-linear identity

$$(Hf(x))^2 = 2H(f \cdot Hf)(x) + f(x)^2, \tag{4.7}$$

which is satisfied by the Hilbert transform, T. Gokhberg and N.Y. Krupnik show that $||H||_{p,p} = n_p$ when $p = 2^n \quad (n = 1, 2, \dots)$. Their argument uses induction, from the case $p = 2$, to the case $p = 4$, and so on doubling the exponents. The next key lemma and the following theorem are due to I.E. Verbitsky who sketched for us, a few years ago, how to extend the Gokhberg-Krupnik strategy to the case of G. We have:

Theorem 4.4

Let G *be the linear operator defined in* (4.6) *and let* J *be the operator defined by*

$$(Jf)(x) = \frac{1}{\pi^2}\sum_{k\neq 0}\frac{f(x-k)}{k^2}. \tag{4.8}$$

Then for any $f \in C_c^\infty(\mathbb{R})$ *we have*

$$(Gf(x))^2 = 2G(f \cdot Gf)(x) + J(f^2)(x) + (2f \cdot Jf)(x). \tag{4.9}$$

Proof. We compute explicitly (omitting "p.v." before the sums)

$$\pi^2\left[(Gf(x))^2 - 2G(f\cdot Gf)(x)\right] = \sum_{\substack{k\neq 0\\ h\neq 0}} \frac{f(x-k)f(x-h)}{kh} - 2G\left(f(x)\sum_{j\neq 0}\frac{f(x-j)}{j}\right)$$
$$= \sum_{\substack{k\neq 0\\ h\neq 0}} \frac{f(x-k)f(x-h)}{kh} - 2\sum_{\substack{i\neq 0\\ j\neq 0}}\frac{f(x-i)f(x-i-j)}{ij}$$
$$= \sum_{\substack{k\neq 0\\ h\neq 0\\ h\neq k}} f(x-k)f(x-h)\left(\frac{1}{kh} - \frac{2}{k(h-k)}\right)$$
$$+\sum_{k\neq 0}\frac{f(x-k)^2}{k^2} - 2\sum_{k\neq 0}\frac{f(x-k)f(x)}{k(0-k)}.$$

Using the definition (4.8), we have shown that

$$(Gf(x))^2 - 2G(f\cdot Gf)(x) = J(f^2)(x) + (2f\cdot Jf)(x)$$
$$-\frac{1}{\pi^2}\sum_{\substack{k\neq 0\\ h\neq 0\\ h\neq k}} f(x-k)f(x-h)\frac{h+k}{kh(h-k)},$$

but the p.v. of the third term on the r.h.s. of this identity is zero, because its generic term changes sign when we exchange h and k. □

Now we can state and prove the following partial answer to Conjecture 4.1.

Theorem 4.5

Let D be the Discrete Hilbert transform defined in (4.4). *We have*

$$||D||_{p,p} = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & if\ \ p = \frac{2^n}{2^n-1} \quad (n = 1, 2, \dots), \\ \cot\left(\frac{\pi}{2p}\right) & if\ \ p = 2^n \quad (n = 1, 2, \dots). \end{cases}$$

Proof. By duality it suffices to prove the result for $p = 2^n$. By Theorem 4.2 we can consider the operator G instead of D. By Lemma 4.4 and Minkowski's inequality

$$||Gf||_{2p}^2 = ||(Gf)^2||_p \le 2||G(f\cdot Gf)||_p + 2||f\cdot Jf||_p + ||J(f^2)||_p$$
$$\le\ 2||G||_{p,p}\left(\int |f|^p\ |Gf|^p\right)^{1/p} + 2\left(\int |f|^p\ |Jf|^p\right)^{1/p} + ||J||_{p,p}||f^2||_p$$
$$\le 2||G||_{p,p}\ ||f||_{2p}\ ||Gf||_{2p} + 2||f||_{2p}\ ||Jf||_{2p} + ||J||_{p,p}||f||_{2p}^2.$$

We have shown that

$$||Gf||_{2p}^2\ \le 2||G||_{p,p}||G||_{2p,2p}\ ||f||_{2p}^2 + 2||J||_{2p,2p}||f||_{2p}^2 + ||J||_{p,p}||f||_{2p}^2. \tag{4.10}$$

Now we observe that the operator J has a *positive* convolution kernel in $\ell^1(\mathbb{Z})$, therefore by Minkowski's inequality its norms $||J||_{p,p}$ are bounded by the constant

$$\frac{1}{\pi^2}\sum_{k\neq 0}\frac{1}{k^2} = 1/3.$$

Setting $d_p = ||G||_{p,p} = ||D||_{p,p}$, plugging the above estimate in (4.10) and using Theorem 4.3 to get a lower estimate for d_{2p} , we obtain that

$$\cot^2\frac{\pi}{4p} \leq d_{2p}^2 \leq 2\ d_p\ d_{2p} + 1. \tag{4.11}$$

We conclude by induction on $p = 2^n$. For $n = 1$ we have $d_2 = 1$, we then use (4.11) together with the trigonometric identity

$$\cot^2\frac{a}{2} = 2\cot\frac{a}{2}\cdot\cot a + 1,$$

which allows us to "feed" the l.h.s. inequality of (4.11) into the r.h.s., as we keep doubling the exponents p. □

Remarks. It is not needed in the above proof, but actually $||J||_{p,p} = 1/3$ for all $1 \leq p \leq \infty$, because translation invariant linear operators with positive and integrable convolution kernels have (p,p) norm equal to the integral of the kernel. Note that such a direct application of Minkowski's inequality fails in studying the (p,p) norm of operators like H_+ with a positive, *but non-integrable* convolution kernel. *A fortiori*, it fails for singular integrals like H, or singular sums like D.

It is not difficult to see that (4.11) implies that if $d_p = \cot\frac{\pi}{2p}$ for any given p, then $d_{2p} = \cot\frac{\pi}{4p}$. In particular, it would suffice to prove conjecture (4.1) for $2 \leq p \leq 4$, and (4.11) would "bootstrap" the result to all $p \in [2, +\infty)$, while duality would take care of the range $p \in (1, 2]$.

Riesz-Thorin interpolation, together with Theorem 4.3, gives us upper and lower bounds for $||D||_{p,p}$ when $2^n < p < 2^{n+1}$, namely

$$\cot\frac{\pi}{2p} \leq ||D||_{p,p} \leq \left(\cot\frac{\pi}{2^{n+1}}\right)^{1-t}\left(\cot\frac{\pi}{2^{n+2}}\right)^{t}, \tag{4.12}$$

where $\frac{1}{p} = \frac{t}{2^n} + \frac{1-t}{2^{n+1}}$. Note that the putative discrepancy implied by (4.12) gets bigger for large values of p between two large powers of 2. The following theorem compares the norms of D and the norms of the Truncated Hilbert transform giving a different kind of discrepancy bound, independent of p.

Theorem 4.6

Let D be the Discrete Hilbert transform defined in (4.4) *and let $H^{(1/2)}$ be the Truncated Hilbert transform* (3.9) *with $\epsilon = 1/2$. Then, for $1 < p < \infty$ we have*

$$||D||_{p,p} - ||H^{(1/2)}||_{p,p} \leq c,$$

where $c > 0$ is an absolute constant.

Proof. Given a sequence $b = \{b_n\}_{n\in\mathbb{Z}}$ let us define an extension operator E in the following way

$$E : b \in \ell^p(\mathbb{Z}) \to Eb \in L^p(\mathbb{R}) \quad \text{where} \quad Eb(x) = b_n \text{ if } |x - n| < 1/2. \tag{4.13}$$

Note that $||E||_{p,p} = 1$. Assuming that $t \in (-1/2, 1/2)$ the discrete Hilbert transform D applied to b, without the normalization constant $1/\pi$, is now equal to

$$\begin{aligned}\sum_{k\neq 0} \frac{b_{n-k}}{k} &= \sum_{k\neq 0} \frac{Eb(n-k+t)}{k} = \sum_{k\neq 0} \int_{-1/2}^{1/2} \frac{Eb(n-k+t)}{k}\, dt \\ &= \sum_{k\neq 0} \int_{-1/2}^{1/2} \frac{Eb(n-k+t)}{k-t}\, dt + \sum_{k\neq 0} \int_{-1/2}^{1/2} Eb(n-k+t)\left[\frac{1}{k} - \frac{1}{k-t}\right] dt \\ &= \int_{|y|\geq 1/2} \frac{Eb(n-y)}{y}\, dy + \sum_{k\neq 0} b_{n-k} \int_{-1/2}^{1/2} \left[\frac{1}{k} - \frac{1}{k-t}\right] dt.\end{aligned} \tag{4.14}$$

We have shown that $(Db)_n = (H^{(1/2)}Eb)_n + (Ab)_n$ where

$$(Ab)_n = \sum_{k\neq 0} b_{n-k} a_k,$$

and $a_k \sim c_1/k^2$ for $k \to \infty$, as is easily checked by evaluating the integral inside the last sum of (4.14). This implies that $||A||_{p,p} = c_2$ (both $c_1 > 0$ and $c_2 > 0$ are absolute constants, independent of p). We have

$$||Db||_p \leq ||H^{(1/2)}Eb||_p + ||A||_{p,p}||b||_p, \tag{4.15}$$

and it might be tempting to hastily deduce from this that

$$||D||_{p,p} \leq ||H^{(1/2)}||_{p,p} + c. \tag{4.16}$$

Such deduction is correct, but it requires some more work, because in (4.15) the term

$$||H^{(1/2)}Eb||_p = \left\{ \sum_n \left| \int_{|y|\geq 1/2} \frac{Eb(n-y)}{y}\, dy \right|^p \right\}^{1/p} \tag{4.17}$$

is a norm on the sequence space $\ell^p(\mathbb{Z})$, and we cannot say immediately that this quantity is bounded by $||H^{(1/2)}||_{L^p(\mathbb{R}),L^p(\mathbb{R})}||Eb||_{L^p(\mathbb{R})}$. On the other hand we have

$$\begin{aligned}||H^{(1/2)}Eb||_p &= \left\{ \sum_n \int_{-1/2}^{1/2} \left| \int_{|y|\geq 1/2} \frac{Eb(n-y)}{y}\, dy \right|^p dv \right\}^{1/p} \\ &\leq \left\{ \sum_n \int_{-1/2}^{1/2} \left[\sup_{|t|\leq 1} \left| \int_{|y|\geq 1/2} \frac{Eb(n+v+t-y)}{y}\, dy \right|^p \right] dv \right\}^{1/p} \\ &\leq \left\{ \int_{\mathbb{R}} \left[\sup_{|t|\leq 1} \left| \int_{|y|\geq 1/2} \frac{Eb(x+t-y)}{y}\, dy \right| \right]^p dx \right\}^{1/p}.\end{aligned} \tag{4.18}$$

We now define a new operator $\tilde{H}$, acting on functions on the real line, by

$$\tilde{H}f(x) = \sup_{|t|\leq 1} \left| \int_{|y|\geq 1/2} \frac{f(x+t-y)}{y}\, dy \right| = \sup_{|t|\leq 1} \left| H^{(1/2)}f(x+t) \right|. \tag{4.19}$$

We *claim* that $\tilde{H}$ satisfies, for $1 < p < \infty$ the uniform estimate

$$||(\tilde{H} - H^{(1/2)})f||_{L^p(\mathbb{R})} \leq c_3||f||_{L^p}, \tag{4.20}$$

with $c_3 > 0$ absolute constant. Assuming this claim, and using (4.18), we obtain

$$\begin{aligned} ||H^{(1/2)}Eb||_{\ell^p(\mathbb{Z})} &\leq \left\{ \int_{\mathbb{R}} |\tilde{H}Eb(x)|^p \, dx \right\}^{1/p} \\ &\leq ||H^{(1/2)}Eb||_{L^p(\mathbb{R})} + ||(\tilde{H} - H^{(1/2)})Eb||_{L^p(\mathbb{R})} \\ &\leq ||H^{(1/2)}||_{p,p}||Eb||_{L^p(\mathbb{R})} + ||(\tilde{H} - H^{(1/2)})||_{p,p}||Eb||_{L^p(\mathbb{R})} \\ &\leq ||H^{(1/2)}||_{p,p}||Eb||_{L^p(\mathbb{R})} + c_3||Eb||_{L^p(\mathbb{R})} \\ &= ||H^{(1/2)}||_{p,p}||b||_{l^p(\mathbb{Z})} + c_3||b||_{l^p(\mathbb{Z})}, \end{aligned}$$

and the desired norm inequality (4.16) follows plugging this last inequality into (4.15).

Having seen this, let us now prove our claim, i.e., inequality (4.20). To do that, we define another operator

$$H[t]f(x) = \int_{|y| \geq 1/2} \frac{f(x + t(x) - y)}{y} \, dy = H^{(1/2)}f(x + t(x)), \tag{4.21}$$

where $t = t(x)$ is a bounded function of $x \in \mathbb{R}$, such that $|t(x)| \leq 1$. We have

$$\begin{aligned} |\{H[t] - H^{(1/2)}\}f(x)| \leq & \int_{|y| \geq 2} |f(x - y)| \left| \frac{1}{y - t(x)} - \frac{1}{y} \right| dy \\ & + 4 \int_{|y| \leq 3} |f(x - y)| dy \leq \int_{\mathbb{R}} |f(x - y)| \, \frac{c_4}{1 + y^2} dy, \end{aligned} \tag{4.22}$$

where $c_4 > 0$ is another absolute constant. Since (4.22) is true for any bounded $|t(x)| \leq 1$, we can take suprema on the l.h.s. over all such t's. Remembering the definition (4.19) of $\tilde{H}$, observing that $\frac{c_4}{1+y^2}$ is a positive convolution kernel belonging to $L^1(\mathbb{R})$, and taking L^p norms of both sides of this last inequality, we obtain (4.20). □

5. Fourier multipliers periodized in frequency

Studying D and D_α, because of Theorem 4.2, we have been led to consider convolution operators defined by a weighted sum of translated functions

$$Mf(x) = \sum_{k \in \mathbb{Z}} a_k f(x - k), \tag{5.1}$$

where $\{a_k\}$ is a sequence of coefficients in $\ell^\infty(\mathbb{Z})$, while $f \in L^p(\mathbb{R})$. These kind of operators can also be seen as Fourier multipliers

$$Mf(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x \xi} d\xi, \tag{5.2}$$

and are characterized by the fact that $m(\xi)$ is 1-periodic, with Fourier series given by

$$m(\xi) = \sum_{k \in \mathbb{Z}} a_k e^{-2\pi i k \xi}. \tag{5.3}$$

It can be difficult, in general, to compute $||M||_{p,p} = |||m|||_p$ exactly, but the following proposition collects some simple general facts about these norms.

Proposition 5.1

Let M be the Fourier multiplier operator given in (5.1) and (5.2), associated with the sequence of coefficients $\{a_k\} \in \ell^\infty(\mathbb{Z})$. Then, for $1 \le p \le \infty$, we have

$$\left(\sum_{k\in\mathbb{Z}} |a_k|^p\right)^{1/p} \le |||m|||_p \le \sum_{k\in\mathbb{Z}} |a_k|. \tag{5.4}$$

We also have

$$|||m|||_2 = ||m||_{L^\infty([0,1])} = \left|\left|\sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k \cdot}\right|\right|_{L^\infty([0,1])}.$$

Proof. The second equality follows from Plancherel's theorem and the definition of a Fourier multiplier. The inequality on the r.h.s. of (5.4) follows from Minkowski's inequality applied to (5.1). The inequality on the l.h.s. of (5.4) is obtained applying M to a function f compactly supported in $[-\epsilon, \epsilon]$, with $\epsilon < 1/2$ and such that $\int_{\mathbb{R}} |f(x)|^p dx = 1$. □

Remarks. If only a finite number of coefficients $\{a_k\}$ are non-zero, and if these coefficients are non-negative (actually, it suffices $a_k = c\ b_k$ with $b_k \ge 0$ and c complex constant), equality holds on the r.h.s. of (5.4). Namely, we have

$$\left|\left|\sum_{k=-N}^{N} a_k f(\cdot - k)\right|\right|_{p,p} = \sum_{k=-N}^{N} |a_k|.$$

In particular, the operator norm does not depend on p in this case. Note that M applied to the functions $f(x) = \chi_{[-K,K]}(x)$, where $K > 0$ is chosen much larger than N (fixed), approaches the norm

$$\sum_{k=-N}^{N} |a_k|, \quad \text{as} \quad K \to +\infty.$$

Such an equality, in particular cases, can hold also when the coefficients have variable sign. An example is given by $Mf(x) = f(x-1) - f(x+1)$, which is, in some sense, a first-order approximation of the operator (4.6). In fact, applying this M to the truncated sinus function $\chi_{[-4K,4K]}(x)\sin(\frac{\pi\xi}{2})$ we see that, except for two "tails" which become negligible as $K \to +\infty$, we have $|Mf(x)| = 2|f(x)|$, and therefore the norm of M is equal to 2, for all $1 \le p \le \infty$.

When we apply (5.4) to the m corresponding to D, i.e., $a_k = \dfrac{\operatorname{sgn}(k)}{\pi\ k}$, the r.h.s. is a diverging harmonic series so that the upper estimate is empty. The lower estimate is

$$\frac{1}{\pi}\left(\sum_{k\neq 0}(1/|k|)^p\right)^{1/p} = \frac{1}{\pi}\,(2\zeta(p))^{1/p} \le |||m|||_p,$$

for $1 < p < \infty$, where we have denoted Riemann's zeta function by ζ.

A better lower estimate is obtained observing that the Fourier series (5.3) of the m corresponding to D is a sawtooth function, namely the 1-periodic function that coincides with $-i(1-2\xi)$ for $\xi \in (0,1)$. Also, the m corresponding to D_α is the 1-periodic function that coincides with $e^{-\pi i\alpha}e^{2\pi i\alpha\xi}$ for $\xi \in (0,1)$. In particular, when $\alpha = 1/2$ it coincides with $-ie^{\pi i\xi}$ for $\xi \in (0,1)$.

Note that the dilation limit of $m(\xi)$, both in the case of D and $D_{1/2}$, coincides with the function $-i\,\mathrm{sgn}\,(\xi)$. This gives us a second proof of Theorem 4.3, because of Fatou's Lemma, and because the multiplier function $-i\,\mathrm{sgn}\,(\xi)$ defines the Hilbert transform H.

If we try to evaluate our multiplier norms via approximations with trigonometric polynomials, we have to keep in mind the following:

Proposition 5.2

If the Fourier multiplier operator given in (5.1) and (5.2) is associated with a multiplier function $m(\xi)$ that has a jump discontinuity in some $\xi = \xi_0$, then the norms

$$\left|\left|\left| \sum_{k=-N}^{N} a_k e^{-2\pi i k\cdot} \right|\right|\right|_p$$

of the multiplier operators corresponding to the partial sums of the Fourier series of m, may fail to converge to $|||m|||_p$ as $N \to +\infty$, for $1 < p < \infty$. On the other hand, taking the Cesaro averaged partial sums of (5.3)*, we obtain that*

$$\lim_{N\to+\infty} \left|\left|\left| \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) a_k e^{-2\pi i k\cdot} \right|\right|\right|_p = |||m|||_p.$$

Proof. When $p = 2$ we have

$$\left|\left|\left| \sum_{k=-N}^{N} a_k e^{-2\pi i k\cdot} \right|\right|\right|_2 = \max_{\xi\in[0,1]} \sum_{k=-N}^{N} a_k e^{-2\pi i k\xi},$$

but because of Gibbs' phenomenon the trigonometric polynomials on the r.h.s. of the last equality "always overshoot the mark". If $\sup|m|$ corresponds to the amplitude of the jump in ξ_0 the limit of

$$\max_{\xi\in[0,1]} \sum_{k=-N}^{N} a_k e^{-2\pi i k\xi} \quad \text{as} \quad N \to +\infty,$$

will be $1.08949\ldots$ times the right value.

Taking the Cesaro averaged partial sums eliminates Gibbs' phenomenon and corresponds to a convolution with Fejér's kernel for Fourier Series, an operator with (p,p)-norm equal to 1 for all $1 < p < \infty$. This implies the positive part of our statement. □

There is another way to approximate our multiplier operators (5.2), periodized in frequency, based on the observation that m can be written as an infinite sum of "non-overlapping translations" of its restriction μ to any given interval of length 1. If we define, for example

$$\mu(\xi) = m(\xi)\ \chi_{[-1/2,1/2)}(\xi), \tag{5.5}$$

we have

$$m(\xi) = \sum_{n\in\mathbb{Z}} \mu(\xi + n), \tag{5.6}$$

and the following holds:

Theorem 5.3

Let m be a 1-periodic, $L^\infty(\mathbb{R})$-function and let us represent it as a sum of non-overlapping translations of μ, like in (5.6), with μ supported in $[-1/2, 1/2]$. We then have, for $1 < p < \infty$,

$$r\ |||m|||_p \le |||\mu|||_p \le n_p\ |||m|||_p, \tag{5.7}$$

where n_p is the expression (2.2) and where $0 < r < 1$ is some absolute constant (independent of p and m).

Proof. The r.h.s. inequality follows immediately from our main theorem in [5], which can be stated as $|||\chi_{[-1/2,1/2]}|||_p = n_p$. The l.h.s. inequality has been proven by M. Jodeit in [17]. □

Remarks. The r.h.s. inequality is sharp. The equal sign is attained when we choose $m(\xi) \equiv 1$. (Note the "smoothing effect" of the periodization in this example).

The exact value of r is not known. There are examples where $|||\mu|||_p < |||m|||_p$, and they seem to correspond to cases where μ, in some suitable sense, becomes "rougher" after periodization (one such example is the truncated parabola $\chi_{[-1/2,1/2]}(\xi)(1-4\xi^2)$).

The case of equality $|||\mu|||_p = |||m|||_p$ is also possible. In fact we pose the following:

Conjecture 5.4 Let μ and m be $L^\infty(\mathbb{R})$ functions, related as in (5.6) and corresponding to the Discrete Hilbert transform D. Then $|||\mu|||_p = |||m|||_p$ for all $1 < p < \infty$.

Note that, because of Theorem 3.1 we know that $|||\mu|||_p = n_p$ (one "tooth" of the sawtooth function m has the same norm of the Hilbert transform H). This observation, together with Theorem 5.3 imply that this conjecture is true at least for $p = 2^n$ $(n = 1, 2, \dots)$ and for their dual exponents.

Conjecture 5.5 Let μ and m be $L^\infty(\mathbb{R})$ functions, related as in (5.6) and corresponding to the Discrete Hilbert transform $D_{1/2}$. Then $|||\mu|||_p = |||m|||_p$ for all $1 < p < \infty$.

For a generic α the following holds:

Theorem 5.6

For $1 < p < \infty$ and $\alpha \in (0, 1)$ we have

$$||D_\alpha||_{p,p} \ge ||\cos \pi\alpha\ I + \sin \pi\alpha\ H||_{p,p},$$

where I is the identity operator and H is the Hilbert transform.

Proof. The Discrete Hilbert transform D_α corresponds to the the 1-periodic multiplier whose values for $\xi \in (0,1)$ are given by $m(\xi) = e^{-\pi i\alpha}e^{2\pi i\alpha\xi}$. Dilating this function away from $\xi = 0$ we obtain a step-function equal to $\cos\alpha\pi - i\sin\alpha\pi$ for $\xi > 0$ and equal to $\cos\alpha\pi + i\sin\alpha\pi$ for $\xi < 0$. This multiplier corresponds to $\cos\pi\alpha\, I + \sin\pi\alpha\, H$ and the theorem follows from Fatou's Lemma. □

Conjecture 5.7 For $1 < p < \infty$ and $\alpha \in (0,1)$ the norms $||D_\alpha||_{p,p}$ actually coincide with the norms $||\cos\pi\alpha\, I + \sin\pi\alpha\, H||_{p,p}$.

A "closed" expression for the above quantity is

$$||\cos\pi\alpha\, I + \sin\pi\alpha\, H||_{p,p} = \max_{t\in[0,2\pi]} \left\{ \frac{|\cos(t+\pi\alpha)|^p + |\cos(t+\pi(\alpha+\frac{1}{p}))|^p}{|\cos(t)|^p + |\cos(t+\frac{\pi}{p})|^p} \right\}^{1/p},$$

as it follows from one of the results in [15, page 240].

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