ON A CLASS OF $\ell_{\Phi\varphi}$ OPERATORS

by

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INTRODUCTION.

In a previous paper [6] some classes of linear operators are studied by means of the approximation numbers $\alpha_n(T) = \inf \|T - K\|$, $T, K \in L(E, F)$ $\dim K \leqslant n$.

Since the expresion of the approximation numbers of the tensor product operator $T_1 \otimes T_2$ [6] is not convenient, here we consider a new concept of the approximation number of a linear operator which is more convenient.

By means of these numbers some classes of linear and bounded operator are studied.

1. DYADIC APPROXIMATION NUMBERS ARE THEIR APPLICATIONS.

Let E, F be normed spaces and T a linear and bounded operator

$$T : E \to F (T \in L(E, F)).$$

Definition 1. For all $T \in L(E, F)$ we define the n^{th} dyadic approximation number as follows

$$a_n(T) = \inf \|T - K\|, \quad k \in L(E, F), \quad \dim K < 2^{n-1}, \quad n = 1, 2, \ldots$$

Remark $a_n(T) = \alpha_{2^{n-1}}(T)$, $n = 1, 2, \ldots$

Proposition 1. For all $T_k \in L(E, E)$, $k = 1, 2$, exist the relation:

a) $a_{n+1}(T_1 + T_2) \leqslant a_n(T_1) + a_n(T_2)$

b) $a_{n+1}(T_1 T_2) \leqslant a_n(T_1) \cdot a_n(T_2), \qquad n = 1, 2, \ldots$

Proof. a) From the definition of $a_n(T)$ results that given $\epsilon_1, \epsilon_2 > 0$ there exist

$$K_1, K_2 \ (\dim K_i < 2^{n-1}),\ i = 1, 2$$

such that

$$\|T_1 - K_1\| \leqslant a_n(T_1) + \epsilon_1, \qquad \|T_2 - K_2\| \leqslant a_n(T_2) + \epsilon_2$$

Then

$$a_{n+1}(T_1 + T_2) \leqslant \|(T_1 + T_2) - (K_1 + K_2)\| \leqslant \|T_1 - K_1\| + \|T_2 - K_2\|$$

$$\leqslant a_n(T_1) + \epsilon_1 + a_n(T_2) + \epsilon_2$$

Since $\dim(K_1 + K_2) \leqslant 2^{n-1} + 2^{n-1} = 2^{(n+1)-1}$ and ϵ_1, ϵ_2 are arbitrary results (a).

In a similar way we can prove (b).

Let be $T_i : E_i \to F_i$ $(i = 1, 2)$ and $T_1 \otimes_\theta T_2 : E_1 \otimes_\theta E_2 \to F_1 \otimes_\theta F_2$, whre θ is a tensor norm $\epsilon \leqslant \theta \leqslant \pi$ (see [1], [6]).

Proposition 2. For all $T_k \in L(E_k, F_k)$, $K = 1, 2$, exists the relation

$$a_{2n-1}(T_1 \otimes_\theta T_2) \leqslant a_n(T_1)\,\|T_2\| + \|T_1\|\,a_n(T_2) + a_n(T_1) \cdot a_n(T_2),$$

$$n = 1, 2, 3, \ldots$$

Proof. It is similar to the proof in [6] for classical approximation numbers.

We consider K_1, K_2, $(\dim K_i < 2^{n-1})$, $i = 1, 2$.

$$a_{2n-1}(T_1 \otimes T_2) \leqslant \|T_1 \otimes T_2 - K_1 \otimes K_2\| = \|(T_1 - K_1) \otimes T_2 + K_1 \otimes (T_2 - K_2)\|$$

$$\leqslant \|T_1 - K_1\| . \|T_2\| + \|K_1\| . \|T_2 - K_2\|$$

$$\leqslant (a_n(T_1) + \epsilon_1)\|T_2\| + \|K_1 - T_1 + T_1\|(a_n(T_2) + \epsilon_2)$$

$$\leqslant (a_n(T_1) + \epsilon_1)\|T_2\| + [a_n(T_1) + \epsilon_1 + \|T_1\|](a_n(T_2) + \epsilon_2)$$

Since ϵ_1, ϵ_2 are arbitrary results the relation

$$a_{2n-1}(T_1 \otimes_\theta T_2) \leqslant a_n(T_1)\,\|T_2\| + \|T_1\|\,a_n(T_2) + a_n(T_1)\,.\,a_n(T_2).$$

Remarks. A similar result can be obtained for the n^{th} dyadic Kolmogorov and Gelfand numbers. (For the expresions of the classical numbers see Lemma 3.1 [6]).

Replacing in the definition of the $\mathfrak{L}_{\Phi,\varphi}$ classes of operators [6] the $\alpha_n(T)$ numbers by $a_n(T)$ we obtain a new class $\mathfrak{L}^*_{\Phi,\varphi}$.

It is clear that $\mathfrak{L}_{\Phi,\varphi} \subset \mathfrak{L}^*_{\Phi,\varphi}$ S since $\alpha_n(T) \geqslant a_n(T)$ for all $n \in N$.

Here we considerar a particular class $\mathfrak{L}^*_p(E, F)$, $1 \leqslant p < \infty$,

$$\mathfrak{L}^*_p(E,F) = \left\{T \in L(E,F) \mid \sum_1^\infty a_n^p(T) < \infty\right\}$$

We can prove.

Proposition 2. $\mathfrak{L}^*_p(E, F)$ is a liniar quasi normed space with the quasi-norm

$$\|T\|^*_p = (\Sigma\, a_n^p(T))^{\frac{1}{p}}$$

The proof results in a simple way from the definition of $a_n(T)$, proposition 1 and Minkowski inequality (see the method from [6]).

Proposition 4. If $T_k \in \mathfrak{L}^*_p(E_k, F_k)$, $(k = 1,2)$ then

$$T_1 \otimes_\theta T_2 \in \mathfrak{L}^*_p(E_1 \otimes_\theta E_2, F_1 \otimes_\theta F_2).$$

Proof. Results from the proposition 2 in the following way

$$(\Sigma\, a_n^p(T_1 \otimes T_2))^{\frac{1}{p}} \leqslant 2^{\frac{1}{p}}(\Sigma\, a^p_{2n-1}(T_1 \otimes T_2))^{\frac{1}{p}} \leqslant$$
$$2^{\frac{1}{p}}[\Sigma\,(a_n(T_1)\,\|T_2\| + 2\|T_1\|\,a_n(T_2))^p]^{\frac{1}{p}} \leqslant$$
$$2^{\frac{1}{p}}[\|T_2\|\,.\,\|T_1\|^*_p + 2\,\|T_1\|\,.\,\|T_2\|^*_p] < \infty\ ,\quad p \in [1, \infty).$$

*) The relation $(\Sigma\, n\, d_n^{2p}(T))^{\frac{1}{2p}} \leqslant 2^{\frac{1}{2p}}.(\Sigma\, d_n^p(T))^{\frac{1}{p}}$ results from [2] (see the inclusion relations between Lorentz sequence spaces).

Now a specificaciton relative to the propsoition 3.1 [6] is presented. Recall that $d_{m.n}(T_1 \otimes_\theta T_2) \leqslant d_m(T_1). \|T_2\| + d_n(T_2). \|T_1\|$, where

$$d_n(T) = \inf_M \{\|T_{|M}\|\},\ M \subset E,\ \text{codim } M \leqslant n \text{ are the Gelfand numbers.}$$

$$\overline{\ell}_p(E, F) = \{T \in L(E, F)_| \ \Sigma\, d_n^p(T) < \infty\},\quad 1 \leqslant p < \infty.$$

Proposition 5. If $T_i \in \overline{\ell}_p(E_i, F_i)$, $i = 1, 2$ then

$$T_1 \otimes_\theta T_2 \in \overline{\ell}_{2p}(E_1 \otimes_\theta E_2, F_1 \otimes_\theta F_2).$$

Proof. Since $d_1(T) \geqslant d_2(T_2) \geqslant d_3(T) \geqslant \ldots$, results

$$(\Sigma\, d^{2p}n\,(T_1 \otimes T_2))^{\frac{1}{2p}} \leqslant [\Sigma\,(2n+1)\,(d_{n^2}(T_1 \otimes T_2))^{2p}]^{\frac{1}{2p}} \leqslant$$
$$3^{\frac{1}{2p}} \{\Sigma\, n\, [d_n(T_1). \|T_2\| + \|T_1\|\, d_n(T_2)]^{2p}\}^{\frac{1}{2p}} \leqslant$$
$$\leqslant 3^{\frac{1}{2p}} \{[\Sigma\, n.\, d_n^{2p}(T_1)]^{\frac{1}{2p}} \|T_2\| + \|T_1\| [\Sigma\, n.\, d_n^{2p}(T_2)]^{\frac{1}{2p}}\}$$
$$\leqslant 6^{\frac{1}{2p}} [\, \|T_2\| (\Sigma\, d_n^p(T_1))^{\frac{1}{p}} + \|T_1\| (\Sigma\, d_n^p(T_2))^{\frac{1}{p}}] < \infty\ ^{(*)}$$

Hence $T_1 \otimes_\theta T_2 \in \overline{\ell}_{2p}(E_1 \otimes E_2, F_1 \otimes F_2)$.

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