SPACES OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

by

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1. INTRODUCTION: Spaces of entire functions endowed with various topologies are considered. First of all, we endow the space of all entire functions represented by Dirichlet Series with four topologies which are shown to be equivalent and then we show that it is a MONTEL Space under one of them. Next, we consider the space of entire functions of finite Ritt-order and finite type, endowed with a certain topology under which it is a FRÉCHET space. On this space we characterize the form of linear continuous functionals. Finally, we give a method of constructing total sets in this space.

2. PRELIMINARIES: Let X denote the set of all entire functions f where

$$f(s) = \sum_{n \geq 1} a_n e^{s\lambda_n}, \; s = \sigma + it, \tag{2.1}$$

and where further

$$1 < \lambda_1 \dots < \lambda_n \dots ; \; (\lambda_n \to \infty \text{ as } n \to \infty) \tag{2.2}$$

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = D < \infty; \; \underline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = h > 0, \tag{2.3}$$

and that

$$\lim_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty. \tag{2.4}$$

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It is to be noted that (2.4) is an «if and only if» condition for $f(s)$ in (2.1) to be an entire function represented by the Dirichlet series in (2.1).

If u is a topology on X, the notation X'_u stands for the family of all continuous linear functionals on X_u.

To recall a few results obtained earlier by one of us [5], assume that $\{r_i\}$ is a non-decreasing sequence of positive numbers, $r_i \to \infty$ with i. Let for each $f \in X$

$$(2.5) \qquad ||f||_{r_i} = \sum |a_n| \, e^{r_i \lambda_n}; \; i = 1, 2, \ldots$$

then $||f||\, r_i$ exists for each i on account of (2.4) and it is easily seen that this is a norm on X. It is clear that $||f||_{r_i} \leq ||f||_{r_{i+1}}$ for all $i \geq 1$. With these countable norms $||f||_{r_i}$ $(i \geq 1)$ we define a metric topology on X, with metric d:

$$(2.6) \qquad d\,(f, g) = \sum_{k \geq 1} \frac{1}{2^k} \frac{||f-g||\, r_k}{1 + ||f-g||\, r_k}; \; f, g \in X.$$

Since $||f||\, r_i \leq ||f||\, r_{i+1}$ for each i, it is clear that the metric topology defined by d is the sup topology which is locally convex and the following result is a consequence of a known fact, namely

$$(2.7) \qquad X'_d = \bigcup_{i \geq 1} X'_{||\ldots||r_i}.$$

3. *Various topologies on X:* Let us recall that the function $f \in X$ is the additive zero of X if and only if $a_n = 0$ for each $n \geq 1$. Now define for each $f \in X$, the following functions:

$$(3.1) \qquad p\,(f) = \sup_{n \geq 1} |a_n|^{1/\lambda_n};$$

$$(3.2) \qquad ||f||_i = \sup_{n \leq i} \{|a_n|^{1/\lambda_n}\}.$$

Observe that (3.1) is defined in view of (2.4). Then the functions $p\,(f)$ and $||f||_i$ are paranorms on X. Let us further denote by s the metric generated by the para norms in (3.2) as in (2.6). Then as $||f||_i \leq ||f||_{i+1}$, the topology given by s is the sup topology. The three topologies given by d, s, and p are equivalent as shown below:

LEMMA 1: We have $d = s = p$.

Proof: First we show that $d = p$. Let $\{f_m\} \subset X$ and that $f_m \to f$ in the paranorm p, where $f \in X$. Then if

$$f_m(s) = \sum_{n\geq 1} a_n^{(m)} e^{s\lambda_n},\ f(s) = \sum_{n\geq 1} a_n e^{s\lambda_n},$$

we have for an arbitrarily large k and all $n \geq 1$,

$$| a_n^{(m)} - a_n |^{1/\lambda_n} < \frac{1}{k}\ , \text{ for all } m \geq m_0 = m_0(k)$$

$$\Rightarrow \| f_m - f \| r_i < \sum_{n\geq 1} \exp \{ (r_i - \log k)\lambda_n \}\ ;\ m \geq m_0,\ i \geq 1$$

$$< O(1)\ e^{-\lambda_1 \log \sqrt{k}}$$

$$\to O \text{ as } k \to \infty, \text{ for each } i \geq 1.$$

Therefore $f_m \to f$ under each norm $\| f \| r_i$, and hence $f_m \to f$ in the metric d. On the other hand, suppose $f_m \to f$ in the metric $d \Rightarrow f_m \to f$ under each norm $\| f \| r_i$ and therefore for a given $\varepsilon > 0$

$$| a_n^{(m)} - a_n | < \varepsilon^{1/\lambda_n} e^{-ri};\ m \geq m_0(\varepsilon)\ ;\ i,\ n \geq 1$$

$$\sup_n | a_n^{(m)} - a_n |^{1/\lambda_n} < e^{-ri} \sup_n (\varepsilon^{1/\lambda_n}),\ m \geq m_0,\ i \geq 1$$

$$\leq \varepsilon,\ m \geq m_0,$$

by choosing i sufficiently large. Consequently $f_m \to f$ in the paranorm p. Hence $d = p$.

To prove the other part of the lemma, let $f_m \to f$ in the paranorm p. Then for arbitrarily large k

$$| a_n^{(m)} - a_n | < k^{-\lambda_n},\ m \geq m_0,\ n \geq 1$$

$$\sup_{n\leq i} | a_n^{(m)} - a_n |^{1/\lambda n} < \sup_{n\leq i} (k^{-\lambda 1}, k^{-\lambda i})$$

$$< k^{-1},\ m \geq m_0,\ i \geq 1,$$

$$\Rightarrow f_m \to f \text{ in the metric } s$$

$$\Rightarrow s \subset p.$$

Similarly $p \supset s$, $\Rightarrow p = s$. This completes the proof of the lemma. The convergence under $p(f)$ is equivalent to convergence in every finite vertical strip, for:

LEMMA 2: A sequence $\{f_m\} \subset X$ converges to an $f \epsilon X$ under $p\,(f)$ if and only if it converges to f on every finite rectangle.

Proof: Let $\{f_m\}$ converge to f on every finite rectangle. Then for a given $\varepsilon > 0$, there corresponds an $m_0 = m_0(\varepsilon)$, such that

$$|f_m(s) - f(s)| < \varepsilon,\ m \geq m_0$$

for all s in that rectangle. Then

$$|(a_n^{(m)} - a_n)\, e^{\lambda_n R1(s)}| < \varepsilon,\ m \geq m_0\,,\ n \geq 1$$

$$\Rightarrow |a_n^{(m)} - a_n| < \varepsilon\, e^{-\lambda_n R1(s)},\ m \geq m_0,\ n \geq 1$$

$$\text{or, } p\,(f_m - f) < \varepsilon,\ m \geq m_0. \tag{3.3}$$

Let now (3.3) hold. Then for all $n \geq 1$

$$|a_n^{(m)} - a_n| < \varepsilon^{\lambda_n},\ m \geq m_0$$

or, $$|a_n^{(m)} - a_n| < \varepsilon^{-\lambda_n k},\ k \text{ is arbitrarily large.}$$

Therefore for $m \geq m_0$ and s in a finite rectangle

$$|f_m(s) - f(s)| < \sum_{n \geq 1} \exp\,\{(R1\,(s) - k)\,\lambda_n\}$$

$$< e^{-\frac{k\lambda 1}{2}} \sum_{n \geq 1} \exp\left\{\{R1\,(s) - \frac{k}{2}\}\,\lambda_n\right\}$$

$$= O\,(1)\, \exp\left(\frac{-k\,\lambda_1}{2}\right),\ \text{(Since } k \text{ is arbitrarily large).}$$

Consequently, if $\varepsilon > 0$ is given in advance, then we can determine an $m_0\,(\varepsilon)$, such that for all $m \geq m_0$, $|f_m(s) - f(s)| < \varepsilon$, for all s in a finite rectangle. Thus $f_m \to f$ uniformly on every finite rectangle. This completes the proof.

We now show the completeness of the space X under various topologies established above. Observe that it is sufficient to establish the completeness under one of them, in view of Lemma 1.

LEMMA 3: The space X_s is complete.

Proof: Let $\{f_p\}$ be a s-Cauchy sequence in X. Then to a given $1 > \varepsilon > 0$, there exists a $Q = Q(\varepsilon)$, such that

$$\sum_{k \geq 1} \frac{1}{2^k} \frac{\| f_p - f_q \|_k}{1 + \| f_p - f_q \|_k} < \varepsilon, \text{ for all } p, q \geq Q.$$

Then

$$| a_m^{(p)} - a_n^{(q)} |^{1/\lambda_n} < \varepsilon_1 ; \; p, q \geq Q, \; n \leq k, \; k \geq 1..$$

Obviously ε_1 is related with ε and $\varepsilon_1 \to 0$ as $\varepsilon \to 0$.

Hence the sequence $\{a_n^p\}$ in p tends to a_n for each $n \geq 1$. Now

$$\begin{aligned} | a_n | &\leq | a_n^{(p)} - a_n | + | a_n^{(p)} | \\ &\leq \varepsilon_1 + | a_n^{(p)} | ; \; p \geq Q, \; n \geq 1 \\ &\leq \varepsilon_1 + e^{-k\lambda_n}, \; n \geq n_0(k) \\ \Rightarrow \; &| a_n |^{1/\lambda_n} \leq e^{-k}, \; n \geq n_0(k). \end{aligned}$$

Thus if

$$f(s) = \sum_{n \geq 1} a_n e^{s\lambda_n},$$

then f is an entire function. Hence

$$\| f_p - f \|_i = \sup_{n \leq i} | a_n^{(p)} - a_n |^{1/\lambda_n} < \varepsilon, \text{ all } p \geq Q.$$

Hence

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\| f_p - f \|_i}{1 + \| f_p - f \|_i} < \frac{\varepsilon}{1 + \varepsilon}, \; p \geq Q,$$

where $f \in X$. The result is, therefore, proved.

COROLLARY 1: The spaces X_d, X_s, X_p, and the space X endowed with the compact open topology (as in Lemma 2) are FRÉCHET spaces and consequently are barrelled spaces (see [2], p. 19).

COROLLARY 2: X_s is a separable FRÈCHET space.

Proof— Since the exponential polynomials with rational coefficients are dense in X_s, the corollary follows.

COROLLARY 3: Every $\sigma(X'_s, X_s)$ — bounded sequence in X'_s contains a convergent subsequence (see [3]).

COROLLARY 4: Every $\sigma(X'_s, X_s)$ — bounded set in X'_s is relatively sequentially compact.

Proposition 1— The dual of X_p is the inductive limit of X'_i, where each X_i is endowed with norm topology $\| \ldots \|_{r_i}$.

Proof— Since $\| f \|_{r_i} \leq \| f \|_{r_{i+1}}$ for any entire function $f \in X$ and each $i \geq 1$, it follows that $X'_i \subset X'_{i+1}$.

Hence $\bigcup\limits_{i=1}^{\infty} X'_i$ is a linear space which is the limit of X'_i's.

Hence by a well known result ([1], p. 218)

$$X'_p = \bigcup_{i=1}^{\infty} X'_i.$$

Proposition 2— The space X_p is a MONTEL Space.

Proof— For the definition of a MONTEL space, see for instance P. 32 [2]. In view of remark 2 following lemma 2, it is sufficient to prove that each uniformly bounded subset $F \subset X$ on a finite rectangle is equi-continuous. So let D be a strip of width Δ and length T. Let $\delta > 0$ be an arbitrary fixed number. Let D' denote the rectangle of width $\Delta + 2\delta$ and length $T + 2\delta$.

Clearly $D \subset D'$. Now by hypothesis

$$| f(s) | \leq k_D$$

for all $f \in F$ and $s \in D$, where k_D is a constant depending on D.

Let $s \in D$ and γ be the circle with centre at s and radius δ.

Then $\gamma \subset D$ and

$$|f'(s)| = \left| \frac{1}{2\pi_i} \int_\gamma \frac{f(\omega)\, d\omega}{(\omega - s)^2} \right| \leq \frac{1}{2\pi\delta^2} \int_\gamma |f(\omega)|\, |d\omega|$$

$$\leq \frac{k_{D'}}{\delta}, \; f \in F.$$

Thus for s_1, $s_2 \in D$

$$|f(s_1) - f(s_2)| = |(f(s) + \varepsilon)(s_1 - s_2)|$$

$$\leq \frac{(k_{D'} + \varepsilon)}{\delta} |s_1 - s_2|, \text{ for all } f \in F.$$

This shows that F is equi-continuous. Now by a well-known argument we can select a subsequence of F which converges uniformly on D to a function f. But each member of this sequence is analytic in D and so is f (by Weierstrass Theorem) and since D is an arbitrary strip it follows that f is entire and is representable by a Dirichlet Series since X_p is complete (See lemmas 2 and 3).

4. *Functions of finite order and type*

In the preceding results, the entire functions of finite proximate order and proximate type do not occupy a privileged position. It is, therefore, the purpose of this section to throw some light on the space of entire Dirichlet functions having finite proximate order and proximate type (for definitions of proximate order and type see [4]).

Let now X be the space of those entire Dirichlet functions f,

f being represented by (2.1), for which

$$\overline{\lim_{\sigma \to \infty}} \frac{\log m(\sigma)}{\sigma^{\sigma\varrho(\sigma)}} \leq A < \infty, \tag{4.1}$$

where $z(\sigma)$ is the proximate function satisfying the conditions

(i) $\varrho(\sigma) \to \varrho$ as $\sigma \to \infty$, $0 < \varrho < \infty$

(ii) $$\sigma \varrho'(\sigma) \to 0 \text{ as } \sigma \to \infty;$$

and that

$$m(\sigma) = \sup_{-\infty < t < \infty} |f(\sigma + it)|.$$

The following result is known [4], namely (4.1) implies and is implied by

$$\overline{\lim_{n\to\infty}} \varphi(\lambda_n) |a_n|^{1/\lambda_n} \leq (A \varrho e)^{1/\varrho}, \tag{4.2}$$

where $\varphi(t)$ is the single solution (for $t < t_0$) of the equation

$$t = e.\ \varrho(\log \varphi) \log \varphi$$

Let now for each $f \in X$,

$$||f||_q = \sum_{n\geq 1} |a_n| \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n}$$

where $q = 1, 2, 3, \ldots$ Clearly $||f||_r$ exists on account of (4.2) and represents a norm on X for each $q \geq 1$. We further note that $q_1 \geq q_2 \Rightarrow ||f||\, q_2 \to ||f||_{q_1} \geq ||f||_{q_2}$. As before the family $\{ ||f||_q : q = 1, 2, \ldots \}$ induces on X a unique topology such that X becomes a locally convex topological vector space and this topology is given by the metric λ, where

$$\lambda(f, g) = \sum_{q\geq 1} \frac{1}{2^q} \frac{||f-g||_q}{1 + ||f-g||_q}. \tag{4.3}$$

We write this space as X_λ. Then one has the following

THEOREM 1: The space X_λ is a FRÉCHET space.

Proof— It is sufficient to show that X_λ is complete. Let therefore $\{f_\alpha\}$ be a λ-CAUCHY Sequence in X and so for a given $\varepsilon > 0$ there corresponds an $m_0 = m_0(\varepsilon)$, such that $||f_\alpha - f_\beta||_q\ \varepsilon$ for all

$\alpha, \beta \geq m_0$ and all $q \geq 1$; consequently for these values of α, β and q, we have

$$(4.4) \qquad \sum_{n \geq 1} | a_n^{(\alpha)} - a_n^{(\beta)} | \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n} < \varepsilon,$$

and this shows that $\{ a_n^{(\alpha)} \}$ is a Cauchy sequence in the complex plane for each $n \geq 1$ and therefore $a_n^{(\alpha)} \to a_n$ as $\alpha \to \infty$, $(n \geq 1)$ and so letting $\beta \to \infty$ in (4.4) one has for all $\alpha \geq m_0$

$$(4.5) \qquad \sum_{n \geq 1} | \alpha_n^{(\alpha)} - a_n | \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n} \leq \varepsilon,$$

and consequently taking $\alpha = m_0$ in (4.5) we get for a fixed q

$$| a_n | \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n} \leq |a_n^{(m_0)}| \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n} + \varepsilon$$

$$\leq \left\{ \frac{A + \frac{1}{p}}{A + \frac{1}{q}} \right\}^{\lambda_n} + \varepsilon,$$

where $q < p$ and n is sufficiently large; and as ε is arbitrary, one finds the *an* satisfies (4.2) and so f, with its representation as in (2.1), belongs to X_λ. Hence using (4.5) again, we see that

$$\lambda(f_\alpha, f) < \varepsilon, \text{ for all } \alpha \geq m_0,$$

where $f \in X_\lambda$ and the result is proved.

5. *Characterisation of continuous linear functionals on X_λ:*

This section is devoted to characterising the forms of linear continuous functionals on the space X_λ.

Precisely, we have the following:

THEOREM 2: A continuous linear functional x on X_λ is of the form

$$x(f) = \sum_{n \geq 1} a_n c_n ; \; f = f(s) = \sum_{n \geq 1} a_n e^{s\lambda_n}$$

if and only if

$$| c_n | \leq K \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) e \varrho \right)^{1/\varrho}} \right\}^{\lambda_n},$$

for all $n \geq 1$, $q \geq 1$, where K is a finite positive number and λ_1 is sufficiently large.

Proof: Let $x \in X'_\lambda$, clearly means if $\alpha_m \to \alpha$ in X_λ, then $x(\alpha_m) \to x(\alpha)$. Now Let

$$\alpha(s) = \sum_{n \geq 1} a_n e^{s\lambda_n},$$

where a_n satisfies (4.2). Suppose

$$\mathcal{L}_m(s) = \sum_{n=1}^{M} a_n e^{s\lambda_n}$$

then we claim that $a_m \to \alpha$ in X_λ (observe that $\alpha_m \in X_\lambda$). To ascertain this, it is sufficient to prove that $\alpha_m \to \alpha$ as $m \to \infty$ under the norm $|| \ldots ||_q$ for each $q \geq 1$. Let q be a fixed integer. Choose ε such that $0 < \varepsilon < \frac{1}{q}$.

Then we can determine an integer $m = m(\varepsilon)$ such that

$$| a_n | \leq \left\{ \frac{((A + \varepsilon) \varrho e)^{1/\varrho}}{\varphi(\lambda_n)} \right\}^{\lambda n}, \text{ for all } n \geq m + 1, \text{ (from 4,2)},$$

and it follows that

$$\left\| a - \sum_{n=1}^{m} a_n e^{s\lambda_n} \right\|_q \leq \sum_{n \geq m+1} \left\{ \frac{A + \varepsilon}{A + \frac{1}{q}} \right\}^{\lambda_n/\varrho}$$

$< \varepsilon$, for sufficiently large m, and this ascertains our claim. Hence from the continuity of x, $\lim_{m\to\infty} x(\alpha_m) = x(\alpha)$ in the topology ind uced by λ.

Note that

$$x(\alpha_m) = \sum_{n=1}^{m} a_n\, x(e^{s\lambda_n}).$$

Let

$$c_n = x(e^{s\lambda_n}),$$

then $|c_n| = x(e^{s\lambda_n})|$. Since x is continuous on $X_{\|\dots\|_q}$ for each $q \geq 1$, there exists a $K > 0$ (independent of q)

$$|c_n| \leq K\, \|f\|_q,\ (q \geq 1),$$

where f is given by $f(s) = e^{s\lambda_n}$ and so using the definition of the norm $\|f\|_q$, we get

$$|c_n| \leq K \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q})\, \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n} q \geq 1, n \geq 1. \tag{5.1}$$

Hence

$$x(\alpha) = \sum_{n\geq 1} a_n c_n$$

where c_n satisfies (5.1).

To prove the other part, let now c_n satisfy (5.1), then we have

$$x(\alpha) \leq K \sum_{n\geq 1} |a_n| \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q})\, \varrho e \right)^{1/\varrho}} \right\}^{\lambda_n},\ q \geq 1$$

and so

$$|x(\alpha)| \leq K\, \|\alpha\|_q,\ q \geq 1$$

and therefore $x \in X'\ \|\dots\|_q$ $(q \geq 1)$ and from (2.7) $x \in X'_\lambda$.

This completes the proof of Theorem 2.

6. *Construction of total sets in* X_λ.

Definition— If X is a locally convex topological vector space, then a set $E \subset X$ is said to be *total* if and only if $x \in X'$, such that $x(E) = \{0\}$ implies that $x \equiv 0$.

We give in this last section a method of constructing total sets in X_λ.

Proposition 3— Consider the space X_λ of Section 5. Let $f \in X$ and that

$$f(s) = \sum_{n \geq 1} a_n e^{s\lambda_n}, \ a_n \neq 0 \text{ for } n \geq 1.$$

Suppose G is a subset of complex plane which has at least one finite limit point. Define f_μ by

$$f_\mu(s) = \sum_{n \geq 1} a_n e^{(s+\mu)\lambda_n} = \sum_{n \geq 1} (a_n e^{\mu\lambda_n}) e^{s\lambda_n}.$$

Let $E = \{f_\mu : \mu \in G\}$. Then E is total in X_λ.

Proof: Is sesily follows that on account of (2.4) f is an entire function represented by Dirichlet series; further

$$\overline{\lim_{n \to \infty}} \varphi(\lambda_n) \mid a_n e^{\mu\lambda_n} \mid^{1/\lambda_n} \leq (A \varrho e)^{1/\varrho} e^{R1\mu},$$

and so

$$\overline{\lim_{\sigma \to \infty}} \frac{\log M_\mu(\sigma)}{e^{\sigma\varrho(\sigma)}} \leq A \exp\{\varrho R1(\mu)\} < \infty,$$

where $M_\mu(\sigma) = \sup_{-\infty < t < \infty} |f_\mu(s)|$, and therefore $f_\mu \in X_\lambda$. Let now x be a continuous linear functional on X_λ, such that $x(f_\mu) = 0$. Then there exists a sequence $\{c_n\}$, such that (see Theorem 2)

$$x(g) = \sum_{n \geq 1} a_n c_n, \ g(s) = \sum_{n \geq 1} a_n e^{s\lambda_n}$$

where

$$| c_n | \leq K \left\{ \frac{\varphi(\lambda_n)}{\left((A + \frac{1}{q}) \varrho e \right)^{1/\varrho}} \right\}^{1/\lambda_n}, n \geq 1, q \geq 1$$

and where further K is a constant and λ_1 is sufficiently large.

Now suppose that for every $\mu \in G$

$$x(f_\mu) = \sum_{n \geq 1} a_n c_n e^{\mu \lambda_n} = 0.$$

Next consider the function L, where

$$L(s) = \sum_{n \geq 1} a_n c_n e^{s \lambda_n},$$

then ([4], p. 277) for large n

$$\frac{1}{\varrho(\sigma)} \log \left[\frac{\lambda_n}{A_1 \varrho} \right] = \log \varphi(\lambda_n)$$

and consequently

$$\frac{\log | a_n c_n |}{\lambda_n} \to - \infty \text{ as } n \to \infty,$$

and so L is an entire function represented by Dirichlet series.

Furthermore,

$$\varphi(\lambda_n) | a_n c_n |^{1/\lambda_n} \leq (A \varrho e)^{1/\varrho} + \varepsilon) (1 + 0(1)), \text{ for all } n \geq n_0$$

and consequently $L \in X_\lambda$. Also $L(\mu) = 0$ for each $\mu \in G$. This shows that $L(s) = 0$ for all s in the complex plane. Hence $a_n c_n = 0$ for all $n \geq 1$ and as $a_n \neq 0$ for $n \geq 1$, $\Rightarrow c_n = 0$ for $n \geq 1$. Hence $x \equiv 0$. This completes the proof.

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