SOME FINITE AND INFINITE INTEGRALS INVOLVING H-FUNCTION AND GAUSS' HYPERGEOMETRIC FUNCTIONS

By

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1. FOX [4, p. 408] introduced the H-function in the form of MELLIN-BARNES type integral as

$$(1.1) \qquad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s\, ds,$$

where x is not equal to zero and empty product is interpreted as unity; p, q, m and n are integers satisfying $1 \leq m \leq q$, $0 \leq n \leq p$; $\alpha_j\,(j=1,\ldots,p)$, $\beta_j\,(j=1,\ldots,q)$ are positive numbers and $a_j\,(j=1,\ldots,p)$; $b_j\,(j=1,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_h-\beta_h s)\,(h=1,\ldots,m)$ concides with any pole of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,\ldots,n)$, i.e.

$$(1.2) \qquad \alpha_i\,(b_h+\nu) \neq (a_i-\eta-1)\,\beta_h$$

$$(\nu,\eta = 0,1,\ldots,;\ h=1,\ldots,m;\ i=1,\ldots,n).$$

Further the contour T runs from $\sigma - i\,\infty$ to $\sigma + i\,\infty$ such that the points:

$$(1.3) \qquad s = \frac{b_h+\nu}{\beta_h}\ (h=1,\ldots,m;\ \nu=0,1,\ldots),$$

which are poles of $\Gamma(b_h-\beta_h\, s)$ $(h=1,\ldots,m)$ lie on the right and the points:

$$(1.4) \qquad s = \frac{(\alpha_i-\eta-1)}{\alpha_i}\ (i=1,\ldots,n;\ \eta=0,1,\ldots,)$$

which are the poles of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,\ldots,m)$ lie to the left of T. Such a contour is possible on account of (1.2). These assumptions for the H-function will be adhered to through-out this paper.

Recently GUPTA and JAIN [5] have denoted (1.1) symbolically by

$$\text{(1.5)} \qquad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

and in a more compact form by

$$\text{(1.6)} \qquad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right],$$

where $\{(f_r, \gamma_r)\}$ stands for set of the parameters $(f_1, \gamma_1), \ldots, (f_r, \gamma_r)$.

Accoding to BRAAKSMA [3, p. 278]

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = 0\,(|\chi|^{\alpha}) \text{ for small } x,$$

where $\sum_1^p \alpha_j - \sum_1^q \beta_j \leq 0$ and $\alpha = \min. \operatorname{Re}\left(\frac{b_h}{\beta_h}\right)$ $(h=1,\ldots,m)$

and

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = 0(|\chi|^{\beta}) \text{ for large } x,$$

where $\sum_1^p \alpha_j - \sum_1^q \beta_j < 0$, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$

and $\beta = \max. \, Re\left(\frac{a_i - 1}{\alpha_i}\right)$ $(i=1,\ldots,n)$.

The object of this paper is to evaluate some finite and in finite integrals, involving product of the H-function and GAUSS' hypergeometric functions. As the H-function is a very general function, we get, on specializing the parameters, many cases, some of which are known and others are balieved to be new.

2. In this section we state the properties and results, given by GUPTA and JAIN [5] which will be used in our present work.

The H-function is symmetric in the pairs $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$ likewise $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$; in $(b_1, \beta_1), \ldots, (b_m, \beta_m)$ and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$.

If one of (a_j, α_j) $(j = 1, \dots, n)$ is the same as one of (b_h, β_h) $(h = m + 1, \dots, q)$ or one of (b_h, β_h) $(h = 1, \dots, m)$ is the same as one of (a_j, α_j) $(j = n + 1, \dots, p)$ then the H-function reduces to one of lower order i.e. each of p, q and n or m decreases by unity.

$$(2.1) \quad \chi^{\sigma} H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}\{(a_p + \sigma\alpha_p, \alpha_p)\}\\ \{(b_q + \sigma\beta_q, \beta_q)\}\end{matrix}\right].$$

$$(2.2) \quad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv H_{q,\,p}^{n,\,m}\left[\frac{1}{x} \,\middle|\, \begin{matrix}\{(1 - b_q, \beta_q)\}\\ \{(1 - a_p, \alpha_p)\}\end{matrix}\right].$$

$$(2.3) \quad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv c H_{p,\,q}^{m,\,n}\left[x^{C} \,\middle|\, \begin{matrix}\{(a_p, c\alpha_p)\}\\ \{(b_q, c\beta_q)\}\end{matrix}\right],$$

where $c > 0$.

$$(2.4) \quad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv (2\pi)^{(1-t)\,(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\mu} \times$$

$$\times H_{tp,\,tq}^{tm,\,tn}\left[x t^{\tau} \,\middle|\, \begin{matrix}\left\{\left(\Delta\,(t, a_p), \dfrac{\alpha_p}{t}\right)\right\}\\ \left\{\left(\Delta\,(t, b_q), \dfrac{\beta_q}{t}\right)\right\}\end{matrix}\right],$$

where μ; τ and $(\Delta\,\{(t, f_r), \gamma_r)\}$ stand for the quantities

$\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q$; $\sum_1^p \alpha_j - \sum_1^q \beta$ and $\left\{\left(\frac{f_r}{t}, \gamma_r\right)\right\}$ $\left\{\left(\frac{f_r+1}{t}, \gamma_r\right)\right\}$, $\dots$, $\left\{\left(\frac{f_r + t - 1}{t}, \gamma_r\right)\right\}$ respectively, provided that t is a positive integer greater than 2.

$$(2.5) \quad H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, 1)\}\\ \{(b_q, 1)\}\end{matrix}\right] \equiv G_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_q\end{matrix}\right].$$

$$(2.6) \quad H_{q+1,\,p}^{p,\,1}\left[x \,\middle|\, \begin{matrix}(1,1), \{(b_q, 1)\}\\ \{(a_p, 1)\}\end{matrix}\right] \equiv E\left(a_1, \dots, a_p; b_1, \dots, b_q; x\right).$$

$$(2.7) \quad H_{0,2}^{2,0}\left[\frac{1}{4}x^2 \,\middle|\, (\tfrac{1}{2}l - \tfrac{1}{2}\nu, 1), (\tfrac{1}{2}l + \tfrac{1}{2}\nu, 1)\right] \equiv 2^{1-l} x^{l} K_{\nu}(x),$$

where $K_{\nu}(x)$ is a modified Bessel function.

$$(2.8) \quad H^{2,0}_{1,2}\left[\varkappa \middle| \begin{matrix} (l-\lambda+1,1) \\ (l+u+\frac{1}{2},1),(l-u+\frac{1}{2},1) \end{matrix}\right] \equiv \varkappa^{l} e^{-\frac{1}{2}\varkappa} W_{\lambda,u}(\varkappa),$$

where $W_{\lambda,u}(x)$ is a whittaker function.

$$(2.9) \quad H^{1,p}_{p,q+1}\left[\begin{matrix} \{(1-a_p,\ \alpha_p)\} \\ (0,1),\{(1-b_q,\beta_q)\} \end{matrix}\right] \equiv \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j r)} \cdot \frac{(-\varkappa)^r}{r!};$$

the above series was studied by WRIGHT [7, p. 287] and has been called as Wright's generalised hypergeometric function and is denoted by the symbol:

$${}_p\Psi_q\left[\begin{matrix} \{(a_p,\ \alpha_p)\} \\ \{(b_q,\ \beta_q)\} \end{matrix};-\varkappa\right].$$

$$(2.10) \quad H^{1,0}_{0,2}\left[\varkappa \middle| (0,1),(-\nu,u)\right] \equiv \sum_{r=0}^{\infty} \frac{(-\varkappa)^r}{r!\,\Gamma(1+\nu+ur)} \equiv J_\nu^u(\varkappa),$$

where $J_\nu^u(x)$ is Maitland's generalised Bessel function [8, p. 257].

$$(2.11) \quad H^{1,1}_{1,1}\left[\varkappa \middle| \begin{matrix} (1-\nu,\ 1) \\ (0,\ 1) \end{matrix}\right] \equiv \Gamma(\nu)\,(1+\varkappa)^{-\nu} \equiv \Gamma(\nu)\,{}_1F_0\,(\nu;-\varkappa).$$

3. In our discussion, because of large number of parameters the notation $\left(\Delta\left(\delta,\ a \pm \begin{vmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{vmatrix}\right),1\right)$ will stand for set of the parameters $(\Delta(\delta,\ a \pm r_1),\ 1),\ldots,(\Delta(\delta,\ a \pm r_n),\ 1)$.

The following integrals will be evaluated in this section:

$$(3.1) \quad \int_0^1 \varkappa^{\varrho-1}(1-\varkappa)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;\varkappa)\, H^{m,n}_{p,q}\left[z\varkappa^{\delta/t} \middle| \begin{matrix} \{(a_p,\ \alpha_p)\} \\ \{(b_q,\ \beta_q)\} \end{matrix}\right] d\varkappa$$

$$= (2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\, \delta^{-\sigma}\, \Gamma(\sigma) \times$$

$$\times \sum_{r=0}^{\infty} \frac{(\alpha)_r\,(\beta)_r}{(\gamma)_r \cdot r!} \times$$

$$\times H^{tm,\,tn+\delta}_{tp+\delta,\,tq+\delta}\left[(zt^{\tau})^{t} \middle| \begin{matrix} (\Delta(\delta,\ 1-\varrho-r),\ 1),\ \{(\Delta(t,\ a_p),\ \alpha_p)\} \\ \{(\Delta(t,\ b_q),\ \beta_q)\},\ (\Delta(\delta,\ 1-\varrho-\sigma-r),\ 1) \end{matrix}\right],$$

where δ and t are positive integers, $Re(\sigma) > 0$, $Re(\gamma + \sigma - \alpha - \beta) > 0$,

$$\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leq 0, \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0, |\arg z| > \tfrac{1}{2} \lambda\pi$$

and $Re\left(\varrho + \frac{\delta}{t} \cdot \frac{b_h}{\beta_h}\right) > 0$ $(h = 1, \ldots, m)$.

(3.2)

$$\int_0^1 \varkappa^{\varrho-1} (1 - \varkappa)^{\sigma-1} {}_2F_1(\alpha, \beta; \gamma; \varkappa) \mathsf{H}_{p,\ q}^{m,\ n}\left[z\left(\frac{\varkappa}{1-\varkappa}\right)^{d/t} \middle| \begin{matrix}\{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}\end{matrix}\right] d\varkappa$$

$$= (2\pi)^{(1-\delta)+(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\ t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\ \delta^{\sigma+\varrho-1} \times$$

$$\times \frac{1}{\Gamma(\varrho + \sigma)} \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r \delta^r}{(\gamma)_r (\sigma + \varrho)_r . r!} \times$$

$$\times \mathsf{H}_{tp+\delta,\ tq+\delta}^{tm+\delta,\ tn+\delta}\left[(zt^{\tau})^t \middle| \begin{matrix}(\Delta(\delta, 1-\varrho-r), \{(\Delta(t, a_p), \alpha_p)\} \\ (\Delta(\delta, \sigma), 1), \{(\Delta(t, b_q), \beta_q)\}\end{matrix}\right],$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leq 0$,

$$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0, |\arg z| < \tfrac{1}{2}\lambda\pi, Re\left(\varrho + \frac{\delta}{t}\frac{b_h}{\beta_h}\right) > 0$$

$$(h = 1, \ldots, m),\ Re\left(\sigma - \frac{\delta}{t} \cdot \frac{a_i - 1}{\alpha_i}\right) > 0 \text{ and}$$

$$Re\left(\gamma + \sigma - \alpha - \beta - \frac{\delta}{t} \cdot \frac{a_i - 1}{\alpha_i}\right) > 0 \quad (i = 1, \ldots, n).$$

(3.3)

$$\int_0^1 \varkappa^{\varrho-1} (1 - \varkappa)^{\sigma-1} {}_2F_1(\alpha, \beta; \gamma; \varkappa) \mathsf{H}_{p,\ q}^{m,\ n}\left[z\varkappa^{\delta}(1-\varkappa)^{\lambda} \middle| \begin{matrix}\{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}\end{matrix}\right] d\varkappa$$

$$= \sqrt{(2\pi)}\ \lambda^{\sigma-\frac{1}{2}}\ \delta^{\varrho-\frac{1}{2}}\ (\lambda + \delta)^{\frac{1}{2}-\varrho-\sigma} \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r \delta^r}{(\gamma)_r (\lambda + \delta)^r . r!} \times$$

$$\times \mathsf{H}_{p+\lambda+\delta,\ q+\lambda+\delta}^{m,\ n+\lambda+\delta}\left[\frac{z\delta^{\delta}\lambda^{\lambda}}{(\lambda + \delta)^{(\lambda+\delta)}} \middle| \begin{matrix}(\Delta(\lambda, 1-\sigma), 1), (\Delta(\delta, 1-\varrho-r), 1), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, (\Delta(\lambda+\delta, 1-\varrho-\sigma-r), 1)\end{matrix}\right],$$

where δ is a positive integer,

$$\sum_1^p \alpha_j - \sum_1^q \beta_j \leq 0,\ \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \mu > 0$$

$$|\arg z| < \tfrac{1}{2}\mu\pi,\ Re\left(\varrho + \delta . \frac{b_h}{\beta_h}\right) > 0,\ Re\left(\sigma + \lambda . \frac{b_h}{\beta_h}\right) > 0 \text{ and}$$

$$Re\left(\gamma + \sigma - \alpha - \beta + \lambda . \frac{b_h}{\beta_h}\right) > 0\ (h = 1, \ldots, m).$$

(3.4)

$$\int_0^1 \varkappa^{\gamma-1} (1-\varkappa)^{\varrho-1} e^{-z\varkappa}\, {}_2F_1 (\alpha, \beta; \gamma; \varkappa)\, \mathbf{H}_{p,\, q}^{m,\, n}\left[\xi (1-\varkappa)^{\delta/t} \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right] d\varkappa$$

$$= (2\pi)^{(1-t)\,(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\, \delta^{-\gamma}\, \sigma^{-z}\, \Gamma(\gamma) \sum_{r=0}^{\infty} \frac{z^r}{r!} \times$$

$$\times\, \mathbf{H}_{tp+2\delta,\, tq+2\delta}^{tm,\, tn+2\delta}\left[(zt^{\tau})^t \middle| \begin{matrix} \left(\Delta\left(\delta, 1-\varrho-r+ \begin{vmatrix} 0 \\ \alpha+\beta-\gamma \end{vmatrix}, 1\right), \{(\Delta(t, a_p), \alpha_p)\} \right. \\ \{(\Delta(t, b_q), \beta_q)\}, \left(\Delta\left(\delta, 1-\varrho-\gamma-r+ \begin{vmatrix} \alpha \\ \beta \end{vmatrix}\right), 1\right) \end{matrix}\right],$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leq 0,$

$$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,\ |\arg \xi| < \tfrac{1}{2}\lambda\pi,\ Re\ (\gamma) > 0,$$

$$Re\left(\varrho + \frac{\delta}{t} . \frac{b_h}{\beta_h}\right) > 0,$$

$$\text{and } Re\left(\gamma + \varrho - \alpha - \beta + \frac{\delta}{t} . \frac{b_h}{\beta_h}\right) > 0\ (h = 1, \ldots, m).$$

(3.5)

$$\int_0^1 \varkappa^{\varrho-1} (1-\varkappa)^{\sigma-1}\, {}_2F_1 (\alpha, \beta; \gamma; z\varkappa)\, \mathbf{H}_{p,\, q}^{m,\, n}\left[\xi \varkappa^{\lambda} (1-\varkappa)^{\delta} \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right] d\varkappa$$

$$= \sqrt{(2\pi)}\ \lambda^{\varrho-\frac{1}{2}}\, \delta^{\sigma-\frac{1}{2}}\, (\lambda+\delta)^{\frac{1}{2}-\varrho-\sigma} \sum_{r=0}^{\infty} \frac{(\alpha)_r\, (\beta)_r}{(\lambda)_r .\ r!} . \left(\frac{\lambda z}{\delta}\right)^r \times$$

$$\times \mathsf{H}^{m,\, n+\lambda+2\delta}_{p+\lambda+2\delta,\, q+\lambda+2\delta}\left[\frac{\xi\,\delta^{\delta}\,\lambda^{\lambda}}{(\lambda+\delta)^{(\lambda+\delta)}}\left|\begin{matrix}\left(\Delta\left(\delta, 1-\left|\begin{matrix}\sigma\\ \sigma\end{matrix}\right|\right), 1\right), (\Delta(\lambda, 1-\varrho-r), 1), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\{, (\Delta(\lambda+\delta, 1-\varrho-\sigma), 1), (\Delta(\delta, 1-\sigma-r), 1)\end{matrix}\right.\right],$$

where δ and λ are positive integers, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv u > 0$,

$|\arg\xi| < \frac{1}{2}u\pi$, $|\arg(1-z)| < \pi$, $Re\left(\varrho + \lambda.\dfrac{b_h}{\beta_h}\right) > 0$, $Re\left(\sigma + \delta.\dfrac{b_h}{\beta_h}\right) > 0$

$(h = 1, \ldots, m)$ and $\sum_1^p \alpha_j - \sum_1^q \beta_j \leq 0$.

(3.6)

$$\int_0^{\infty} \varkappa^{\gamma-1}(\varkappa+z)^{-\sigma}\, {}_2F_1(\alpha, \beta; \gamma; -\varkappa)\, \mathsf{H}^{m,\, n}_{p,\, q}\left[\xi(z+\varkappa)^{\delta/t}\left|\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] d\varkappa$$

$$= (2\pi)^{(1-t)\,(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\, \delta^{-\gamma}\, \Gamma(\gamma) \sum_{r=0}^{\infty} \frac{\delta^r (1-z)^r}{r!} \times$$

$$\times \mathsf{H}^{tm+2\delta,\, tn}_{tp+2\delta,\, tq+2\delta}\left[(\xi t^{\tau})^t\left|\begin{matrix}\{(\Delta(t, a_p), \alpha_p)\}, \left(\Delta\left(\delta. \sigma + \left|\begin{matrix}0\\ \alpha\end{matrix}\right. + \beta - \gamma + r\right|\right), 1\right)\\ \left(\Delta\left(\delta, \sigma + r - \gamma + \left|\begin{matrix}\alpha\\ \beta\end{matrix}\right|\right), 1\right), \{(\Delta(t, b_q), \beta_q)\}\end{matrix}\right.\right],$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leq 0$,

$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg \xi| < \frac{1}{2}\lambda\pi$, $Re(\gamma) > 0$,

$|\arg z| < \pi$, $Re\left(\alpha - \gamma + \sigma - \dfrac{\delta}{t}.\dfrac{a_i - 1}{\alpha_i}\right) > 0$, $Re\left(\beta - \gamma + \sigma - \dfrac{\delta}{t}.\dfrac{a_i - 1}{\alpha_i}\right) > 0$

$(i = 1, \ldots, n)$.

PROOF: Initially we start with $t = 1$ in the integrand of (3.1). Expressing the H-function in its integrand in the form of MELLÍN-BARNES type of integral (1.1) and interchanging the order of integration, which is permissible under the conditions, stated in (3.1), we get:

$$(3.7)\qquad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, z^s\, ds \times$$

$$\times \int_0^1 \varkappa^{s\delta+\varrho-1}(1-\varkappa)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;\varkappa)\,d\varkappa.$$

After evaluating the x-integral with the help of the result [2, p. 399 (5)] and using the GAUSS' multiplication theorem for Gamma-functions [6, p. 26], (3.7) reduces to :

$$(3.8)\qquad \delta^{-\sigma}\,\Gamma(\sigma)\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\cdot r!}\times$$

$$\times\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)\prod_{R=0}^{\delta-1}\Gamma\left(\frac{\varrho+r+R}{\delta}+s\right)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{R=0}^{\delta-1}\Gamma\left(\frac{\varrho+\sigma+r+R}{\delta}+s\right)}\, z^s d\, s.$$

Therefore in accodance with the definition (1.1) of the H-function, (3.8) yields the value of the integral (3.1) with $t = 1$ as :

$$(3.9)\qquad \delta^{-\sigma}\,\Gamma(\sigma)\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\cdot r!}\times$$

$$\times\, H^{m,\,n+\delta}_{p+\delta,\,q+\delta}\left[z\,\middle|\,\begin{matrix}(\Delta(\delta,1-\varrho-r),1),\ \{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\},\ (\Delta(\delta,1-\varrho-\sigma-r),1)\end{matrix}\right].$$

In (2.4) replacing x by $(z\, x^{\delta/t})$ and using (2.3) we get :

$$(3.10)\qquad H^{m,\,n}_{p,\,q}\left[z\varkappa^{\delta/t}\,\middle|\,\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] \equiv (2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum_1^q b_j-\sum_1^p a_j+\frac{1}{2}p-\frac{1}{2}q+1}\times$$

$$\times\, H^{tm,\,tn}_{tp,\,tq}\left[(zt^{\tau})^t\varkappa^{\delta}\,\middle|\,\begin{matrix}\{(\Delta(t,\alpha_p),\alpha_p)\}\\ \{(\Delta(t,b_q),\beta_q)\}\end{matrix}\right],$$

where τ stands for $\left(\sum_1^p \alpha_j-\sum_1^q \beta_j\right)$.

By virtue of (3.10) the integral (3.1) can easily be deduced from (3.9) on making proper substitutions.

Following the same procedure as done above, we can establish the integrals (3.2) and (3.3) by using the result [2, p. 399 (5)] and the integrals (3.4), (3.5) and (3.6) by using the results [2, p. 400 (8)], [2, p. 399 (7)] and [2, p. 400 (10)] respectively.

4. Particular cases : In view of the section 2, by setting the parameters suitably the H-function and Gauss' hypergeometric functions, involved in the integrals, evaluated above in the section 3, will yield many simple functions and thus so many integrals may be obtained as their special cases. However, few interesting cases, some of wich give generalisation of certain known integrals, are mentioned here.

Taking $\alpha_j = \beta_h = 1$ $(j = 1, \ldots, p\,;\ h = 1, \ldots, q)$ in (3.1) we come to a known case, recently obtained by Bazpaí [1] as:

$$(4.1)\quad \int_0^1 \varkappa^{\varrho-1}(1-\varkappa)^{\sigma-1}\,{}_2F_1(\alpha, \beta; \gamma; \varkappa)\, G^{m,\, n}_{p,\, q}\left[z\varkappa^{\delta/t}\middle|\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}\right] d\varkappa$$

$$= (2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\sum\limits_1^q b_j - \sum\limits_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\, \delta^{-\sigma}\, \Gamma(\sigma) \times$$

$$\times \sum_{r=0}^{\infty} \frac{(\alpha)_r\, (\beta)_r}{(\gamma)_r \cdot r!} \times$$

$$\times\, G^{tm,\, tn+\delta}_{tp+\delta,\, tq+\delta}\left[z^t\, t^{t(p-q)}\middle|\begin{matrix} \Delta(\delta, 1-\varrho-r), \Delta(t, a_1), \ldots, \Delta(t, a_p) \\ \Delta(t, b_1), \ldots, \Delta(t, b_q), \Delta(\delta, 1-\varrho-\sigma-r)\end{matrix}\right],$$

where δ and t are positive integers, $\Delta(t, f)$ represents set of the parameters $\frac{f}{t}, \frac{f+1}{t}, \ldots, \frac{f+t-1}{t}$, $p + q < 2\,(m + n)$,

$|\arg z| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\,\pi$, $Re\,(\sigma) > 0$, $Re\,(\gamma + \sigma - \alpha - \beta) > 0$

and $Re\left(\varrho + \frac{\delta}{t} \cdot b_h\right) < 0$ $(h = 1, \ldots, m)$.

In (3.1) replacing m, n, p, q by p, 1, $q+1$, p respectively and setting the other parameters suitably in view of (2.6) we obtain:

$$(4.2)\quad \int_0^1 \varkappa^{\varrho-1}(1-\varkappa)^{\sigma-1}\,{}_2F_1(\alpha, \beta; \gamma; \varkappa)\, E\,(a_1, \ldots, a_p\,; b_1, \ldots, b_q\,; z\varkappa^{\delta/t})\, d\varkappa$$

$$= (2\pi)^{\frac{1}{2}(1-t)(p-q+1)}\, t^{\sum_1^p a_j - \sum_1^q b_j + \frac{1}{2}q - \frac{1}{2}p + \frac{1}{2}}\, \delta^{-\sigma}\, \Gamma(\sigma) \times$$

$$\times \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r .\, r!} \times$$

$$\times G^{tp,\, t+\delta}_{tq+t+\delta,\, tp+\delta}\left[(zt^{\tau})^t \,\middle|\, \begin{matrix} \Delta(\delta, 1-\varrho-r), \Delta(t,1), \Delta(t,b_1), \ldots, \Delta(t,b_q) \\ \Delta(t,a_1), \ldots, \Delta(t,a_p), \Delta(\delta, 1-\varrho-\sigma-r) \end{matrix}\right]$$

provided that δ and t are positive integers, $Re\ (\sigma) > 0$; $Re\,(\gamma+\sigma-\alpha-\beta) > 0$, $q-p+1 \equiv \tau \leq 0$, $p-q+1 \equiv \lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and $Re\ (\varrho + \frac{\delta}{t}\alpha_j) > 0$ $(j = 1, \ldots, p)$.

Putting $m = q = 2$, $n = p = 0$, $b_1 = \frac{1}{2}l - \frac{1}{2}\nu$, $b_\cdot = \frac{1}{2}l + \frac{1}{2}\nu$ and $\beta_1 = \beta_2 = 1$, the integral (3.1), in view of (2.7), reduces to:

$$(4.3) \qquad \int_0^1 x^{\frac{l\delta}{2t}+\varrho-1}(1-x)^{\sigma-1}\, {}_2F_1(\alpha, \beta; \gamma; x)\, K_\nu(2\sqrt{z}\, x^{\frac{\delta}{2t}})\, dx$$

$$= (2t)^{)1-t)} . \tfrac{1}{2} .\ z^{-l/2}\, t^l\, \delta^{-\sigma}\, \Gamma(\sigma) \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r .\, r!} \times$$

$$\times G^{2t,\, \delta}_{\delta,\, 2t+\delta}\left[\left(\frac{z}{t^2}\right)^t \,\middle|\, \begin{matrix} \Delta(\delta, 1-\varrho-r) \\ \Delta(l, \frac{1}{2}l \pm \frac{1}{2}\nu), \Delta(\delta, 1-\varrho-\sigma-r) \end{matrix}\right],$$

where δ and t are positive integers, $Re\ (\sigma) > 0$, $Re\,(\gamma + \sigma - \alpha - \beta) > 0$ $|\arg z| < \pi$ and $Re\left(\varrho + \frac{\delta}{t} . \frac{l \pm \nu}{2}\right) > 0$.

Taking $m = q = 2$, $n = 0$, $p = 1$, $a_1 = l - \lambda + 1$, $b_1 = \frac{1}{2} + l\,\mu$, $b_2 = \frac{1}{2} + l - \mu$, and $\alpha_1 = \beta_1 = \beta_2 = 1$ in (3.1) and using (2.8) we get:

$$(4.4) \quad \int_0^1 x^{\frac{\delta l}{t}+\varrho-1}(1-x)^{\sigma-1}\, {}_2F_1(\alpha, \beta; \gamma; x)\, W_{\lambda, u}(zx^{\delta/t})\, e^{-\frac{1}{2}zx^{\delta/t}}\, dx$$

$$= 2\pi)^{\frac{1}{2}(1-t)}\, t^{l+\lambda+\frac{1}{2}}\, z^{-l}\, \delta^{-\sigma}\, \Gamma(\sigma) \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r .\, r!} \times$$

$$\times G^{2t,\, \delta}_{t+\delta,\, 2t+\delta}\left[\left(\frac{z}{t}\right)^t \,\middle|\, \begin{matrix} \Delta(\delta, 1-\varrho-r), \Delta(t, l-\lambda+1) \\ \Delta(t, \frac{1}{2}+l \pm \mu), \Delta(\delta, 1-\varrho-\sigma-r) \end{matrix}\right],$$

provided that t and δ are positive integers, $Re\,(\sigma)>0$,

$Re\,(\gamma+\sigma-\alpha-\beta)>0$, $|\arg z|<\frac{1}{2}\pi$, and $Re\left[\varrho+\frac{\delta}{t}\left(\frac{1}{2}+l\pm\nu\right)\right]>0$.

In (3.1) replacing m, n, q by $1, p, q+1$ respectively and choosing the other parameters suitably in view of (2.9), we obtain:

(4.5) $$\int_0^1 x^{\varrho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,{}_p\Psi_q\left[\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix};-zx^{\delta/t}\right]dx$$

$$=(2\pi)^{\frac{1}{2}(1-t)(p-q+1)}\,t^{\sum_1^p a_j-\sum_1^q b_j+\frac{1}{2}q-\frac{1}{2}p+\frac{1}{2}}\,\delta^{-\sigma}\,\Gamma(\sigma)\times$$

$$\times\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\cdot r!}\times$$

$$\times H^{t,\,tp+\delta}_{tp+\delta,\,tq+t+\delta}\left[(z^{t\tau})^t\,\middle|\,\begin{matrix}(\Delta(\delta,1-\varrho-r),1),\{(\Delta(t,1-a_p),\alpha_p)\}\\ (\Delta(t,0),1),\{(\Delta(t,1-b_q),\beta_q)\},(\Delta(\delta,1-\varrho-\sigma-r),1)\end{matrix}\right],$$

where δ and t are positive integers, $Re\,(\sigma)>0$, $Re\,(\gamma+\sigma-\alpha-\beta)>0$,

$\sum_1^p\alpha_j-\sum_1^q\beta_j-1\equiv\tau\leq 0$, $\sum_1^p\alpha_j-\sum_1^q\beta_j+1\equiv\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$ and

$Re\,(\varrho)>0$.

In (3.1) setting $m=1$, $n=p=0$, $q=2$, $b_1=0$, $b_2=-\nu$, $\beta_1=1$ and $\beta_2=u$, it, by virtue of (2.10), reduces to:

(4.6) $$\int_0^1 x^{\varrho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,J_\nu^u(zx^{\delta/t})\,dx$$

$$=t^{-\nu}\,\delta^{-\sigma}\,\Gamma(\sigma)\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\cdot r!}\times$$

$$\times H^{t,\,\delta}_{\delta,\,2t+\delta}\left[\left(\frac{z}{t^{u+1}}\right)^t\,\middle|\,\begin{matrix}(\Delta(\delta,1-\varrho-r),1)\\ (\Delta(t,0)1),(\Delta(t,-\nu),u),(\Delta(\delta,1-\varrho-\sigma-r),1)\end{matrix}\right],$$

provided that δ and t are positive integers, $Re\,(\sigma)>0$,
$Re\,(\gamma+\sigma-\alpha-\beta)>0$, $-1\leq u<1$, $|\arg z|<\frac{1}{2}(1-u)\pi$ and
$Re\,(\varrho)>0$.

With $m = n = p = q = 1$, $a_1 = 1 - \nu$, $b_1 = 0$, $\alpha_1 = \beta_1 = 1$ in (3.1) and using (2.11) we get:

$$(4.7) \quad \int_0^1 x^{\varrho-1} (1-x)^{\sigma-1} {}_2F_1(\alpha, \beta; \gamma; x) (1 + z x^{\delta/t})^{-\nu} dx.$$

$$= (2\pi)^{(1-t)} t^{\nu} \delta^{-\sigma} \Gamma(\sigma) \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r . r!} \times$$

$$\times G^{t,\, t+\delta}_{t+\delta,\, t+\delta} \left[z^t \middle| \begin{matrix} \Delta(\delta, 1-\varrho-r), \Delta(t, 1-\nu) \\ \Delta(t, 0), \Delta(\delta, 1-\varrho-\sigma-r) \end{matrix} \right],$$

where δ and t are positive integers, $Re(\sigma) > 0$, $Re(\gamma + \sigma - \alpha - \beta) > 0$ $|\arg z| < \pi$ and $Re(\varrho) > 0$.

Proceeding on similar lines as above and using the results of the section 2, the integrals (3.2) to (3.6) will also yield many integrals as their particular cases.

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