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Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and their secant varieties

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Abstract

In this paper we compute the dimension of all the s^{th} higher secant varieties of the Segre-Veronese embeddings $Y_{\underline{d}}$ of the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the projective space $\mathbb{P}^N$ via divisors of multi-degree $\underline{d} = (a, b, c)$ $(N = (a + 1)(b + 1)(c + 1) - 1)$. We find that $Y_{\underline{d}}$ has no deficient higher secant varieties, unless $\underline{d} = (2, 2, 2)$ and $s = 7$, or $\underline{d} = (2h, 1, 1)$ and $s = 2h + 1$, with defect 1 in both cases.

0. Introduction

In this paper we will consider embeddings of the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^{\underline{n}}$, $\underline{n} = (1, 1, 1)$, given by complete linear systems, i.e. by sheaves $\mathcal{O}_{\mathbb{P}^{\underline{n}}}(\underline{d})$, where $\underline{d} = (d_1, d_2, d_3)$. These

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embeddings of the product, in $\mathbb{P}^N$, $N = [\Pi_{i=1}^3(d_i+1)] - 1$, are called *Segre-Veronese embeddings* and will be denoted by $X_{\underline{n},\underline{d}}$.

We determine the dimension of **all** the higher secant varieties for such embeddings. It turns out that only the embeddings corresponding to $(d_1, d_2, d_3) = (2h, 1, 1)$ and $(d_1, d_2, d_3) = (2, 2, 2)$ have defective secant varieties (see Theorem 3.1).

It is worth noting that L. Chiantini and C. Ciliberto, in [9], have classified all defective threefolds. Thus, in theory, our Theorem 3.1 could be deduced from their Theorem 0.1. But, the way their classification is made makes the deduction, from it, of our Theorem 3.1, far from obvious. In any case, our methods of dealing with this question are different from theirs.

We use Terracini's Lemma (as in [4, 5, 6]) in order to translate the problem of determining the dimension of secant varieties into that of determining the value (at (d_1, d_2, d_3)) of the Hilbert function of the $\mathbb{N}^3$-graded homogeneous coordinate ring of generic sets of 2-fat points in $\mathbb{P}^{\underline{n}}$, $\underline{n} = (1, 1, 1)$. Then we show, by passing to an affine chart of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and then homogenizing (in order to pass to $\mathbb{P}^3$), that this amounts to computing the Hilbert function of very particular subschemes of $\mathbb{P}^3$ in a specific degree.

Problems concerning the higher secant varieties of Segre varieties have attracted the interest of researchers for over a century. There are also significant interactions of these problem with questions from different areas of mathematics; in fact such problems are strongly connected to questions in representation theory, coding theory and algebraic complexity theory (see our paper [4] for some recent results as well as a summary of known results, and also [2]) and even in algebraic statistics (e.g. see [12, 13]).

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§1. Preliminaries, the multiprojective-affine-projective method

Let us recall the notion of higher secant variety.

DEFINITION 1.1 Let $X \subseteq \mathbb{P}^N$ be a closed irreducible and non-degenerate projective variety of dimension n; the s^{th} *higher secant variety* of X, denoted by X^s (or sometimes $Sec_{s-1}(X)$), is the closure of the union of all linear spaces spanned by s independent points of X.

We shall be considering Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ almost exclusively in this paper. Recall these are the varieties obtained from the compositions

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{(\nu_{d_1}, \nu_{d_2}, \nu_{d_3})} \mathbb{P}^{d_1} \times \mathbb{P}^{d_2} \times \mathbb{P}^{d_3} \to \mathbb{P}^N$$

where the maps ν_d are the d-Veronese (or *d-uple*) embeddings of $\mathbb{P}^1$ and the last map is the usual Segre embedding of the product space. We denote by $Y_{\underline{d}} = X_{(1,1,1),(d_1,d_2,d_3)} \subset$

$\mathbb{P}^N$ the image of such a map, where $\underline{d} = (d_1, d_2, d_3)$ and $N = [\prod_{i=1}^{3}(d_i + 1)] - 1$. In this note we want to study the dimension of $Y^s_{\underline{d}}$.

There is an expected dimension for $Y^s_{\underline{d}}$. Since $\dim Y_{\underline{d}} = 3$, one expects that

$$\dim Y^s_{\underline{d}} = \min\{N, 3s + (s-1)\}.$$

When $Y^s_{\underline{d}}$ does not have the expected dimension, then $Y^s_{\underline{d}}$ is said to be $(s-1)$-*defective*, and the positive integer

$$\delta_{s-1} := \min\{N, 4s-1\} - \dim Y^s_{\underline{d}}$$

is called the $(s-1)$-*defect* of $Y_{\underline{d}}$.

Undoubtedly the most famous classical result about secant varieties is Terracini's Lemma (see [14]):

Lemma 1.2 (Terracini)

Let $(X, \mathcal{L})$ be a polarized, integral scheme and suppose $\mathcal{L}$ embeds X into $\mathbb{P}^N$, then:

$$T_P(X^s) = \langle T_{P_1}(X), \dots, T_{P_s}(X)\rangle,$$

where $P_1, \dots, P_s$ are s generic points on X, and P is a generic point of $\langle P_1, \dots, P_s\rangle$; here $T_{P_i}(X)$ is the projectivized tangent space of X in $\mathbb{P}^N$.

Let mP denote an ***m-fat point*** on X with support at the point P i.e. the scheme defined by the ideal sheaf $\mathcal{I}^m_P \subset \mathcal{O}_X$. Let $Z \subseteq X$ be a scheme of s generic 2-*fat points*, i.e. a scheme defined by the ideal sheaf $\mathcal{I}_Z = \mathcal{I}^2_{P_1} \cap \cdots \cap \mathcal{I}^2_{P_s} \subseteq \mathcal{O}_X$, where $P_1, \dots, P_s$ are s generic points of X. Then, since there is a bijection between hyperplanes of the space $\mathbb{P}^N$ containing the subspace $\langle T_{P_1}(X), \dots, T_{P_s}(X)\rangle$ and the elements of $H^0(X, \mathcal{I}_Z(\mathcal{L}))$, we have:

Corollary 1.3

With X, $\mathcal{L}$, Z as above; then

$$\dim X^s = \dim\langle T_{P_1}(X), \dots, T_{P_s}(X)\rangle = N - \dim H^0(X, \mathcal{I}_Z(\mathcal{L})).$$

If $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $Y_{\underline{d}} \subseteq \mathbb{P}^N$ is the embedding of X given by $\mathcal{L} = \mathcal{O}_X(d_1, d_2, d_3)$ then applying the corollary above to our case, we get

$$\dim Y^s_{\underline{d}} = H(Z, \underline{d}) - 1,$$

where $Z \subseteq X$ is a set of s generic 2-fat points, and where $\forall \underline{j} \in \mathbb{N}^3$, $H(Z, \underline{j})$ is the $\mathbb{N}^3$-graded Hilbert function of Z. I.e.,

$$\dim Y^s_{\underline{d}} + 1 = H(Z, \underline{d}) = \dim R_{\underline{d}} - \dim H^0(X, \mathcal{I}_Z(\underline{d})),$$

where $R = k[x_{0,1}, x_{1,1}, x_{0,2}, x_{1,2}, x_{0,3}, x_{1,3}]$ is the multihomogeneous coordinate ring of $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Now consider the birational map

$$g : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 - -- \to \mathbb{A}^3,$$

where:

$$((x_{0,1},x_{1,1}),(x_{0,2},x_{1,2}),(x_{0,3},x_{1,3})) \longmapsto \left(\frac{x_{1,1}}{x_{0,1}},\frac{x_{1,2}}{x_{0,2}},\frac{x_{1,3}}{x_{0,3}}\right),$$

which is defined in the open subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by $\{x_{0,1}x_{0,2}x_{0,3} \neq 0\}$.

Let $S = k[z_0, z_1, z_2, z_3]$ be the coordinate ring of $\mathbb{P}^3$ and consider the embedding $\mathbb{A}^3 \to \mathbb{P}^3$ whose image is the chart $\mathbb{A}^3_0 = \{z_0 \neq 0\}$. By composing the two maps above we get:

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 - - - \to \mathbb{P}^3,$$

with

$$((x_{0,1},x_{1,1}),(x_{0,2},x_{1,2}),(x_{0,3},x_{1,3})) \longmapsto \left(1,\frac{x_{1,1}}{x_{0,1}},\frac{x_{1,2}}{x_{0,2}},\frac{x_{1,3}}{x_{0,3}}\right).$$

If $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is any zero-dimensional scheme which is contained in the affine chart $\{x_{0,1}x_{0,2}x_{0,3} \neq 0\}$ then, if we let A_0, A_1, A_2, A_3 be the coordinate points of $\mathbb{P}^3$ we obtain:

Theorem 1.4

Let f, Z, A_i be as above, $\underline{d} = (d_1, d_2, d_3)$ and $d = d_1 + d_2 + d_3$. Let $W = (d - d_1)A_1 + (d - d_2)A_2 + (d - d_3)A_3 + f(Z) \subseteq \mathbb{P}^3$. Then we have:

$$\dim(I_Z)_{(d_1,d_2,d_3)} = \dim(I_W)_d.$$

Proof. See [5, Theorem 1.5] for the proof in a more general setting; (see also [11]). □

When Z is given by s generic 2-fat points, we have the obvious corollary:

Corollary 1.5

Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a generic set of s 2-fat points, let $\underline{d} = (d_1, d_2, d_3) \in \mathbb{N}^3$, $d = d_1 + d_2 + d_3$ and $a_j = \sum_{i \neq j} d_i$.

If we set $W = a_1A_1 + a_2A_2 + a_3A_3 + 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^3$ (as in Theorem 1.4), then we have:

$$\dim Y^s_{\underline{d}} = H(Z, (d_1, d_2, d_3)) - 1 = \Pi^3_{i=1}(d_i + 1) - 1 - \dim(I_W)_d.$$

Now we give some preliminary lemmata and observations.

Since we will make use of Castelnuovo's inequality and of J. Alexander and A. Hirschowitz's *Lemme d'Horace différentiel* several times in the next sections, we recall them here in a form more suited to our use (for notation and proofs we refer to [1, Section 2 and Corollary 9.3]).

Lemma 1.6 (Castelnuovo's inequality)

Let $\mathcal{D} \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d, and let $Z \subseteq \mathbb{P}^n$ be a zero-dimensional scheme. The scheme Z' defined by the ideal $(I_Z : I_{\mathcal{D}})$ is called the **residual of Z with respect to $\mathcal{D}$**, *and denoted by $\mathrm{Res}_{\mathcal{D}}Z$; the schematic intersection $T = Z \cap \mathcal{D}$ is called the* **trace of Z on $\mathcal{D}$**, *and denoted by $Tr_{\mathcal{D}}Z$. Then for $t \geq d$*

$$\dim(I_{Z,\mathbb{P}^n})_t \leq \dim(I_{Z',\mathbb{P}^n})_{t-d} + \dim(I_{T,\mathcal{D}})_t.$$

Lemma 1.7 (Lemme d'Horace différentiel)

Let $H \subseteq \mathbb{P}^n$ *be a hyperplane,* $P_1, \ldots, P_r$ *generic points in* $\mathbb{P}^n$*, and* $\widetilde{Z}$ *be a zero-dimensional scheme. Let* $Z = \widetilde{Z} + 2P_1 + \cdots + 2P_r \subseteq \mathbb{P}^n$*,* $\widetilde{Z'} = \mathrm{Res}_H \widetilde{Z}$*, and* $\widetilde{T} = \mathrm{Tr}_H \widetilde{Z}$*.*

Let $P'_1, \ldots, P'_r$ *be generic points in* H*. Let* $D_{2,H}(P'_i) = 2P'_i \cap H$*, and* $Z' = \widetilde{Z'} + D_{2,H}(P'_1) + \cdots + D_{2,H}(P'_r)$*,* $T = \widetilde{T} + P'_1 + \cdots + P'_r$*.*

Then $\dim(I_Z)_t = 0$ *if the following two conditions are satisfied:*

Degue $\quad \dim(I_{Z'})_{t-1} = \dim(I_{\widetilde{Z'} + D_{2,H}(P'_1) + \cdots + D_{2,H}(P'_r)})_{t-1} = 0;$

Dime $\quad \dim(I_T)_t = \dim(I_{\widetilde{T} + P'_1 + \cdots + P'_r})_t = 0.$

The following remark is quite immediate.

Remark 1.8 Let Z, $Z' \subseteq \mathbb{P}^n$, be zero-dimensional schemes such that $Z' \subseteq Z$. Then

i) if Z imposes independent conditions to the hypersurfaces of I_t, then so also does the smaller scheme Z';

ii) if $\dim(I_{Z'})_t = 0$, then the bigger scheme Z has $\dim(I_Z)_t = 0$;

iii) if in the support of Z there is a generic point P, not lying in the support of Z', and $\dim(I_{Z'})_t = 1$, then the bigger scheme Z has $\dim(I_Z)_t = 0$.

The following simple lemma gives a criterion for adding, to a zero-dimensional scheme $Z \subseteq \mathbb{P}^n$, a set of reduced points which lie on a smooth hypersurface $\mathcal{D} \subseteq \mathbb{P}^n$ and which impose independent conditions to forms of a given degree in the ideal of Z (see also [7, Lemma 4]).

Lemma 1.9

Let $Z \subseteq \mathbb{P}^n$ *be a zero dimensional scheme. Let* $\mathcal{D} \subseteq \mathbb{P}^n$ *be a smooth hypersurface of degree* d *and* $P_1, \ldots, P_s$ *be generic points on* D*; let* $Z' = \mathrm{Res}_{\mathcal{D}} Z$*.*

i) If $\dim(I_{Z+P_1+\cdots+P_{s-1}})_t > \dim(I_{Z'})_{t-d}$*, then* $\dim(I_{Z+P_1+\cdots+P_s})_t = \dim(I_Z)_t - s$*;*

ii) if $\dim(I_{Z'})_{t-d} = 0$ *and* $\dim(I_Z)_t \le s$*, then* $\dim(I_{Z+P_1+\cdots+P_s})_t = 0$*.*

Proof. *i)* By induction on s. If $s = 1$, and $\dim(I_Z)_t > \dim(I_{Z'})_{t-d}$, then there is a hypersurface in $(I_Z)_t$ not containing $\mathcal{D}$, hence P_1 imposes one condition to the hypersurfaces of $(I_Z)_t$ and we get $\dim(I_{Z+P_1})_t = \dim(I_Z)_t - 1$.

Now let $s > 1$. Since $\dim(I_{Z+P_1+\cdots+P_{s-1}})_t > \dim(I_{Z'})_{t-d}$, there is a hypersurface in $(I_{Z+P_1+\cdots+P_{s-1}})_t$ not containing $\mathcal{D}$, hence $\dim(I_{Z+P_1+\cdots+P_{s-1}+P_s})_t = \dim(I_{Z+P_1+\cdots+P_{s-1}})_t - 1$. So, by the induction hypothesis, we get

$$\dim(I_{Z+P_1+\cdots+P_{s-1}+P_s})_t = (\dim(I_Z)_t - (s-1)) - 1 = \dim(I_Z)_t - s.$$

ii) Obvious if $\dim(I_Z)_t = 0$. If $\dim(I_Z)_t = v > 0$, then $\dim(I_{Z+P_1+\cdots+P_{v-1}})_t > 0 = \dim(I_{Z'})_{t-d}$. So by *i)* we get $\dim(I_{Z+P_1+\cdots+P_v})_t = \dim(I_Z)_t - v = 0$, and since $s \ge v$ it follows that $\dim(I_{Z+P_1+\cdots+P_s})_t = 0$. $\square$

2. Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ and fat point schemes in $\mathbb{P}^2$

Before we begin our investigations of the dimensions of the higher secant varieties of the Segre-Veronese varieties $Y_{\underline{d}}$, we need to recall some of our results about fat point schemes in $\mathbb{P}^2$ and also establish another result about such schemes. Our results about fat point schemes in $\mathbb{P}^2$ are related to the study of the higher secant varieties of the Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ and can be found in [5]. We recall our main results from [5] (see [5, Theorem 2.7 and Corollary 2.3]).

Theorem 2.1

Let $\{A, B, P_1, \ldots, P_s\}$ be a generic set of $s+2$ points in $\mathbb{P}^2$ and let W be the subscheme of $\mathbb{P}^2$

$$W = d_1A + d_2B + 2P_1 + \cdots + 2P_s.$$

Suppose that $d_1 \geq d_2 \geq 2$, and define (given a positive integer t) the integers

$$r = \max\{d_1 + d_2 - t; 0\}, \quad \text{and} \quad \tilde{r} = \max\{d_1 + 2 - t; 0\}.$$

Then

$$\text{i)} \qquad H(W,t) = \begin{cases} \binom{t+2}{2} & \text{for } t < d_1 \\ \min\left\{\binom{t+2}{2};\ \deg W - \binom{r}{2} - s\binom{\tilde{r}}{2}\right\} & \text{for } t \geq d_1 \end{cases}$$

except for

$$t = d_1 + 2, \quad d_1 - d_2 \text{ even }, \quad s = d_1 - d_2 + 3,$$

in which case $H(W,t) = \binom{t+2}{2} - 1$;

ii) the Hilbert function of W is **NOT** *maximal $\Leftrightarrow$ either*

$$s < \max\left\{\frac{d_1 - d_2 + 1}{2}, \frac{(d_1-1)(d_2-1)}{3}\right\} = \begin{cases} \frac{d_1-d_2+1}{2} & \text{for } d_2 = 2 \\ \frac{(d_1-1)(d_2-1)}{3} & \text{for } d_2 > 2, \end{cases}$$

or $s = d_1 - d_2 + 3$ and $d_1 - d_2$ is even.

Proposition 2.2

Let d_1, d_2, h, s be positive integers. As in Theorem 2.1, let

$$W = d_1A + d_2B + 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^2.$$

Then the Hilbert function of W in degree $d_1 + d_2$ is maximal, that is

$$H(W, d_1 + d_2) = \min\left\{\binom{d_1 + d_2 + 2}{2}, \binom{d_1+1}{2} + \binom{d_2+1}{2} + 3s\right\},$$

and

$$\dim(I_W)_{(d_1+d_2)} = \max\{0; (d_1+1)(d_2+1) - 3s\}$$

except when,

$$(d_1, d_2) \in \{(2h, 2), (2, 2h)\} \text{ and } s = 2h+1$$

in this case $H(W, d_1 + d_2)$ is 1 less than expected, and we have $\dim(I_W)_{(d_1+d_2)} = 1$.

We can use the results above to show the following, which will be helpful in the proof of our main theorem.

Proposition 2.3

Let a, b, c be positive integers. Let $W = aA + bB + cC + 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^2$, where $A, B, C, P_1, \ldots, P_s$ are generic points in $\mathbb{P}^2$ ($s \geq 0$). Then the Hilbert function of W is maximal in degree $a+b+c$, that is

i) $H(W, a+b+c) = \min\{\binom{a+b+c+2}{2}, \binom{a+1}{2} + \binom{b+1}{2} + \binom{c+1}{2} + 3s\}$; *equivalently,*

ii) $\dim(I_W)_{a+b+c} = \max\{0; (a+1)(b+1)(c+1) - abc - 3s\}$.

Proof. The equivalence of $i)$ and $ii)$ follows from the equality

$$\binom{a+b+c+2}{2} - \binom{a+1}{2} - \binom{b+1}{2} - \binom{c+1}{2} = (a+1)(b+1)(c+1) - abc.$$

We prove $ii)$.

Let

$$e = \left\lfloor \frac{(a+1)(b+1)(c+1) - abc}{3} \right\rfloor, \qquad r = (a+1)(b+1)(c+1) - abc - 3e,$$
$$e^* = \left\lceil \frac{(a+1)(b+1)(c+1) - abc}{3} \right\rceil.$$

We have to prove that $\dim(I_W)_{a+b+c} = \max\{0; r + 3(e-s)\}$, that is

$$\dim(I_W)_{a+b+c} = \begin{cases} r + 3(e-s) & \text{for } s \leq e \\ 0 & \text{for } s \geq e^*. \end{cases}$$

By Remark 1.8 it suffices to prove that

$$\dim(I_W)_{a+b+c} = \begin{cases} r & \text{for } s = e \\ 0 & \text{for } s = e^* \text{ and } r = 2. \end{cases}$$

We may assume $a \geq b \geq c$. When $b = c = 1$ the statement follows from Theorem 2.1.

We proceed by induction on $a+b+c$, noting that the initial steps of the induction are covered by the previous case.

If $a \geq b \geq 2$, and $c = 1$, we can apply a quadratic transformation centered on A, B, and P_1, which gives us a scheme

$$W' = a'A' + b'B' + c'C' + 2P'_1 + \cdots + 2P'_{s-2} + P'_{s-1} + P'_s,$$

where $a' = a-1$, $b' = b-1$, $c' = 2$. Since for $W' - \{P'_{s-1} + P'_s\}$ we can use the induction hypothesis, it is easy to check that the statement holds for W.

Let $a \geq b \geq c \geq 2$, and let $\Gamma \subseteq \mathbb{P}^2$ be a smooth cubic curve. We specialize A, B, C and $a+b+c$ of the P_i's, to generic points of Γ (notice that we can do that, since $a \geq b \geq c \geq 2$ implies $e^* \geq e \geq a+b+c$).

Let

$$Z = \mathrm{Res}_\Gamma W = (a-1)A+(b-1)B+(c-1)C+P_1+\cdots+P_{a+b+c}+2P_{a+b+c+1}+\cdots+2P_s,$$

and consider the exact sequence of ideal sheaves:

$$0 \to \mathcal{I}_Z(a+b+c-3) \to \mathcal{I}_W(a+b+c) \to \mathcal{I}_{W\cap\Gamma,\Gamma}(a+b+c) \to 0.$$

We have

$$\begin{aligned} &H^0(\mathbb{P}^2, \mathcal{I}_{W\cap\Gamma,\Gamma}(a+b+c)) \\ &= H^0(\Gamma, \mathcal{O}_\Gamma((a+b+c)H - aA - bB - cC - 2P_1 - \cdots - 2P_{a+b+c}) = 0, \end{aligned}$$

since $(a+b+c)H - aA - bB - cC - 2P_1 - \cdots - 2P_{a+b+c}$ is a divisor of degree 0 on Γ (which is not a canonical divisor since the points are generic on Γ - here H is a line section of Γ). Hence $\dim(I_W)_{a+b+c} = \dim(I_Z)_{a+b+c-3}$.

Now consider

$$\begin{aligned} \check{Z} &= Z - (P_1 + \cdots + P_{a+b+c}) \\ &= (a-1)A + (b-1)B + (c-1)C + 2P_{a+b+c+1} + \cdots + 2P_s. \end{aligned}$$

By induction, $\check{Z}$ has maximal Hilbert function in degree $a+b+c-3$, so we easily get

$$\dim(I_{\check{Z}})_{a+b+c-3} = \begin{cases} a+b+c+r & \text{for} \quad s=e \\ a+b+c-1 & \text{for} \quad s=e^* \quad \text{and} \quad r=2. \end{cases} \tag{$*$}$$

We are done if we can prove that d generic points $\{P_1, \ldots, P_d\}$ lying on Γ impose independent conditions to the curves of $(I_{\check{Z}})_{a+b+c-3}$ (where $d = a+b+c$ for $s = e$, while $d = a+b+c-1$, for $s = e^*$ and $r = 2$) or, equivalently, that the curves defined by the forms of $(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3}$ do not have Γ as a fixed component (see Lemma 1.9).

Since

$$\begin{aligned} &\check{Z} + P_1 + \cdots + P_{d-1} \\ &\quad = (a-1)A + (b-1)B + (c-1)C + P_1 + \cdots + P_{d-1} + 2P_{a+b+c+1} + \cdots + 2P_s, \end{aligned}$$

we have to prove that $\dim(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3} > \dim(I_{Z'})_{a+b+c-6}$, where

$$\begin{aligned} Z' = (a-2)A + (b-2)B + (c-2)C + 2P_{a+b+c+1} \\ + \cdots + 2P_s = \mathrm{Res}_\Gamma(\check{Z} + P_1 + \cdots + P_{d-1}). \end{aligned}$$

By (*) we get

$$\dim(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3} \geq \begin{cases} 1+r & \text{for} \quad s=e \\ 1 & \text{for} \quad s=e^* \quad \text{and} \quad r=2. \end{cases}$$

For $a=b=c=2$, we have $e=6$, $r=1$, $Z'=\emptyset$, hence

$$\dim(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3} \geq 2 > \dim(I_{Z'})_0 = 1.$$

For $a > b = c = 2$, the curves of $(I_{Z'})_{a+b+c-6}$ are formed by $a-2$ lines through A, in fact we have

$$Z' = (a-2)A + 2P_{a+5} + \cdots + 2P_s,$$

and hence $\dim(I_{Z'})_{a-2} = \max\{0; a-2+1-2(s-(a+5)+1)\} = \max\{0; 3a+7-2s\}$, and we easily get

$$\dim(I_{Z'})_{a-2} = \begin{cases} 1 & \text{for} \quad a=4 \quad \text{and} \quad s=e \\ 0 & \text{for} \quad a\neq 4 \quad \text{and} \quad s=e \\ 0 & \text{for} \quad s=e^* \quad \text{and} \quad r=2. \end{cases}$$

Since for $a=4$, we have $r=2$, then for any $a > b = c = 2$ we get

$$\dim(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3} > \dim(I_{Z'})_{a+b+c-6}.$$

For $a \geq b > c = 2$,

$$Z' = (a-2)A + (b-2)B + 2P_{a+b+3} + \cdots + 2P_s,$$

hence by Proposition 2.2 we get

$$\dim(I_{Z'})_{a+b-4} = \max\{0; (a-1)(b-1) - 3(s-(a+b+3)+1)\},$$

except when $a-2$ is even, $b-2=2$ and $s-(a+b+3)+1 = a-1$.

Since it is easy to check that $s-(a+b+3)+1 > a-1$, by direct computations we get

$$\dim(I_{Z'})_{a+b-4} = \begin{cases} \max\{0; 4+r-a-b\} = 0 & \text{for} \quad s=e \\ \max\{0; 3-a-b\} = 0 & \text{for} \quad s=e^* \quad \text{and} \quad r=2\,, \end{cases}$$

and hence $\dim(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3} > \dim(I_{Z'})_{a+b+c-6}$.

Finally, for $a \geq b \geq c \geq 3$, using the induction hypothesis (and a simple computation) we get that

$$\dim(I_{Z'})_{a+b+c-6} = \begin{cases} \max\{0; 6-(a+b+c)+r\} = 0 & \text{for} \quad s=e \\ \max\{0; 6-(a+b+c)-1\} = 0 & \text{for} \quad s=e^* \quad \text{and} \quad r=2, \end{cases}$$

and hence we again have $\dim(I_{\check{Z}+P_1+\cdots+P_{d-1}})_{a+b+c-3} > \dim(I_{Z'})_{a+b+c-6}$. $\square$

3. Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

We are now ready to consider the case $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and its Segre-Veronese embeddings, that is the varieties $Y_{\underline{d}} = X_{(1,1,1),(d_1,d_2,d_3)}$ where $\underline{d} = (d_1,d_2,d_3) = (a,b,c)$. Our main result is the following:

Theorem 3.1

Let a, b, c, h, s be positive integers. Let $Y_{\underline{d}} = X_{(1,1,1),(a,b,c)}$. Then $Y_{\underline{d}}^s$ has the expected dimension, except for

$$(a, b, c) = (2, 2, 2), \text{ and } \; s = 7\,, \tag{3.1}$$

and, modulo permutations,

$$(a, b, c) = (2h, 1, 1), \text{ and } \; s = 2h + 1. \tag{3.2}$$

In those cases $Y_{\underline{d}}^s$ is defective, with $\delta_{s-1} = 1$ in both cases. I.e.,

$$\dim Y_{\underline{d}}^s = \begin{cases} 25 & \text{in case (3.1)} \\ 4s-2 & \text{in case (3.2).} \end{cases}$$

In order to compute the dimension of $Y_{\underline{d}}^s$, we will study the scheme of fat points

$$W_s = (b+c)A + (a+c)B + (a+b)C + 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^3,$$

where A, B, C, $P_1, \dots, P_s$ are generic points. We will prove Theorem 3.1 in its equivalent formulation (see Corollary 1.5):

Theorem 3.1*

Let a, b, c, h, s be positive integers and let

$$W_s = (b+c)A + (a+c)B + (a+b)C + 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^3,$$

be a scheme of generic fat points. Let

$$N = (a+1)(b+1)(c+1) - 1$$
$$e = \left\lfloor \frac{(a+1)(b+1)(c+1)}{4} \right\rfloor \quad e^* = \left\lceil \frac{(a+1)(b+1)(c+1)}{4} \right\rceil.$$

Then:

$$\dim(I_{W_s})_{a+b+c} = \begin{cases} N+1-4s & \text{for } \; s \le e \\ 0 & \text{for } \; s \ge e^*, \end{cases}$$

except in the following cases:

$$(a, b, c) = (2, 2, 2) \quad \text{and} \quad s = e^* = 7, \tag{3.1*}$$

and, modulo permutations,

$$(a, b, c) = (2h, 1, 1) \quad \text{and} \quad s = e = e^* = 2h + 1, \tag{3.2*}$$

where $\dim(I_{W_s})_{a+b+c} = 1$ instead of 0.

Before proving the theorem, we give some preliminary lemmata and observations. By Theorem 1.4 and Remark 1.8 we get the following:

Remark 3.2 Notation as in the statement of the theorem; let

$$W = (b+c)A + (a+c)B + (a+b)C \subseteq \mathbb{P}^3.$$

Then:

i) $\dim(I_W)_{a+b+c} = (a+1)(b+1)(c+1) = N+1$, and, for $s \le e$, $\dim(I_{W_s})_{a+b+c} \ge N+1-4s$;

ii) if $\dim(I_{W_s})_{a+b+c} = N+1-4s$, then $\dim(I_{W_t})_{a+b+c} = N+1-4t$, for any $1 \le t < s$;

iii) if $\dim(I_{W_s})_{a+b+c} = 0$, then $\dim(I_{W_t})_{a+b+c} = 0$, for any $t > s$;

iv) if $\dim(I_{W_s})_{a+b+c} = 1$, then $\dim(I_{W_{s+1}})_{a+b+c} = 0$.

Notice that $i)$ is immediate from Theorem 1.4 (putting $Z = \emptyset$). The other parts are also clear after recalling Remark 1.8.

Lemma 3.3

Let $\mathcal{Q}$ be a smooth quadric in $\mathbb{P}^3$; let A, B, C, P_i $(i = 1,\ldots,s)$ be generic points lying on $\mathcal{Q}$, and A', B', C',P_i' $(i = 1,\ldots,s)$ be generic points in $\mathbb{P}^2$.

Consider the following subscheme of $\mathcal{Q}$:

$$T = \Big((b+c)A + (a+c)B + (a+b)C + \sum_{i=1}^{s} m_i P_i\Big) \cap \mathcal{Q} \subseteq \mathbb{P}^3$$

and the following scheme of fat points in $\mathbb{P}^2$:

$$T' = aA' + bB' + cC' + \sum_{i=1}^{s} m_i P_i' \subseteq \mathbb{P}^2.$$

Then

$$\dim(I_{T,\mathcal{Q}})_{a+b+c} = \dim(I_{T',\mathbb{P}^2})_{a+b+c}.$$

Proof. Let $\phi(T)$ be the image of T in the isomorphism $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then, by the analogue to Theorem 1.4 for the case $\mathbb{P}^1 \times \mathbb{P}^1$ (see e.g. [5, Theorem 1.5]) we have that

$$\dim(I_{T,\mathcal{Q}})_{a+b+c} = \dim(I_{\phi(T),\mathbb{P}^1\times\mathbb{P}^1})_{(a+b+c,a+b+c)} = \dim(I_{T^*,\mathbb{P}^2})_{2(a+b+c)},$$

where

$$T^* = (a+b+c)D_1 + (a+b+c)D_2 + (a+b)D_3 + (a+c)D_4 + (b+c)D_5 + \sum_{i=1}^{s} m_i E_i$$

and the $D_i{}'s$ and the $E_i{}'s$ are generic points in $\mathbb{P}^2$. By applying two quadratic transformations, we get

$$\dim(I_{T^*,\mathbb{P}^2})_{2(a+b+c)} = \dim(I_{T^{**},\mathbb{P}^2})_{a+b+2c} = \dim(I_{T',\mathbb{P}^2})_{a+b+c},$$

where

$$T^{**} = cF_1 + cF_2 + (a+c)F_3 + (b+c)F_4 + \sum_{i=1}^{s} m_i G_i$$

and the $F_i{}'s$ and $G_i{}'s$ are generic points in $\mathbb{P}^2$. □

Lemma 3.4

Let W be as in Remark 3.2, i.e.:

$$W = (b+c)A + (a+c)B + (a+b)C \subseteq \mathbb{P}^3.$$

Let H be a plane through B and C, and let $\mathcal{Q}$ be a smooth quadric through A, B, C. Then
i) the line BC is a fixed component, of multiplicity a, for the curves of degree $a+b+c$ lying on H and passing through $\mathrm{Tr}_H W = W \cap H$;
ii) the surfaces of degree $a+b+c-2$ through $\mathrm{Res}_{\mathcal{Q}} W$ contain the plane ABC as a fixed component.

Proof. i) Obvious.

ii) Since $\mathrm{Res}_{\mathcal{Q}} W = (b+c-1)A + (a+c-1)B + (a+b-1)C \subseteq \mathbb{P}^3$ we obtain, after an easy computation, that the plane ABC is a fixed component. □

Since $\dim(I_W)_{a+b+c} = N+1$ (W as in Lemma 3.4), Theorem 3.1* affirms that, except in case (3.2*), e generic double points impose independent conditions to the surfaces of $(I_W)_{a+b+c}$, (and hence so do $s \le e$ double points), and that, except in cases (3.1*) and (3.2*), there are no surfaces in $(I_{W_s})_{a+b+c}$ through e^* double points. In other words, the theorem asserts that, except for "few" triples (a,b,c), $\dim(I_{W_s})_{a+b+c}$ is as expected.

Method of proof

In order to prove that the ideal of the scheme

$$W_s = (b+c)A + (a+c)B + (a+b)C + 2P_1 + \cdots + 2P_s$$

has dimension r in degree $t = a+b+c$, usually we will proceed as follows:
I) we show that

$$\dim(I_{W_s})_t \ge r;$$

II) we add r generic simple points to W_s and get a scheme $\widetilde{W}_s$ such that

$$\dim(I_{\widetilde{W}_s})_t = 0 \Longleftrightarrow \dim(I_{W_s})_t \le r;$$

III) we specialize $\widetilde{W}_s$ to a scheme Z, so that

$$\dim(I_Z)_t = 0 \Longrightarrow \dim(I_{\widetilde{W}_s})_t = 0;$$

IV) we "cut" Z by a suitable divisor D of degree d, (a plane H through the points B, C, or a smooth quadric $\mathcal{Q}$ through A, B, C) and we obtain two schemes, $Z' \subseteq \mathbb{P}^3$ and $T \subseteq D$, so that either by applying Lemma 1.6 or Lemma 1.7 we get

$$\dim(I_{Z'})_{t-d} = \dim(I_T)_t = 0 \Longrightarrow \dim(I_Z)_t = 0.$$

V) In case $D = H$, to compute $\dim(I_{Z'})_{t-1}$ and $\dim(I_T)_t$, we introduce three further schemes, $\check{Z}$, Z'' and T' as follows: let $\check{Z}$ be the scheme obtained from Z' by taking away the simple points lying on H; let L be the line BC and set

$$Z'' = \mathrm{Res}_H \check{Z} \subseteq \mathbb{P}^3 \qquad\qquad T' = \mathrm{Res}_{aL} T \subseteq H.$$

By the induction hypothesis we get that $\dim(I_{\check{Z}})_{t-1}$ is the expected one, and $\dim(I_{Z''})_{t-2} = 0$. So by Lemma 1.9 we have $\dim(I_{Z'})_{t-1} = 0$.

By Lemma 3.4 *i)* and Proposition 2.2 we have $\dim(I_T)_t = \dim(I_{T'})_{t-a} = 0$.

VI) In case $D = \mathcal{Q}$, in order to prove that $\dim(I_{Z'})_{t-2} = \dim(I_T)_t = 0$, we introduce the following schemes:

$$Z'' = \mathrm{Res}_\Pi(Z') \subseteq \mathbb{P}^3 \qquad\qquad Z''' = \mathrm{Res}_\Pi(\mathrm{Res}_\mathcal{Q}(Z'')) \subseteq \mathbb{P}^3,$$

where Π is the plane ABC. Let $\check{Z}$ be the scheme obtained from Z'' by taking away the simple points lying on $\mathcal{Q}$. By Lemma 3.4 *ii)* we get $\dim(I_{Z''})_{t-3} = \dim(I_{Z'})_{t-2}$; by the induction hypothesis we have that $\dim(I_{\check{Z}})_{t-3}$ is the expected one and $\dim(I_{Z'''})_{t-6} = 0$, then, by Lemma 1.9, we get that $\dim(I_{Z''})_{t-3} = 0$. Finally we prove that $\dim(I_T)_t = \dim(I_{T'})_t = 0$, where $T' \subseteq \mathbb{P}^2$ is the scheme associated to T by Lemma 3.3.

Remark 3.5 Before beginning the proof of Theorem 3.1* we observe that if $\min\{a, b, c\} = 1$ then the assertion of the theorem follows from [10, Theorem 5.1 (3)] together with Terracini's Lemma on Grassmann defectivity (see [15] or [10, Theorem 2.3]), and the fact that the Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ via divisors of multi-degree $(a, 1)$ are rational normal scrolls $S(a, a)$. Although our methods can be used to handle this case we will skip the proof.

Proof of Theorem 3.1:* We split the proof into five steps. In the first two steps we deal with the exceptional cases (3.1*) and (3.2*). Then we prove the theorem by induction on $a + b + c$.

Step 1: $(a, b, c) = (2, 2, 2)$ (see also [9, pg. 134, Case (3)a]).

By Remark 3.2 *ii*), *iii*), *iv*) it suffices to prove that

$$\dim(I_{W_s})_6 = \begin{cases} 3 & \text{for} \quad s = e = 6 \\ 1 & \text{for} \quad s = e^* = 7. \end{cases}$$

Two times the cubic surface through the scheme $2A + 2B + 2C + P_1 + \cdots + P_7$ gives a surface of degree 6 through W_7, hence $\dim(I_{W_7})_6 \geq 1$, while $\dim(I_{W_6})_6 \geq 3$ follows from Remark 3.2 *i*). So we need only prove that the ideals of the following schemes

$$\widetilde{W}_6 = 4A + 4B + 4C + 2P_1 + \cdots + 2P_6 + G_1 + G_2 + G_3,$$
$$\widetilde{W}_7 = 4A + 4B + 4C + 2P_1 + \cdots + 2P_7 + G_1,$$

where the $G_i's$ are generic points, are zero in degree 6.

Now we specialize the schemes $\widetilde{W}_6, \widetilde{W}_7$, by moving the points $A, B, C, P_1, \ldots, P_6, G_1$ onto a smooth quadric $\mathcal{Q}$, and we get the new schemes Z_6 and Z_7. Let

$$Z'_s = \mathrm{Res}_\mathcal{Q} Z_s \qquad s = 6, 7$$

$$T = \mathrm{Tr}_\mathcal{Q} Z_6 = \mathrm{Tr}_\mathcal{Q} Z_7\ .$$

If we prove that $\dim(I_{Z'_s})_4 = \dim(I_{T,Q})_6 = 0$ $(s = 6, 7)$, then by Lemma 1.6 we have the conclusion. Let

$$Z''_6 = \mathrm{Res}_\Pi Z'_6 = 2A + 2B + 2C + P_1 + \cdots + P_6 + G_2 + G_3 \subseteq \mathbb{P}^3,$$
$$Z''_7 = \mathrm{Res}_\Pi Z'_7 = 2A + 2B + 2C + P_1 + \cdots + P_6 + 2P_7 \subseteq \mathbb{P}^3,$$

where Π is the plane ABC, and let

$$T' = 2A' + 2B' + 2C' + 2P'_1 + \cdots + 2P'_6 + G'_1 \subseteq \mathbb{P}^2$$

where $A', B', C', P'_1, \ldots, P'_6, G'_1$ are generic points of $\mathbb{P}^2$.

By Lemma 3.4 *ii*) and Lemma 3.3 it follows that

$$\dim(I_{Z'_s})_4 = \dim(I_{Z''_s})_3, \qquad s = 6, 7,$$
$$\dim(I_{T,Q})_6 = \dim(I_{T',\mathbb{P}^2})_6 \ .$$

Since it is well known that $\dim(I_{Z''_s})_3 = 0$ and $\dim(I_{T',\mathbb{P}^2})_6 = 0$, by Lemma 1.6 we are done.

Step 2: $\min\{a, b, c\} = 1$. See Remark 3.5.

Now we prove the theorem by induction on $a + b + c$.

Step 3: the starting steps of the induction, i.e. $a + b + c = 3, 4, 5$.

Since $a + b + c \leq 5$ implies that $\min\{a, b, c\} = 1$, see Remark 3.5.

Step 4: The general step of the induction for $(a, b, c) \neq (2\alpha, 2\beta, 2)$ (modulo permutations).

We fix (or recall) the following notation:

$$W = (b + c)A + (a + c)B + (a + b)C \subseteq \mathbb{P}^3$$
$$W_s = W + 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^3$$

$$e = \left\lfloor \frac{(a+1)(b+1)(c+1)}{4} \right\rfloor \qquad e^* = \left\lceil \frac{(a+1)(b+1)(c+1)}{4} \right\rceil$$
$$r = (a + 1)(b + 1)(c + 1) - 4e = N + 1 - 4e.$$

By Remark 3.2, it suffices to prove the theorem for $s = e$, $0 \leq r \leq 3$, and for $s = e^* = e + 1$, $2 \leq r \leq 3$. Since, by Remark 3.2 *i*), it follows that $\dim(I_{W_e})_{a+b+c} \geq r$, we have to prove that:

$$\dim(I_{\widetilde{W}_e})_{a+b+c} = 0 \ \text{ and } \ \dim(I_{W_{e^*}})_{a+b+c} = 0,$$

where

$$\widetilde{W}_e = W + 2P_1 + \cdots + 2P_e + G_1 + \cdots + G_r \subseteq \mathbb{P}^3, \quad W_{e^*} = W + 2P_1 + \cdots + 2P_{e^*} \subseteq \mathbb{P}^3,$$

and the $G_i{}'s$ are generic points in $\mathbb{P}^3$. By **Steps 1, 2, 3** we may assume that $a+b+c > 5$, $(a, b, c) \neq (2, 2, 2)$, and $\min\{a, b, c\} > 1$.

Now let $\overline{n}$ denote the class of the integer n (mod 3) and set

$$e_H = \left\lfloor \frac{(b+1)(c+1)}{3} \right\rfloor \quad \text{and} \quad r_H = (b+1)(c+1) - 3e_H,$$

and, if necessary, rearrange the triple (a, b, c) by the following rules:

• In case $\min\{a, b, c\} = 2$:

Assume $c = 2$; if only one among a and b is odd, assume that b is odd; if both a and b are even, or both a and b are odd, assume $a \geq b$.

• In case $\min\{a, b, c\} \geq 3$:

If $\overline{2} \in \{\overline{a}.\overline{b}, \overline{c}\}$, assume $\overline{c} = 2$; if $\overline{2} \notin \{\overline{a}, \overline{b}, \overline{c}\}$, and at least two among $\overline{a}, \overline{b}, \overline{c}$ are $\overline{0}$, assume $\overline{b} = \overline{c} = \overline{0}$; if $\overline{2} \notin \{\overline{a}, \overline{b}, \overline{c}\}$, and at least two among $\overline{a}, \overline{b}, \overline{c}$ are $\overline{1}$, assume $\overline{b} = \overline{c} = \overline{1}$.

With these conventions we are reduced to considering the following nine cases:

If $\min\{a, b, c\} = 2$:

Case	a	b	c	r_H	r	
1	even	even	2	0	$1, 3$	$a \geq 4$; $a \geq b \geq 2$
2	2	odd	2	0	$0, 2$	$b \geq 3$
3	even	odd	2	0	$0, 2$	$a \geq 4$; $b \geq 3$
4	odd	odd	2	0	0	$a \geq b \geq 3$

If $\min\{a, b, c\} \geq 3$:

Case	$\overline{a}$	$\overline{b}$	$\overline{c}$	r_H	r	
5	$\overline{0}, \overline{1}, \overline{2}$	$\overline{0}, \overline{1}, \overline{2}$	$\overline{2}$	0	$0, 1, 2, 3$	$a \geq b \geq 3$, $c \geq 5$
6	$\overline{0}, \overline{1}$	$\overline{0}$	$\overline{0}$	1	$1, 2, 3$	$a, b, c \geq 3$
7	$\overline{0}, \overline{1}$	$\overline{1}$	$\overline{1}$	1	$1, 2, 3$	$a \geq 3$; $b, c \geq 4$
8	$\overline{0}, \overline{1}$	$\overline{0}$	$\overline{0}$	1	0	$a, b, c \geq 3$
9	$\overline{0}, \overline{1}$	$\overline{1}$	$\overline{1}$	1	0	$a \geq 3$; $b, c \geq 4$

• *Claim 1: If we are not in* **Case 1**, *then* $\dim(I_{\widetilde{W}_e})_{a+b+c} = 0$.

Proof. Let H be a generic plane through B and C, let P'_i, G'_i be generic points on H, and let L be the line BC. Let Z be the scheme obtained by specializing, onto H, some of the points of $\widetilde{W}_e$: more precisely, in **Cases 2** to **7** we specialize e_H double points and r_H simple points, while in **Cases 8, 9** we specialize e_H double points. Now we define the schemes Z' , T, T' (see the *method of proof* above).

Let

$$W' = \mathrm{Res}_H W = (b + c)A + (a + c - 1)B + (a + b - 1)C.$$

Cases 2 to **7:**

In these cases we have $r \geq r_H$ and

$$Z = W + 2P_1' + \cdots + 2P_{e_H}' + 2P_{e_H+1} + \cdots + 2P_e + G_1' + \cdots + G_{r_H}' + G_{r_H+1} + \cdots + G_r \subseteq \mathbb{P}^3,$$

and we define:

$$\begin{aligned} Z' &= \mathrm{Res}_H Z = W' + P_1' + \cdots + P_{e_H}' + 2P_{e_H+1} + \cdots + 2P_e + G_{r_H+1} + \cdots + G_r \subseteq \mathbb{P}^3, \\ T &= \mathrm{Tr}_H Z = ((a+c)B + (a+b)C + 2P_1' + \cdots + 2P_{e_H}' + G_1' + \cdots + G_{r_H}') \cap H, \\ T' &= \mathrm{Res}_{aL} T = (cB + bC + 2P_1' + \cdots + 2P_{e_H}' + G_1' + \cdots + G_{r_H}') \cap H. \end{aligned}$$

Cases 8, 9:

In these cases we have $r = 0$, $r_H = 1$ and

$$Z = W + 2P_1' + \cdots + 2P_{e_H}' + 2P_{e_H+1} + \cdots + 2P_e \subseteq \mathbb{P}^3,$$

and we define:

$$\begin{aligned} Z' &= \mathrm{Res}_H(W + 2P_1' + \cdots + 2P_{e_H}' + 2P_{e_H+1} + \cdots + 2P_{e-1}) + D_{2,H}(P_e') \\ &= W' + P_1' + \cdots + P_{e_H}' + 2P_{e_H+1} + \cdots + 2P_{e-1} + D_{2,H}(P_e') \subseteq \mathbb{P}^3, \\ T &= \mathrm{Tr}_H(W + 2P_1' + \cdots + 2P_{e_H}' + 2P_{e_H+1} + \cdots + 2P_{e-1}) + P_e' \\ &= ((a+c)B + (a+b)C + 2P_1' + \cdots + 2P_{e_H}' + P_e') \cap H, \\ T' &= \mathrm{Res}_{aL} T = (cB + bC + 2P_1' + \cdots + 2P_{e_H}' + P_e') \cap H. \end{aligned}$$

If $\dim(I_{Z'})_{a+b+c-1} = \dim(I_T)_{a+b+c} = 0$, then by Lemma 1.6 or Lemma 1.7, we will get $\dim(I_Z)_{a+b+c} = 0$. From Lemma 3.4 *i)* it follows that $\dim(I_T)_{a+b+c} = \dim(I_{T'})_{b+c}$. By Proposition 2.2, e_H generic double points impose independent conditions to the curves of $(I_{cB+bC})_{b+c}$, except for b, c both even and $\min\{b, c\} = 2$. Since we are not in **Case 1**, then $(I_{T'})_{b+c}$ has the expected dimension, that is

$$\dim(I_{T'})_{b+c} = (b+1)(c+1) - 3e_H - r_H = 0.$$

Hence in **Cases 2** to **9** we will be done if we prove that $\dim(I_{Z'})_{a+b+c-1} = 0$.

Let $\check{Z} = Z' - (P_1' + \cdots + P_{e_H}')$ be the scheme obtained from Z' by taking away the simple points P_i'. First we compute the dimension of $(I_{\check{Z}})_{a+b+c-1}$ by the induction hypothesis. Then we prove that the points P_i' impose the maximum possible number of independent conditions to the surfaces of $(I_{\check{Z}})_{a+b+c-1}$, and so we compute the dimension of $(I_{Z'})_{a+b+c-1}$. In order to prove that the points P_i' impose the expected number of conditions to $(I_{\check{Z}})_{a+b+c-1}$, we have to prove that $\dim(I_{\mathrm{Res}_H Z'})_{a+b+c-2} = 0$ (see Lemma 1.9). We have

$$\check{Z} = \begin{cases} W' + 2P_{e_H+1} + \cdots + 2P_e + G_{r_H+1} + \cdots + G_r & \text{in } \textbf{Cases 2} \text{ to } \textbf{7} \\ W' + 2P_{e_H+1} + \cdots + 2P_{e-1} + D_{2,H}(P_e') & \text{in } \textbf{Cases 8, 9}. \end{cases}$$

Let

$$e_1 = \left\lfloor \frac{a(b+1)(c+1)}{4} \right\rfloor \qquad r_1 = a(b+1)(c+1) - 4e_1,$$

that is, e_1 is the maximum number of generic double points that we expect impose independent conditions to the surfaces of $(I_{W'})_{a+b+c-1}$. Since we are dealing with cases for which $(a-1,b,c)$ is not of type (3.2), then, by the induction hypothesis, e_1 generic double points impose independent conditions to the surfaces of $(I_{W'})_{a+b+c-1}$. So, if $e_1 \geq (e-e_H)$, then we may compute $\dim(I_{\check{Z}})_{a+b+c-1}$. In fact in **Cases 2** to **7** the scheme $\check{Z}$ is the union of W' and $e-e_H$ generic double points, while in **Cases 8, 9**, the scheme $\check{Z}$ is contained in a scheme formed by W' and $e-e_H$ generic double points. Now:

$$e_1-(e-e_H)=\frac{a(b+1)(c+1)-r_1}{4}-\frac{(a+1)(b+1)(c+1)-r}{4}$$
$$+\frac{(b+1)(c+1)-r_H}{3}=\frac{(b+1)(c+1)+3(r-r_1-r_H)-r_H}{12},$$

and we easily get $e_1 \geq (e-e_H)$. So, since $G_{r_H+1}+\cdots+G_r \in \mathbb{P}^3$ are generic points, we have:

$$\dim(I_{\check{Z}})_{a+b+c-1}=\begin{cases} a(b+1)(c+1)-4(e-e_H)-r+r_H=e_H & \text{in } \textbf{Cases 2} \text{ to } \textbf{7} \\ a(b+1)(c+1)-4(e-e_H-1)-3=e_H & \text{in } \textbf{Cases 8},\ \textbf{9}. \end{cases}$$

Now let

$$W''=\mathrm{Res}_H W'=(b+c)A+(a+c-2)B+(a+b-2)C,$$

$$Z''=\mathrm{Res}_H\check{Z}=\begin{cases} W''+2P_{e_H+1}+\cdots+2P_e+G_{r_H+1}+\cdots+G_r & \text{in } \textbf{Cases 2} \text{ to } \textbf{7} \\ W''+2P_{e_H+1}+\cdots+2P_{e-1} & \text{in } \textbf{Cases 8},\ \textbf{9} \end{cases}$$

$$e_2=\left\lfloor\frac{(a-1)(b+1)(c+1)}{4}\right\rfloor \qquad {e^*}_2=\left\lceil\frac{(a-1)(b+1)(c+1)}{4}\right\rceil$$

$$r_2=(a-1)(b+1)(c+1)-4e_2.$$

By the induction hypothesis, if $(a-2,b,c)$ is not of type (3.1) or (3.2), then there are no surfaces in $(I_{W''})_{a+b+c-2}$ through e_2^* double points, while if $(a-2,b,c)$ is of type (3.1) or (3.2), there are no surfaces in $(I_{W''})_{a+b+c-2}$ through e_2^*+1 double points. Now we have

$$(e-e_H)-e_2^*=\frac{(a+1)(b+1)(c+1)-r}{4}-\frac{(b+1)(c+1)-r_H}{3}$$
$$-\frac{(a-1)(b+1)(c+1)-r_2}{4}-\left\lceil\frac{r_2}{4}\right\rceil$$
$$=\frac{2(b+1)(c+1)+3(r_2+r_H-r)+r_H}{12}-\left\lceil\frac{r_2}{4}\right\rceil;$$
$$(e-1-e_H)-e_2^*=\frac{2(b+1)(c+1)+3(r_2+r_H-r)+r_H}{12}-1-\left\lceil\frac{r_2}{4}\right\rceil.$$

In **Cases 3** to **7** we easily get $(e-e_H)\geq e_2^*$. Since $(a-2,b,c)$ is not of type (3.1) or (3.2), then $\dim(I_{Z''})_{a+b+c-2}=0$.

In **Cases 8, 9** (where $r = 0$, $r_H = 1$, and Z'' is the union of W'' and $(e-1-e_H)$ generic double points) we easily get $(e-1-e_H) \geq e_2^*$. Now $(a-2, b, c)$ is not of type (3.1) or (3.2), thus also in **Cases 8, 9** we get $\dim(I_{Z''})_{a+b+c-2} = 0$.

In **Case 2** we have $a = c = 2$ and b odd, hence

$$W'' = \mathrm{Res}_H W' = (b+2)A + 2B + bC.$$

It follows that the surfaces of $(I_{Z''})_{a+b+c-2}$ are cones with vertex in A, so we have

$$\begin{aligned}\dim(I_{Z''})_{a+b+c-2} &= \dim(I_{(b+2)A+2B+bC+2P_{e_H+1}+\cdots+2P_e+G_{r_H+1}+\cdots+G_r})_{b+2} \\ &= \dim(I_{(2B'+bC'+2P'_{e_H+1}+\cdots+2P'_e+G'_{r_H+1}+\cdots+G'_r)\cap M})_{b+2},\end{aligned}$$

where $B', C', P'_{e_H+1}, \ldots, P'_e, G'_{r_H+1}, \ldots, G'_r$ are generic points in $M \cong \mathbb{P}^2$. Since b is odd, by Proposition 2.2 we get

$$\begin{aligned}&\dim(I_{(2B'+bC'+2P'_{e_H+1}+\cdots+2P'_e+G'_{r_H+1}+\cdots+G'_r)\cap M})_{b+2} \\ &= \min\{3(b+1) - 3(e-e_H) - (r-r_H); 0\} \\ &= \min\{-3e - r; 0\} = 0,\end{aligned}$$

and so $\dim(I_{Z''})_{a+b+c-2} = 0$.

Hence in **Cases 2** to **9**, by Lemma 1.9 we can compute the dimension of $(I_{Z'})_{a+b+c-1}$:

$$\dim(I_{Z'})_{a+b+c-1} = \dim(I_{\check{Z}+(P'_1+\cdots+P'_{e_H})})_{a+b+c-1} = \dim(I_{\check{Z}})_{a+b+c-1} - e_H = 0.$$

That finishes the proof of *Claim 1*. □

• *Claim 2:* If $r \in \{2, 3\}$ and we are not in **Case 1**, then $\dim(I_{W_{e^*}})_{a+b+c} = 0$.

Proof. We have to prove that in **Cases 2, 3, 5, 6, 7**, with $2 \leq r \leq 3$, then $\dim(I_{W_{e^*}})_{a+b+c} = 0$, where $e^* = e+1$. As in the proof of *Claim 1*, let H be a generic plane through B and C, let P'_i be generic points on H, and let L be the line BC. Let Z be the scheme obtained by specializing e_H double points of W_{e^*} onto H. Now we define the schemes Z', T, T'. Let W' be as above, that is

$$W' = \mathrm{Res}_H W = (b+c)A + (a+c-1)B + (a+b-1)C.$$

Cases 2, 3, 5:
In these cases we have $r_H = 0$ and

$$Z = W + 2P'_1 + \cdots + 2P'_{e_H} + 2P_{e_H+1} + \cdots + 2P_{e^*}$$

and we define

$$\begin{aligned}Z' &= \mathrm{Res}_H Z = W' + P'_1 + \cdots + P'_{e_H} + 2P_{e_H+1} + \cdots + 2P_{e^*}, \\ T &= \mathrm{Tr}_H Z = ((a+c)B + (a+b)C + 2P'_1 + \cdots + 2P'_{e_H}) \cap H, \\ T' &= \mathrm{Res}_{aL} T = (cB + bC + 2P'_1 + \cdots + 2P'_{e_H}) \cap H.\end{aligned}$$

Cases 6, 7:
In these cases we have $r_H = 1$ and

$$Z = W + 2P'_1 + \cdots + 2P'_{e_H} + 2P_{e_H+1} + \cdots + 2P_{e^*}$$

and we define

$$\begin{aligned} Z' &= \mathrm{Res}_H(W + 2P'_1 + \cdots + 2P'_{e_H} + 2P_{e_H+1} + \cdots + 2P_e) + D_{2,H}(P'_{e^*}) \\ &= W' + P'_1 + \cdots + P'_{e_H} + 2P_{e_H+1} + \cdots + 2P_e + D_{2,H}(P'_{e^*}), \\ T &= \mathrm{Tr}_H(W + 2P'_1 + \cdots + 2P'_{e_H} + 2P_{e_H+1} + \cdots + 2P_e) + P'_{e^*} \\ &= ((a+c)B + (a+b)C + 2P'_1 + \cdots + 2P'_{e_H} + P'_{e^*}) \cap H, \\ T' &= \mathrm{Res}_{aL} T = (cB + bC + 2P'_1 + \cdots + 2P'_{e_H} + P'_{e^*}) \cap H. \end{aligned}$$

By Lemma 3.4 *i)*, we have that $\dim(I_T)_{a+b+c} = \dim(I_{T'})_{b+c}$, and analogously to the proof of *Claim 1*, since (b,c) is not an exceptional case of Proposition 2.2, $(I_{T'})_{b+c}$ has the expected dimension. I.e.

$$\dim(I_{T'})_{b+c} = \begin{cases} (b+1)(c+1) - 3e_H = 0 & \text{in } \textbf{Cases 2, 3, 5} \\ (b+1)(c+1) - 3e_H - 1 = 0 & \text{in } \textbf{Cases 6, 7}. \end{cases}$$

So by Lemma 1.6 or Lemma 1.7 we get the conclusion if we can prove that $\dim(I_{Z'})_{a+b+c-1} = 0$.

Let $\check{Z}$ be the scheme obtained from Z' by taking away the simple points P'_i.

We have

$$\check{Z} = \begin{cases} Z' - (P'_1 + \cdots + P'_{e_H}) = W' + 2P_{e_H+1} + \cdots + 2P_{e^*} & \text{in } \textbf{Cases 2, 3, 5} \\ Z' - (P'_1 + \cdots + P'_{e_H}) = W' + 2P_{e_H+1} + \cdots + 2P_e + D_{2,H}(P'_{e^*}) & \text{in } \textbf{Cases 6, 7}. \end{cases}$$

It is easy to check that $e_1 \geq (e^* - e_H)$ (recall that $e_1 - (e - e_H) = \frac{(b+1)(c+1)+3(r-r_1-r_H)-r_H}{12}$). So, since $(a-1,b,c)$ is not of type (3.2), we may compute the dimension of $(I_{\check{Z}})_{a+b+c-1}$ by the induction hypothesis. We get:

$$\dim(I_{\check{Z}})_{a+b+c-1} = \begin{cases} a(b+1)(c+1) - 4(e+1-e_H) = e_H + r - 4 \geq 0 & \text{in } \textbf{Cases 2, 3, 5} \\ a(b+1)(c+1) - 4(e-e_H) - 3 = e_H + r - 4 \geq 0 & \text{in } \textbf{Cases 6, 7}. \end{cases}$$

In order to apply Lemma 1.9 for computing the dimension of $(I_{Z'})_{a+b+c-1}$, we have to prove that $\dim(I_{Z''})_{a+b+c-2} = 0$, where

$$Z'' = \mathrm{Res}_H \check{Z} = \begin{cases} W'' + 2P_{e_H+1} + \cdots + 2P_{e^*} & \text{in } \textbf{Cases 2, 3, 5} \\ W'' + 2P_{e_H+1} + \cdots + 2P_e + D_{2,H}(P'_{e^*}) & \text{in } \textbf{Cases 6, 7}, \end{cases}$$

and W'' is as above. I.e. $W'' = (b+c)A + (a+c-2)B + (a+b-2)C$. Since $(a-2,b,c)$ is not of type (3.1) or (3.2), it suffices to show that Z'' is the union of W'' and at least

e_2^* generic double points. Now:

$$\begin{aligned}(e-e_H)-e_2^* &= \frac{(a+1)(b+1)(c+1)-r}{4} - \frac{(b+1)(c+1)-r_H}{3} \\ &\quad - \frac{(a-1)(b+1)(c+1)-r_2}{4} - \left\lceil \frac{r_2}{4} \right\rceil \\ &= \frac{2(b+1)(c+1)+3(r_2-r)+4r_H}{12} - \left\lceil \frac{r_2}{4} \right\rceil .\end{aligned}$$

So it is easy to see that in **Cases 6, 7** we have $(e-e_H) \geq e_2^*$, while in **Cases 2, 3, 5** we have $(e^*-e_H) \geq e_2^*$. It follows that $\dim(I_{Z''})_{a+b+c-2} = 0$. Now we can compute the dimension of $(I_{Z'})_{a+b+c-1}$. Since $\dim(I_{\check{Z}})_{a+b+c-1} \leq e_H$, by Lemma 1.9 we immediately get that $\dim(I_{Z'})_{a+b+c-1} = 0$, and *Claim 2* is proved. That finishes the proof of **Step 4**.

Step 5: $(a,b,c) = (2\alpha, 2\beta, 2)$ *(modulo permutations).*

This is the only case left to prove (see **Case 1** in the table above), in fact when a,b,c are even and $\min\{a,b,c\} = 2$ we have to proceed in a different way.

We may assume that $a \geq b$ and, by **Step 1**, that $a \geq 4$. Direct computations using CoCoA (see [3]) (or ad hoc specializations of W_s) show that the theorem holds for $(4,2,2)$, so we assume that $(a,b,c) = (2\alpha, 2\beta, 2) \neq (4,2,2)$, $\alpha \geq \beta$.

Let $W, e, r, e^*, \widetilde{W}_e, W_{e^*}$ be as in **Step 4**. By Remark 3.2 it suffices to prove that

$$\dim(I_{\widetilde{W}_e})_{a+b+c} = \dim(I_{W+2P_1+\cdots+2P_e+G_1+\cdots+G_r})_{a+b+c} = 0,$$

and for $2 \leq r \leq 3$:

$$\dim(I_{W_{e^*}})_{a+b+c} = \dim(I_{W+2P_1+\cdots+2P_{e^*}})_{a+b+c} = 0.$$

Let $\mathcal{Q}$ be a smooth quadric through A, B, C. Recall that

$$e = \left\lfloor \frac{(2\alpha+1)(2\beta+1)(3)}{4} \right\rfloor, \; r = (2\alpha+1)(2\beta+1)(3) - 4e,$$

and fix the following notation

$$e_{\mathcal{Q}} = \left\lfloor \frac{3(a+1)(b+1)-2ab}{3} \right\rfloor \qquad r_{\mathcal{Q}} = 3(a+1)(b+1) - 2ab - 3e_{\mathcal{Q}}.$$

Note that, since a and b are even $r \in \{1,3\}$. So either $r \geq r_{\mathcal{Q}}$ or $r = 1$ and $r_{\mathcal{Q}} = 2$.

Now we prove that

$$\dim(I_{\widetilde{W}_e})_{a+b+c} = 0.$$

We specialize the scheme $\widetilde{W}_e$ by moving $e_{\mathcal{Q}}$ double points and $r_{\mathcal{Q}}$ simple points onto the quadric $\mathcal{Q}$, for $r \geq r_{\mathcal{Q}}$, or by moving $e_{\mathcal{Q}}$ double points and r simple points onto $\mathcal{Q}$, for $r = 1$ and $r_{\mathcal{Q}} = 2$. Let Z be the specialized scheme. We define the schemes Z', T, T', Z'', Z''' (see the *method of proof*).

Let P'_i, G'_i be generic points on $\mathcal{Q}$, and let A'', B'', C'', P''_i be generic points in $\mathbb{P}^2$.

For $r \geq r_{\mathcal{Q}}$, we have

$$Z = W + 2P'_1 + \cdots + 2P'_{e_{\mathcal{Q}}} + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + G'_1 + \cdots + G'_{r_{\mathcal{Q}}} + G_{r_{\mathcal{Q}}+1} + \cdots + G_r,$$

and we define

$$\begin{aligned} Z' &= \mathrm{Res}_{\mathcal{Q}} Z = \mathrm{Res}_{\mathcal{Q}} W + P'_1 + \cdots + P'_{e_{\mathcal{Q}}} + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + G_{r_{\mathcal{Q}}+1} + \cdots + G_r, \\ T &= \mathrm{Tr}_{\mathcal{Q}} Z = \mathrm{Tr}_{\mathcal{Q}}(W + 2P'_1 + \cdots + 2P'_{e_{\mathcal{Q}}} + G'_1 + \cdots + G'_{r_{\mathcal{Q}}}), \\ T' &= aA'' + bB'' + 2C'' + 2P''_1 + \cdots + 2P''_{e_{\mathcal{Q}}} + G''_1 + \cdots + G''_{r_{\mathcal{Q}}} \subseteq \mathbb{P}^2. \end{aligned}$$

For $r = 1$, $r_{\mathcal{Q}} = 2$ we have

$$Z = W + 2P'_1 + \cdots + 2P'_{e_{\mathcal{Q}}} + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + G'_1,$$

and we define (note that $e > e_{\mathcal{Q}}$)

$$\begin{aligned} Z' &= \mathrm{Res}_{\mathcal{Q}} W + P'_1 + \cdots + P'_{e_{\mathcal{Q}}} + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e-1} + D_{2,\mathcal{Q}} P'_e, \\ T &= \mathrm{Tr}_{\mathcal{Q}}(W + 2P'_1 + \cdots + 2P'_{e_{\mathcal{Q}}}) + P'_e + G'_1, \\ T' &= aA'' + bB'' + 2C'' + 2P''_1 + \cdots + 2P''_{e_{\mathcal{Q}}} + P''_e + G''_1 \subseteq \mathbb{P}^2. \end{aligned}$$

If $\dim(I_{Z'})_{a+b+2-2} = \dim(I_T)_{a+b+2} = 0$, then by Lemma 1.6, or by the analogue to Lemma 1.7 when instead of the hyperplane $H \subseteq \mathbb{P}^n$ we consider a non singular quadric in $\mathbb{P}^3$, we get $\dim(I_Z)_{a+b+2} = 0$. By Lemma 3.3, $\dim(I_T)_{a+b+2} = \dim(I_{T'})_{a+b+2}$, and by Proposition 2.3 $(I_{T'})_{a+b+2}$ has the expected dimension, that is

$$\dim(I_{T'})_{a+b+2} = 3(a+1)(b+1) - 2ab - 3e_{\mathcal{Q}} - r_{\mathcal{Q}} = 0.$$

Hence we will be done if we prove that $\dim(I_{Z'})_{a+b} = 0$.

Let

$$\begin{aligned} W' = \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} W) &= ((b-1) + (c-1))A + ((a-1) + (c-1)) \\ B + ((a-1) + (b-1))C &= bA + aB + (a+b-2)C, \end{aligned}$$

where Π is the plane ABC, and let

$$\begin{aligned} Z'' &= \mathrm{Res}_{\Pi} Z' \\ &= \begin{cases} W' + P'_1 + \cdots + P'_{e_{\mathcal{Q}}} + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + G_{r_{\mathcal{Q}}+1} + \cdots + G_r & \text{for } r \geq r_{\mathcal{Q}} \\ W' + P'_1 + \cdots + P'_{e_{\mathcal{Q}}} + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e-1} + D_{2,\mathcal{Q}} P'_e & \text{for } r = 1, r_{\mathcal{Q}} = 2, \end{cases} \end{aligned}$$

$$\begin{aligned} \check{Z} &= Z'' - (P'_1 + \cdots + P'_{e_{\mathcal{Q}}}) \\ &= \begin{cases} W' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + G_{r_{\mathcal{Q}}+1} + \cdots + G_r & \text{for } r \geq r_{\mathcal{Q}} \\ W' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e-1} + D_{2,\mathcal{Q}} P'_e & \text{for } r = 1,\ r_{\mathcal{Q}} = 2. \end{cases} \end{aligned}$$

It is easy to prove that the plane Π is a component for any surface defined by a form in $(I_{Z'})_{a+b}$. Hence $\dim(I_{Z'})_{a+b} = \dim(I_{Z''})_{a+b-1}$.

Since $(a-1,b-1,c-1)=(2\alpha-1,2\beta-1,1)$ is not of type (3.2), by the induction hypothesis $\left\lfloor \frac{abc}{4}\right\rfloor = 2\alpha\beta$ generic double points impose independent conditions to the surfaces of $(I_{W'})_{a+b-1}$. Since we have $2\alpha\beta \geq (e-e_{\mathcal{Q}})$, then the dimension of $(I_{\check{Z}})_{a+b-1}$ is as expected. By a direct computation we get

$$\dim(I_{\check{Z}})_{a+b-1} = \begin{cases} 2ab-4(e-e_{\mathcal{Q}})-(r-r_{\mathcal{Q}}) = e_{\mathcal{Q}} & \text{for } \ r \geq r_{\mathcal{Q}} \\ 2ab-4(e-1-e_{\mathcal{Q}})-3 = e_{\mathcal{Q}} & \text{for } \ r=1, \ r_{\mathcal{Q}}=2. \end{cases}$$

Since $Z'' = \check{Z} + (P'_1+\cdots+P'_{e_{\mathcal{Q}}})$, if we prove that $\dim(I_{\mathrm{Res}_{\mathcal{Q}} Z''})_{a+b-3}=0$, then by Lemma 1.9 we get $\dim(I_{Z''})_{a+b-1}=0$.

Any surface defined by a form of $(I_{\mathrm{Res}_{\mathcal{Q}} Z''})_{a+b-3}$ contains the plane Π, so

$$\dim(I_{\mathrm{Res}_{\mathcal{Q}} Z''})_{a+b-3} = \dim(I_{Z'''})_{a+b-4},$$

where

$$\begin{aligned} Z''' &= \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} Z'') \\ &= \begin{cases} W''+2P_{e_{\mathcal{Q}}+1}+\cdots+2P_e+G_{r_{\mathcal{Q}}+1}+\cdots+G_r & \text{for } \ r \geq r_{\mathcal{Q}} \\ W''+2P_{e_{\mathcal{Q}}+1}+\cdots+2P_{e-1} & \text{for } \ r=1, r_{\mathcal{Q}}=2 \end{cases} \end{aligned}$$

and

$$W'' = \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} W') = (b-2)A+(a-2)B+(a+b-4)C.$$

But the surfaces of degree $a+b-4$ passing through W'' are cones with vertex in C, hence

$$\dim(I_{Z'''})_{a+b-4} = \dim(I_{Z^*})_{a+b-4},$$

where

$$Z^* =$$

$$\begin{cases} ((b-2)A^*+(a-2)B^*+2P_1^*+\cdots+2P^*_{e-e_{\mathcal{Q}}}+G_1^*+\cdots+G^*_{r-r_{\mathcal{Q}}}) \cap M, & \text{for } r \geq r_{\mathcal{Q}} \\ ((b-2)A^*+(a-2)B^*+2P_1^*+\cdots+2P^*_{e-1-e_{\mathcal{Q}}}) \cap M, & \text{for } \ r=1, r_{\mathcal{Q}}=2, \end{cases}$$

and where A^*, B^*, the P_i^*'s and the G_i^*'s are generic points in $M \cong \mathbb{P}^2$. By Proposition 2.2 and by noting that for $b=2$ the curves passing through Z^* are cones with vertex in B^*, a direct computation (which we leave to the reader) shows that $\dim(I_{Z^*})_{a+b-4}=0$. That finishes the prove that $\dim(I_{\widetilde{W}_e})_{a+b+c}=0$.

Finally we have to prove that for $r=3$

$$\dim(I_{W_{e^*}})_{a+b+2}=0.$$

We specialize the scheme W_{e^*} by moving $e_{\mathcal{Q}}$ double points points onto the quadric $\mathcal{Q}$, if $0 \leq r_{\mathcal{Q}} \leq 1$, or by moving $e_{\mathcal{Q}}+1$ double points, if $r_{\mathcal{Q}}=2$. Let Z be the specialized scheme. Then, as above, we define the schemes $Z', T, T', Z'', \check{Z}, Z'''$.

More precisely we have

$$Z = \begin{cases} W+2P'_1+\cdots+2P'_{e_{\mathcal{Q}}}+2P_{e_{\mathcal{Q}}+1}+\cdots+2P_{e^*} & \text{for } \ 0 \leq r_{\mathcal{Q}} \leq 1 \\ W+2P'_1+\cdots+2P'_{e_{\mathcal{Q}}+1}+2P_{e_{\mathcal{Q}}+2}+\cdots+2P_{e^*} & \text{for } \ r_{\mathcal{Q}}=2, \end{cases}$$

where the P_i''s are generic points on $\mathcal{Q}$. We define:

$$T = \begin{cases} \mathrm{Tr}_{\mathcal{Q}} Z = \mathrm{Tr}_{\mathcal{Q}}(W + 2P_1' + \cdots + 2P_{e_{\mathcal{Q}}}') & \text{for } r_{\mathcal{Q}} = 0 \\ \mathrm{Tr}_{\mathcal{Q}}(W + 2P_1' + \cdots + 2P_{e_{\mathcal{Q}}}') + P_{e^*}' & \text{for } r_{\mathcal{Q}} = 1 \\ \mathrm{Tr}_{\mathcal{Q}} Z = \mathrm{Tr}_{\mathcal{Q}}(W + 2P_1' + \cdots + 2P_{e_{\mathcal{Q}}+1}') & \text{for } r_{\mathcal{Q}} = 2, \end{cases}$$

$$T' = \begin{cases} aA'' + bB'' + 2C'' + 2P_1'' + \cdots + 2P_{e_{\mathcal{Q}}}'' \subseteq \mathbb{P}^2 & \text{for } r_{\mathcal{Q}} = 0 \\ aA'' + bB'' + 2C'' + 2P_1'' + \cdots + 2P_{e_{\mathcal{Q}}}'' + P_{e^*}'' \subseteq \mathbb{P}^2 & \text{for } r_{\mathcal{Q}} = 1 \\ aA'' + bB'' + 2C'' + 2P_1'' + \cdots + 2P_{e_{\mathcal{Q}}+1}'' \subseteq \mathbb{P}^2 & \text{for } r_{\mathcal{Q}} = 2, \end{cases}$$

where A'', B'', C'', P_i'' are generic points in $\mathbb{P}^2$;

$$Z' = \begin{cases} \mathrm{Res}_{\mathcal{Q}} W + P_1' + \cdots + P_{e_{\mathcal{Q}}}' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 0 \\ \mathrm{Res}_{\mathcal{Q}} W + P_1' + \cdots + P_{e_{\mathcal{Q}}}' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + D_{2,\mathcal{Q}} P_{e^*}' & \text{for } r_{\mathcal{Q}} = 1 \\ \mathrm{Res}_{\mathcal{Q}} W + P_1' + \cdots + P_{e_{\mathcal{Q}}+1}' + 2P_{e_{\mathcal{Q}}+2} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 2, \end{cases}$$

$$Z'' = \begin{cases} \mathrm{Res}_{\Pi} Z' = W' + P_1' + \cdots + P_{e_{\mathcal{Q}}}' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 0 \\ \mathrm{Res}_{\Pi} Z' = W' + P_1' + \cdots + P_{e_{\mathcal{Q}}}' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + D_{2,\mathcal{Q}} P_{e^*}' & \text{for } r_{\mathcal{Q}} = 1 \\ \mathrm{Res}_{\Pi} Z' = W' + P_1' + \cdots + P_{e_{\mathcal{Q}}+1}' + 2P_{e_{\mathcal{Q}}+2} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 2, \end{cases}$$

$$\check{Z} = \begin{cases} W' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 0 \\ W' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e + D_{2,\mathcal{Q}} P_{e^*}' & \text{for } r_{\mathcal{Q}} = 1 \\ W' + 2P_{e_{\mathcal{Q}}+2} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 2, \end{cases}$$

$$Z''' = \begin{cases} \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} Z'') = W'' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 0 \\ \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} Z'') = W'' + 2P_{e_{\mathcal{Q}}+1} + \cdots + 2P_e & \text{for } r_{\mathcal{Q}} = 1 \\ \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} Z'') = W'' + 2P_{e_{\mathcal{Q}}+2} + \cdots + 2P_{e^*} & \text{for } r_{\mathcal{Q}} = 2 \end{cases}$$

where, as above,

$$\begin{aligned} W' &= \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} W) = bA + aB + (a+b-2)C, \quad W'' = \mathrm{Res}_{\Pi}(\mathrm{Res}_{\mathcal{Q}} W') \\ &= (b-2)A + (a-2)B + (a+b-4)C. \end{aligned}$$

Proceeding analogously to the first part of **Step 5**, it can be checked that $\dim(I_{T,\mathbb{P}^3})_{a+b+2} = \dim(I_{T',\mathbb{P}^2})_{a+b+2} = 0$, $\dim(I_{\check{Z}})_{a+b-1}$ is as expected, i.e.:

$$\dim(I_{\check{Z}})_{a+b-1} = \begin{cases} 2ab - 4(e + 1 - e_{\mathcal{Q}}) = e_{\mathcal{Q}} - 1 & \text{for } r_{\mathcal{Q}} = 0 \\ 2ab - 4(e - e_{\mathcal{Q}}) - 3 = e_{\mathcal{Q}} - 1 & \text{for } r_{\mathcal{Q}} = 1 \\ 2ab - 4(e - e_{\mathcal{Q}}) = e_{\mathcal{Q}} + 1 & \text{for } r_{\mathcal{Q}} = 2, \end{cases}$$

and $\dim(I_{Z'''})_{a+b-1} = 0$. So we get $\dim(I_{Z''})_{a+b-1} = \dim(I_{Z'})_{a+b} = 0$. Finally, by Lemma 1.6 or by the analogue to Lemma 1.7 when instead of the hyperplane $H \subseteq \mathbb{P}^n$ we consider a non singular quadric in $\mathbb{P}^3$, since $\dim(I_{Z'})_{a+b} = \dim(I_T)_{a+b+2} = 0$, we have $\dim(I_Z)_{a+b+2} = 0$, and we are done. □

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