GROWTH OF A MEROMORPHIC FUNCTION

by

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(Dedicated to the memory of late Professor F. S. Balaguer)

1 Let $f(z)$ be a meromorphic function in the plane and $n(r, f)$ be thenumber of poles of $f(z)$ in $|z| \leq r$, and let

$$N(r, f) = \int_0^r \frac{n(x, f)}{x} \, dx .$$

In this paper I wish to investigate a few results on the growth of the NEVALINA characteristic function $T(r, f)$ with certain other functions like $N(r, f)$, $N(r, 1/f)$ and $M(r)$, the maximum of the modulus of $f(z)$, whereas in the last article $f(z)$ is taken to be an entire function. The results will be clear from the context. First I start with an alternative proof of the following result given in Hayman's Meromorphic Functions ([2], p. 101):

2. THEOREM A: Let $f(z)$ be meromorphic in the plane and of finite non-integral order ϱ. Then

$$\overline{\lim_{r\to\infty}} \frac{N(r, f) + N(r, f)}{T(r, f)} \geq \frac{(p + 1 - \varrho)\ (\varrho - p)}{2\varrho\ (p + 1)\ (2 + log\ (p + 1))}$$

where $p = [\varrho]$, the integral part of ϱ and less than ϱ.

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REMARK: Theorem A reduces to a result of Shah [5], when $0 < \varrho < 1$.

PROOF OF THEOREM A: We have the representation of $f(z)$ given by

$$f(z) = z^p \, exp \, \{p\,(z)\} \frac{\prod_{\mu} E\,(p,\ z/a_\mu)}{\prod_{\nu} E\,(p,\ z/b_\nu)},$$

where $E\,(p,\ z)$ is the Weierstrass well-known primary factors, $a_\mu\ (\mu = 1, 2 \ldots)$ are the zeros and $b_\nu\ (\nu = 1, 2, \ldots)$ are the poles of $f(z)$ in the plane ($|a_1| \leq |a_2| \leq \ldots \leq |a_\mu| \leq \ldots, |a_\mu| \to \infty$ with μ; $|b_1| \leq |b_2| \leq \ldots \leq |b_\nu| \to \infty$ with ν), $P\,(z)$ being the polynomial of degree at the most p. Therefore

$$(2.1)\quad T\,(r, f) \leq A\,(p) \left\{ r^p \int_0^r \frac{n\,(x)}{x^{p+1}}\, dx + r^{p+1} \int_r^\infty \frac{n\,(x)}{x^{p+2}}\, dx \right\} + O\,(r^p) + O\,(log\, r),$$

where

$$n\,(x) = n\,(x, f) - n\,(o, f) + n\,(x, 1/f) - n\,(o, 1/f);$$

$$A\,(p) = 1,\ if\ p = o;\ A\,(p) = 2\,(p+1)\,(2 + log\,(p+1)),\ if\ p \geq 1.$$

Now we have $n\,(x)\,dx = xd\,N\,(x)$ where $N\,(x) = N\,(x, f) + N\,(x, 1/f)$ and so from (2.1) we find

$$(2.2)\quad T\,(r, f) \leq A\,(p) \left\{ p\,r^p \int_0^r \frac{N\,(x)}{x^{p+1}}\, dx + (p+1)\, r^{p+1} \int_r^\infty \frac{N\,(x)}{x^{p+2}}\, dx \right\} + O\,(r^p).$$

We assert that

$$(2.3)\qquad A = \varrho,$$

where $A = \lim_{r\to\infty} \log N\,(r)/\log r$, for if $A < \varrho$ (obviously $A \not> \varrho$) then from (2.2) one has for all $r \geq r_0$, the inequality

$$T\,(r, f) < Kr^\lambda;\ \lambda < \varrho,\ K \text{ is a constant},$$

and so

$$\overline{\lim_{r\to\infty}} \frac{\log\, T\,(r, f)}{\log\, r} = \lambda < \varrho,$$

and therefore a contradiction and hence (2.3) holds. We notice that* $N(r)$ is non-decreasing and continuous for $r > o$ and so we can construct the proximate order $\varrho(r)$ in terms of $N(r)$, see for instance Theorem 16 ([3], p. 52), to satisfy the following conditions:

(i) $\quad \overline{\lim\limits_{r\to\infty}}\ \varrho(r) = \varrho$; (ii) $N(r) \leq r^{\varrho(r)}$, for all $r \geq r_0$;

(iii) $N(r) \doteq r^{\varrho(r)}$ $r = r_n$ $(n \geq 1) \to \infty$ with n; (iv) $\lim\limits_{r\to\infty} r\varrho'(r) \log r = o$.

Making use of the relations (i) — (iv), we find that

$$\int_0^r \frac{N(x)}{x^{p+1}}\, dx \leq O(1) + \int_{r_0}^r x^{\varrho(x)-p-1}\, dx$$

$$\sim \frac{r^{\varrho(r)}}{\varrho - p} = \frac{N(r)}{\varrho - p}, \quad (r = r_n);$$

also

$$\int_r^\infty \frac{N(x)}{x^{p+2}}\, dx \leq \int_r^\infty x^{\varrho(x)-p-2}\, dx \sim \frac{r^{\varrho(r)}}{p+1-\varrho} = \frac{N(r)}{p+1-\varrho}, \ (r = r_n).$$

Therefore from (2.2), we have for $\gamma = \gamma_1$ the inewuality (valid for large n)

$$T(r, f) \leq A(p) \left\{ \frac{p}{\varrho - p} + \frac{p+1}{p+1-\varrho} \right\} N(r) + O(r^p), \ r = r_n \ (n \to \infty) \tag{2.4}$$

Now one easily verifies that $N(r)/r^{\varrho} \to \infty$ as $r = r_n \to \infty$ and so (2.4) leads to the required result.

Next, I state and prove a result of somewhat different nature (see remark 2)

THEOREM B: Let $f(z)$ be meromorphic in the plane, having a finite non-integral order ϱ, then

$$\overline{\lim_{r\to\infty}}\ \frac{T(r, f)}{\{N_p(r, f) + N_p(r, 1/f)\}\, r^p} \leq \frac{2(p+1)(2 + \log(p+1))}{p+1-\varrho},$$

* The integral representation of $N(r)$ suggests that $N(r)$ is a convex function of *log* r, since $n(r,f) + n(r, 1/f)$ is a step function and tends to infinity with r, and a convex function is non-decreasing and continuous.

where

$$N_p(r, f) = \int_0^r \frac{n(x, f) - n(o, f)}{x^{p+1}} dx,$$

and a similar expression for $N_p(r, 1/f)$ and that the other terms involved in stand as in the previous theorem.

PROOF: Let

$$N_p(r) = N_p(r, f) + N_p(r, 1/f) = \int_0^r \frac{n(x)}{x^{p+1}} dx.$$

Then

$$r^p \int_0^r \frac{n(x)}{x^{p+1}} dx + r^{p+1} \int_r^\infty \frac{n(x)}{x^{p+2}} dx = r^p N_p(r) + r^{p+1} \int_r^\infty \frac{d\, N_p(x)}{x}$$

$$= r^p N_p(r) + r^{p+1} \left\{ \left[\frac{N_p(x)}{x} \right]_r^\infty + \int_r^\infty \frac{N_p(x)}{x^2} dx \right\}$$

$$= r^{p+1} \int_r^\infty \frac{N_p(x)}{x^2} dx,$$

since for large x

$$\frac{N_p(x)}{x} \leq 0(1) + x^{-1} \int_{x_0}^x u^{\varrho+\varepsilon-p-1}\, du = 0(1) + \frac{x^{\varrho+\varepsilon-p-1}}{\varrho+\varepsilon-p} \to 0$$

as $x \to \infty$ $(\varrho < p+1)$. Therefore using (2.1) we get

$$T(r, f) \leq A(p)\, r^{p+1} \int_r^\infty \frac{N_p(x)}{x^2} dx + O(r^p) + O(\log r).$$

Let $\delta > 0$, then

$$N_p(x) = 0(x^{\varrho+\delta-p}), \quad (x \to \infty)$$

and therefore there exisrs a sequence $\{R_n\}$, $R_n \to \infty$ with n, such that

$$N_p(x)\, R_n^{\varrho+\delta-p} \leq N_p(R_n)\, x^{\varrho+\delta-p}, \ (x > R_n).$$

Consequently

$$T(R_n, f) \leq \frac{A(p)\, R_n^{p+1}}{R_n^{\varrho+\delta-p}} \int_{Rn}^{\infty} N_p(R_n)\, x^{\varrho+\delta-p-2}\, dx + O(R_n^p) + O(\log R_n)$$

$$= \frac{(1+0(1))\, A(p)\, N_p(R_n)\, R_n^p}{p+1-\varrho-\delta}$$

and as δ is arbitrary, we get

$$\varlimsup_{r\to\infty} \frac{T(r, f)}{r^p\, N_p(r)} \leq \frac{A(p)}{p+1-\varrho}.$$

REMARK 1: If $0 < \varrho < 1$, then $p = 0$ and so $A(p) = 1$ and the above result once again reduces to a result of SHAH [5].

REMARK 2: Neither of the results, Theorems A and B seem to follow from each other. I, however, offer certain possibilities under which Theorem A follows from Theorem B. Infact, we find that

$$N_p(r) = \int_0^r n(x)/x^{p+1}\, dx = \frac{N(x)}{r^p} + p \int_0^r \frac{N(x)}{x^{p+1}}\, dx.$$

But for $\delta > 0$, $\varlimsup_{x\to\infty} N(x)/x^{\varrho-\delta} = \infty$, and so for $x \leq \sigma_n$ ($\sigma_n \to \infty$ with n), we get

$$N(x)/x^{\varrho-\delta} \leq N(\sigma_n)/\sigma_n^{\varrho-\delta}.$$

Therefore

$$N_p(\sigma_n) \leq \frac{N(\sigma_n)}{\sigma_n^p} + p\, \frac{N(\sigma_n)}{\sigma_n^{\varrho-\delta}} \int_0^{\sigma_n} x^{\varrho-\delta-p-1}\, dx$$

$$= \frac{\varrho-\delta}{\varrho-p-\delta}\, \frac{N(\sigma_n)}{\sigma_n^p}.$$

The probability that $\{\sigma_n\}$ coincides with $\{R_n\}$ above for large n or a subsequence $\{\sigma_{n_q}\}$ of $\{\sigma_n\}$ coincides with $\{R_n\}$ for sufficiently large q, gives that

$$\frac{\lim}{r\to\infty}\frac{T(r,f)}{N(r)} \leq \frac{\varrho}{\varrho-p}\,\frac{\lim}{r\to\infty}\frac{T(r,f)}{r^p N_p(r)}$$

$$\leq \frac{\varrho A(p)}{(\varrho-p)(p+1-\varrho)},$$

which is nothing but Theorem 4.5 of HAYMAN [2]. We have for all $x \leq \sigma_n$ (however n large may be) that

$$\frac{1}{N(\sigma_n)} \leq \frac{\varrho-\delta}{p-p-\delta}\,\frac{1}{x^p N_p(x)}$$

$$\leq \frac{\varrho-\delta}{\varrho-p-\delta}\,\frac{A(p)+\delta}{p+1-\varrho}\,(T(R_n,f))^{-1},$$

and so for Theorem A to hold good we should have

$$\lim_{n\to\infty}\frac{T(R_n,f)}{T(\sigma_n,f)} \leq 1,$$

and which is, therefore, another probability. We would certainly prefer the second probability and it is an open question to deal with.

2(ii). To continue the study further in the theory of meromorphic functions, I proceed to find an upper bound for $m(r,f) + m(r,1/f)$, where $f(z)$ is a meromorphic function and

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}\,|f(re^{i\theta})|\,d\theta.$$

Precisely we have:

THEOREM C: Let $f(z)$ be a meromorphic function of non-integral order ϱ and let q be an integer, $q < \varrho < q+1$, then

$$r^{-q}\{m(r,f)+m(r,1/f)\} \leq \int_0^r \frac{n(x)}{x^{q+1}}\,dx + r\int_r^\infty \frac{n(x)}{x^{q+2}}\,dx + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}$$

$$\left\{\frac{1}{r^{2m}}\int_0^r \frac{n(x)}{x^{q-2m+1}}\,dx + r^{2m+2}\int_r^\infty \frac{n(x)}{x^{2m+q+3}}\,dx\right\} + O(1).$$

PROOF: Since $f(z)$ is of non-integral order, therefore

$$f(z) = z^m \exp(P(z)) \prod_a E(z/a) / \prod_b E(z/b),$$

where $P(z)$ is a polynomial of degree $\leq q$; $q = [\varrho]$; $E(u) = E(u, q)$ is the Weierstrass primary factor, a's and b's are the zeros and poles of $f(z)$. Let $\{d\}$ be the sequence composed of zeros and poles of $f(z)$, such that $0 < |d_1| \leq |d_2| \leq \ldots$, Also

$$\sum_{|dn| \leq r} 1 = n(r) = n(r, 0) + n(r, \infty).$$

Now

$$m(r, f) + m(r, 1/f) \leq \sum_{n=1}^{\infty}\left\{m(r, E(z/d_n)) + m(r, 1/E(z/d_n))\right\} + O(r^q),$$

and now following EDREI and FUCHS ([1], p. 300), we have:

$$m(r, f) + m(r, 1/f) \leq r^q \sum_{n=1}^{\infty}\int_{r_n}^{\infty} t^{-q-1}\varphi(t/r)\,dt + O(r^q),$$

where

$$\varphi(t/r) \leq \frac{r}{t+r}\left\{1 + \frac{2}{\pi}\log\left|\frac{1+t/r}{1-t/r}\right|\right\}.$$

Then

$$m(r, f) + m(r, 1/f) \leq r^q \sum_{r_n < r}\int_{r_n}^{\infty} t^{-q-1}\varphi(t/r)\,dt + r^q \sum_{r_n > r}\int_{r_n}^{\infty} t^{-q-1}\varphi(t/r)dt + O(r^q). \tag{2.5}$$

Now

$$\sum_{rn<r}\int_{r_n}^{\infty} t^{-q-1}\varphi(t/r)\,dt \leq \sum_{r_n<r}\int_{r_n}^{r} t^{-q-1}\left[1 + \frac{Br}{t}\log\left\{\frac{1+t/r}{1-t/r}\right\}\right] dt \tag{2.6}$$

$$+ \sum_{r_n<r}\int_{r}^{\infty} t^{-q-1}\left[\frac{r}{t} + \frac{Br}{t}\log\left\{\frac{1+r/t}{1+r/t}\right\}\right] dt$$

$$= \Sigma_1 + \Sigma_2, \text{ (say)}, \quad B = 2/\pi.$$

But

$$\Sigma_1 = \sum_{r_n<r} \frac{1}{q}(r_n^{-q} - r^{-q}) + \sum_{m=0}^{\infty} \frac{2B}{2m+1} \sum_{r_n<r} \int_{r_n}^{r} \left(\frac{t}{r}\right)^{2m} t^{-q-1}\, dt$$

$$(2.7) \qquad = \int_0^r \frac{n(x)}{x^{q+1}}\, dx + 2B \sum_{m=0}^{\infty} \frac{1}{(2m+1)\, r^{2m}} \int_0^r \frac{n(x)}{x^{q+1-2m}}\, dx.$$

Also

$$\Sigma_2 = \sum_{r_n<r} \int_r^{\infty} t^{-q-1} \left\{\frac{r}{t} + \frac{Br}{t} \log\left\{\frac{1+t/r}{1-t/r}\right\}\right\} dt$$

$$= \sum_{r_n<r} \int_r^{\infty} t^{-q-2}\, dt + Br \sum_{r_n<r} 2 \sum_{m=0} \frac{1}{2m+1} \int_r^{\infty} \left(\frac{r}{t}\right)^{2m+1} t^{-q-2}\, dt$$

$$= \frac{n(r)}{(q+1)\, r^q} + 2Br \sum_{m=0}^{\infty} \frac{r^{2m+1}}{2m+1} \sum_{r_n<r} \int_r^{\infty} t^{-2m-3-q}\, dt$$

$$(2.8) \qquad = \frac{n(r)}{(q+1)\, r^q} + \frac{2B}{r^q} \sum_{m=0}^{\infty} \frac{n(r)}{(2m+1)(2m+q+2)}.$$

Considering the second term, that is the sum involving $r_n < r$, we have:

$$\sum_{r_n<r} \int_{r_n}^{\infty} t^{-q-1}\, \varphi(r/t)\, dt \le \sum_{r_n>r} \int_{r_n}^{\infty} t^{-q-1} \left\{\frac{r}{t} + \frac{Br}{t} \log\left|\frac{1+t/r}{1-t/r}\right|\right\} dt$$

$$= \frac{r}{q+1} \sum_{r_n>r} r_n^{-q-1} + 2B \sum_{m=0}^{\infty} \frac{r^{2m+2}}{2m+1} \sum_{r_n>r} \int_{r_n}^{\infty} t^{-2m-q-3}\, dt$$

$$= \frac{r}{q+1} \int_r^{\infty} x^{-q-1}\, dn(x) + 2B \sum_{m=0}^{\infty} \frac{r^{2m+2}}{(2m+1)(2m+q+2)} \int_r^{\infty} x^{-2m-q-2}\, dn(x)$$

$$= -\frac{n(r)\, r^{-q}}{q+1} + r \int_r^{\infty} \frac{n(x)}{x^{q+2}}\, dx - 2B \sum_{m=0}^{\infty} \frac{r^{-q}\, n(r)}{(2m+1)(2m+q+2)} +$$

$$(2.9) \qquad + 2B \sum_{m=0}^{\infty} \frac{r^{2m+2}}{2m+1} \int_r^{\infty} \frac{n(x)}{x^{2m+q+3}}\, dx.$$

Therefore from (2.6) - (2.8) and (2.9), we get

$$\sum_{r_n<r}\int_{r_n}^{\infty} t^{-q-1}\varphi(r/t)\,dt+\sum_{r_n>r}\int_{r_n}^{\infty} t^{-q-1}\varphi(r/t)\,dt\leq\int_0^r\frac{n(x)}{x^{q+1}}dx+r\int_r^{\infty}\frac{n(x)}{x^{q+2}}dx$$

$$+2B\sum_{m=0}^{\infty}\left\{\frac{r^{-2m}}{2m+1}\int_0^r\frac{n(x)}{x^{q+1-2m}}dx+\frac{r^{2m+2}}{2m+1}\int_r^{\infty}\frac{n(x)}{x^{2m+q+3}}dx.\right\}.$$

The result now follows from the preceding inequality and (2.5).

I shall deal with the various applications of this theorem in a next sequel of my work. The following is of independent interest.

THEOREM D: Let $f(z)$ be meromorphic, of finite non-zero order. Let

$$M(r)=\max_{|z|=r}|f(z)|.$$

Then

$$\underline{\lim}_{r\to\infty}\frac{1}{r\,T(r)\,\varphi(r)}\int_0^r\log M(x)\,dx=0,$$

for all functions, $\varphi(r)\to\infty$ as $r\to\infty$, howerer increasing slowly.

PROOF: We have ([4], p. 26):

$$\frac{1}{r}\int_0^r\log M(x)\,dx\leq C(k)\,T(kr),\ k>1$$

Since $T(r)$ is continuous, non-decreasing for $r>0$ and so a proximate order for $T(r)$ can be constructed, satisfying the conditions (i)-(iii) in § 2, with $N(r)$ replaced by $T(r)$. Now

$$T(kr)\leq(kr)^{\varrho(kr)},\ r\geq r_0$$

$$=k^{\varrho(kr)}\,r^{\varrho(r)}\exp\{(\varrho(kr)-\varrho(r))\log r\}.$$

But,

$$\varrho\ (kr) - \varrho\ (r) = \int_r^{kr} \varrho'(x)\ dx \leq \varepsilon \int_r^{kr} \frac{dx}{x \log x} \leq \frac{\varepsilon}{\log r} \int_r^{kr} \frac{dx}{x}$$

$$\leq \frac{\varepsilon \log k}{\log r}$$

Therefore

$$T\ (kr) \leq k^{\varrho'kr)}\ r^{\varrho(r)} \exp\ (\varepsilon_1),\ \ \varepsilon_1 = \varepsilon \log k$$

$$\sim k^{\varrho(kr)}\ T\ (r),\ \ r = r_n,\ \ r_n \to \infty\ .$$

Therefore

$$\frac{1}{r} \int_0^r \log\ M\ (x)\ dx \leq C\ (k)\ k^{\varrho}\ T\ (r)\ ,$$

for arbitrarily large values of $r \to \infty$. This leads to the desired result.

3. ENTIRE FUNCTIONS: GROWTH OF $T\ (r, f)$:

Suppose now $f\ (z)$ is an entire function and $T\ (r,\ f)$ be the NEVANLINNA characteristic function corresponding to $f\ (z)$. We write $T\ (r, f)$ as $T\ (r)$. It is a well-known result that

$$(3.1) \qquad T\ (r) \leq \log^+\ M\ (r) \leq \frac{R + r}{R - r}\ T\ (R)\ ;\ 0 \leq r < R,$$

where

$$M\ (r) = \underset{|z| = r}{\mathrm{l.\ u.\ b.}}\ |f\ (z)|\ .$$

For a non-constant entire function T. Shimizu [6] has proved the following result:

$$(3.2) \qquad \lim_{r \to \infty} \frac{\log\ M\ (r)}{T\ (r)\ \{\log T\ (r)\}^k} = 0,\ \ k < 1.$$

The term $(log\ T\ (r))^k$ in (3.2) does not seem to be a sharp term, for example see the remark 1 immediately following Theorem F. If

$f(z)$ is an entire function of non-zero finite order, this can be largey improved. Precisely. one has the following

THEOREM E: Let $f(z)$ be an entire function of non-zero finite, order, then

$$\lim_{r \to \infty} \frac{\log M(r)}{T(r)\, \Psi(r)} = 0$$

where $\Psi(r) \to \infty$ as $r \to \infty$ is non-decreasing, however slow growth be of $\Psi(r)$ to increase with r.

PROOF: The proof follows on the lines of Theorem D, since if $R = kr$, $k > 1$, then

$$\log M(\gamma) \leq \frac{k+1}{k-1} T(kr).$$

As $T(r)$ is continuous and non-decreasing and so a corresponding proximate order for $T(r)$ exists with the help of which we find that $T(kr) \leq O(1)\, T(r)$, for arbitrarily large values of $r \to \infty$. Hence the result follows.

REMARK 1: In the above theorem, one can choose $\Psi(r)$ to be much smaller than $(\log T(r))^k$, for example let $f(z) = e^z$, then $\log M(r) = r$; $T(r) = r/\pi$, and so $\log M(r)/T(r)\,(\log T(r))^k = \pi/(1 + 0(1))$ $(\log r)^k \to 0$, $\gamma \to \infty$; while $\log M(r)/T(r)\, \Psi(r) = \pi/\Psi(r)$. Let $\Psi(r) = {}$ $= l_k r$, where k is as large as we please but fixed.

REMARK 2: For a class of entire functions of infinite order the result (3.2) can be easily obtained. For, let

$$0 < \overline{\lim_{r \to \infty}} \frac{\log T(r)}{r} < \infty; \quad \overline{\lim_{r \to \infty}} \frac{\log T(r)}{\log r} = \infty. \tag{3.3}$$

If the relation in (3.3) (the first relation) holds, then we have on following Levin ([3] p. 52) that

$$T(r) \leq e^{r\lambda(r)}, \text{ all } r \geq r_0$$

$$T(r_n) = e^{r_n \lambda r_n}, \; r_n \to \infty$$

$$r\,\lambda'(r) \to 0, \; (r \to \infty)$$

and where

$$\lim_{r\to\infty} \lambda(r) = \overline{\lim_{r\to\infty}} \frac{\log T(r)}{r}.$$

Then

$$T(r+\eta) \leq e^{\eta\lambda(r+\eta)+r\lambda(r)}\ e^{r(\lambda(r+\eta)-\lambda(r))},\ \text{all } r \geq r_0$$

$$\leq A\ T(r)\ e^{r(\lambda(r+\eta)-\lambda(r))},\ r = r_n.$$

Also

$$\lambda(r+\eta) - \lambda(r) \leq \int_r^{r+\eta} \frac{\varepsilon\, dx}{x} \leq \frac{\varepsilon\,\eta}{r}.$$

Therefore for $r = r_n$

$$T(r+\eta) \leq A_1\ T(r).$$

Also $r_n = \log T(r_n)/\lambda(r_n)$. Therefore

$$\log M(r_n) \leq A_1 (1 + 0(1)) \frac{r_n}{\eta} T(r_n).$$

Hence for $k > 1$

$$\frac{\log M(r_n)}{T(r)\ \{\log T(r_n)\}^k} \leq A_2 (1 + \alpha\ 1))\ r_n^{1-k} \to 0,\ n \to \infty,$$

and so

$$\underline{\lim_{r\to\infty}} \frac{\log M(r)}{T(r)\ \{\log T(r)\}^k} = 0.$$

Lastly we examine under what circumstances we can prove that $T(r+k) \sim T(r)$, and $\log T(r) \sim \log\log M(r)$, where k is a constant > 0. In the former case we have:

Theorem F: If $f(z)$ is an antire function of order ϱ and lower order λ, such that $\varrho - \lambda < 1$, then for any positive constant k,

$$T(r+k) \sim T(r),\ r \to \infty.$$

PROOF: As $T(r)$ is convex with respect to $\log r$, we find that

$$T(r) = T(r_0) + \int_{r_0}^{r} \frac{\omega(x)}{x}\,dx\,,$$

where $\omega(x)$ is non-decreasing, tending to infihity with x.

Now

$$T(r) < r^{\varrho+\varepsilon},\ \text{all } r \geq r_0 = r_0(\varepsilon)$$

and so for $\mu > 1$,

$$\int_{r}^{\mu r} \frac{\omega(x)}{x}\,dx < O(1)\, r^{\varrho+\varepsilon},\ r \geq r_0.$$

Now for $R > r$

$$T(R) < T(r) + \omega(R) \log \left[1 + \frac{R-r}{r}\right]$$

$$< T(r) + O(1)\, R^{\varrho+\varepsilon} \frac{R-r}{r}\,.$$

Let $R = r + k\ (k > 0)$. Then

$$T(r+k) < T(r) + O(1)\, r^{\varrho+\varepsilon-1}\, k\ (1 + k\, r^{-1})^{\varrho+\varepsilon}$$

and as $T(r) > r^{\lambda-\varepsilon}$, for all $r \geq r_0$, therefore for all sufficiently large r

$$T(r+k) < \{1 + O(1)\,(1 + k\, r^{-1})^{\varrho+\varepsilon}\, r^{\varrho-\lambda+2\varepsilon-1}\}\ T(r)$$

or

$$T(r+k) < (1 + 0(1)\ T(r),\ r \geq r_0\,,$$

and this, when combined with $T(r+k) \geq T(r)$ yields the desired result.

COROLLARY: If $f(z)$ is of regular growth, then $T(r+k) \sim T(r)$ for every positive k.

Coming to the second problem raised just before Theorem F, we note first of all from the example

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}\,,\ \alpha = \frac{1}{k}\,,\ k > \frac{1}{2}$$

that $T(r) \sim \log M(r)$, $r \to \infty$ need not be necessarily true, for in this case

$$\log M(r) \sim r^k; \; T(r) \sim \frac{r^k}{\pi k}.$$

We, therefore, naturally think if *log log* $M(r) \sim$ *log* $T(r)$ is always true. I have been in a position to give a partial answer to this question, namely

THEOREM G: Let $f(z)$ be an entire function of finite order. Then there exists a sequence $\{r_n\}$, $r_n \to \infty$ with n, such that

$$\log \log M(r_n) \sim \log T(r_n), \; n \to \infty.$$

PROOF: Let $k > 1$ Then following the proof of Theorem D

$$T(kr) \geq k^{\varrho(kr)} T(r)$$

for $r = r_n$, $r_n \to \infty$ as $n \to \infty$. Hence from (3.1)

$$\log M(r) \leq \frac{k+1}{k-1} T(kr)$$

or, $$\log M(r_n) \leq \left[\frac{k+1}{k-1}\right] k^{\varrho(kr_n)} T(r_n)$$

or, $$\log \log M(r_n) \leq (1 + 0(1)) \log T(r_n)$$

and using (3.1) we get the result.

REMARK: This seems still an open question if

$$\log \log M(r) \sim \log T(r)$$

as $r \to \infty$ through all large values, holds for entire functions of all orders.

REFERENCES

[1] EFREI, A. & FUCHS, W.H.J.: *On the growth of meromorphic funcions with several deficient values;* Trans. Amer. Math. Soc., 93, n.º 2, (1959), 292.328.

[2] HAYMAN, W.K. *Meromorphic Functions*, Oxford, (1964).

[3] LEVIN, B.JA.: *Raspredelenei Konei Aselykh Funksii* (Russian), Moscow, (1956).

[4] NEVANLINNA, R.: *Théorème de Picard-Borel et la Théorie des Fonctions Meromorphes*, Paris, (1929)

[5] SHAH, S. M.: *A note on meromorphic functions;* The Math. Student, 12, (1944), 67-70.

[6] SHIMIZU, T.: *On the theory of meromorphic functions;* Jap. J. Math., 6, (1929), 119-171.

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