A DETERMINANTAL INEQUALITY IN HILBERT SPACE

by

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Assume that H is a HILBERT space. Let $h_i\,(i = 1, 2, \ldots, n)$ be any element in it. Following the conventional notation (h_i, h_j) $(j = 1, 2, \ldots, n)$ is defined as the inner or scalar product. The priminary interest of this note is to prove the following ein-lit equality in HILBERT space H.

$$(1)\qquad \left\{\prod_{i \le i_1 < i_2 < \ldots i_u \le n} \operatorname{Det}_{i,j=i_1,\, i_2,\, \ldots,\, i_u} (h_i, h_j)\right\}^{1/\binom{n-1}{u-1}}$$

$$\le \left\{\prod_{i \le < i_1 < i_2 < \ldots i_u < n} \operatorname{Det}_{i,j_1 = i_u, \ldots, i_u} (h_i, h_j)\right\}^{\frac{u-2}{u-1}, \frac{u}{u-1} \cdot \frac{1}{\binom{n-1}{u-1}}} \cdot \left\{\prod_{i=1}^{n} ||h_i||^2\right\}^{\binom{n-1}{u-1}/(u-1)\binom{u-1}{u-2}\binom{n-1}{u-1}}$$

$$\le \ldots \le \left\{\prod_{1 \le i_1 < i_2 < \ldots i_{u-2} \le n} \operatorname{Det}_{i,j = i_1, \ldots, i_{u-2}} (h_i, h_j)\right\}^{\binom{n-u+2}{2}/(u-1)\binom{u-1}{1}\binom{n-1}{u-1}}$$

$$\cdot \left\{\prod_{1 \le i_1 < i_2 < \ldots i_u \le n} \operatorname{Det}_{i,j = i_1, \ldots, i_u} (h_i, h_j)\right\}^{1/2 \cdot u/u-1 \cdot 1/\binom{n-1}{u-1}}$$

$$\le \left\{\prod_{1 \le i_1 < i_2 < \ldots i_{u-1} < n} \operatorname{Det}_{i,j = i_1, \ldots, i_{u-1}} (h_i, h_j)\right\}^{1/\binom{n-1}{u-1}}$$

where i_u represents u different numbers iamong $1, 2, \ldots n$. Actually (1), is a generalization of the following inequality :

$$(2)\quad \prod_{i=1}^{n} ||h_i||^2 \ge \ldots \ge \left\{\prod_{1, \le i_1 < i_2 < \ldots < i_u \le n} \operatorname{Det}_{i,j = i_1, i_2, \ldots i_u} (h_i, h_j)\right\}^{\frac{1}{\binom{n-1}{u-1}}}$$

$$\ge \left\{\prod_{1 \le i_1 < i_2 < \ldots i_{u-1} \le n} \operatorname{Det}_{i,j = i_1, i_2, \ldots i_{n-1}} (h_i, h_j)\right\}^{1/\binom{n-1}{1}}$$

$$\ge \operatorname{Det}_{i,j=1}^{n} (h_i, h_j).$$

But (2) is essentially alternative form of SZASZ's inequality, in terms of elements in HILBERT space!

For simplicity, let us list notations used throughout this paper as follows:

$$D_u : \underset{k,p\ 1}{\overset{n}{\mathrm{Det}}}\ f_{i_k i_p} = \mathrm{Det}\ (h_{i_k},\ h_{i_p}).$$

$$D_u^1 = \begin{vmatrix} D_{i,i}, & \ldots\ldots, & D_{i,iu} \\ \ldots & \ldots & \ldots \\ D_{ui}, & \ldots\ldots, & D_{iuiu} \end{vmatrix}.$$

where $D_{i_k\ i_p}$ is the cofactor of $f_{i_k\ i_p}$.

D_u^r: any r-roweorminor of D_u.

D_*^r: r rowed minor of D_1 which is one to one corresponding to D_u^r.

D^*: the cofactor of D_{u*}^r

A_r: the product of all principal r-rowed minors of D_u^1.

B_r^*: the product of all órincipal r-rowed minors of D_u.

B_u: the product of all principale u-rowed minor of Det f_{ij}.

Now we quote two known results for proving our inequality. First is SZASZ's ineauality:

$$B_1 > B_2^1/\binom{n-1}{1} > B_3^1/\binom{n-1}{2} > \ldots > B_{n-1}^{\ 1}/\binom{n-1}{n-2} < \mathrm{B}_n. \tag{3}$$

The other is

$$D_*^r = (D_u)^{r-1}\ D^*. \tag{4}$$

Based on (4), we readily get a recurrence relation as follows.

$$A_r = (D_u)^{(r-1)}\ \binom{u}{2}\ B_{n-r}^*.$$

By the (3),

$$A_1 > A_2^1/\binom{u-1}{1} > A_3^1/\binom{u-1}{2}2 > \ldots > A_{u-1}^{\ 1}/\binom{u-1}{u-2} < D_u^1.$$

Therefore,

$$(B^*_{u-1}) > (D_u)^{u-\frac{u}{2}} (B^*_{u-1})^{1/\binom{u-1}{1}} > (D_u)^{\frac{2}{3}u} (B^*_{u-3})^{1/\binom{u-1}{2}}$$

$$> \dots > (D_u)^{u-z/u-1.u} (B^*_1)^{1/\binom{u-1}{u-2}} > (D_u)^{u-1}.$$

Then,

$$(5) \quad (B^*_{u-1})^{1/u-1} > (D)^{\frac{1}{2}\cdot\frac{u}{u-1}} (B^*_{u-2})^{1/u-1.\binom{u-1}{1}} > (D_u)^{\frac{2}{3}\frac{u}{u-1}} (B^*_{u-3})^{1/(u-1)\binom{u-1}{2}}$$

$$\dots > (D_u)^{\frac{u-2}{u-1}\cdot\frac{u}{u-1}} (B^*_1)^{1/u-1\binom{u-1}{u-2}} < D_u.$$

Let $i_1, i_2, \dots i_u$ exhaustively run though the different value of 1, 2, 3,... n. Hence, we are to product the corresponding different inequalities (5′ with each other. Since the matrix (f_{ij}) is a positive definite non-diagonal HERMITIAN matrix, all principle minors should be positive. The occurrence of every factor of B^*_{u-1} is $\binom{u-u+1}{1}$, the occurrence of every factor of B^*_{u-2} is $\binom{n-u+2}{2}$, and so on.

Therefore,

$$(B_{u-1})^{\binom{n-u+1}{1}/(u-1)} > (B_u)^{\frac{1}{2}\frac{u}{u-1}} (B_{u-2})^{\binom{n-u+2}{2}/(u-1)\binom{u-1}{1}}$$

$$> (B_u)^{\frac{2}{3}\cdot\frac{u}{u-1}} (B_{u-3})^{\binom{n-u+3}{3}/(u-1)\binom{u-1}{2}} > \dots$$

$$> (B_u)^{\frac{u-2}{u-1}\frac{u}{u-1}} (B_1)^{\binom{n-1}{u-1}/(u-1)\binom{u-1}{u-2}} > (B_u).$$

Hence,

$$(B_{u-1})^{\binom{n-u+1}{1}/(k-1)\binom{n-1}{u-1}} > (B_u)^{\frac{1}{2}\cdot\frac{u}{u-1}\cdot\frac{1}{\binom{n-1}{u-1}}} (B_{u-2})^{\binom{n-u+2}{2}/(u+1)\binom{u-1}{1}\binom{n-1}{u-1}}$$

$$> (B_u)^{\frac{2}{3}\cdot\frac{u}{u-1}\cdot\frac{1}{\binom{n-1}{u-1}}} (B_{u-3})^{\binom{u-n+3}{3}/(u-1)\binom{u-1}{2}\binom{n-1}{u-1}} > \dots$$

$$> (B_u)^{\frac{u-2}{u-1}\cdot\frac{u}{u-1}\cdot\frac{1}{\binom{n-1}{u-1}}} (B_1)^{1/(u-1)\binom{u-1}{u-2}} > (B_u)^{1/\binom{n-1}{u-1}}.$$

This implies

$$(B_{u-1})^{1/\binom{n-1}{u-2}} > (B)^{\frac{1}{2}\cdot\frac{u}{u-1}\cdot\frac{1}{\binom{n-1}{u-1}}} (B_{u-2})^{\binom{n-n+2}{2}/(u-1)\binom{u-1}{1}\binom{n-1}{u-1}}$$

$$> \dots > (B_u)^{1/\binom{n-1}{u-1}}.$$

If the above inequality is interpolated into between $(B_{u-1})^{1/\binom{n-1}{u-2}}$ and $(B_u)^{1/\binom{n-1}{u-1}}$, and u is put to be 1, 2,..., n, then our inequality follows.

Finally, we claim that the elements $(b_{i1}, b_{i2}, \ldots)$ $(i = 1, 2, \ldots n)$ in which $f_i = (h_i, h_j) = \frac{n}{2}\, bisbjs\ s = 1$ $(i, j = 1, 2, \ldots, n)$ is orthogonal, if only if our inequality and (2) become equality. The proof of our proposition is omitted here, and remained to readers.

REFERENCES

1. SZAZS. — *Uber eine Verallgemeinerung des Hadamardschen Determinanten Satzes*. Monatsh. Fur Math. und Physiks, Bd. 28 (1917), 253-257.

L. MIRSKY. — *On a generalisation of Hadamard's determinantal inequality due due to Szasz*. Archiv der Math. Bd VIII (1957), 274-275.

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