SOME THEOREMS ON VARMA TRANSFORM OF FIRST KIND

by

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1. INTRODUCTION

In a paper (4) VARMA introduced a generalization of the classical LAPLACE Transform

$$\Psi(p) = p \int_0^\infty e^{-pt}\ h(t)\ dt \qquad \dots (1.1)$$

in the form

$$\Phi(p) = p \int_0^\infty (2pt)^{-1/4}\, W_{k,m}(2pt)\, h(t)\, dt \qquad \dots (1.2)$$

where $W_{k,m}(x)$ is the whittaker Function.

Relation (1.2) reduces to (1.1) due to the identity

$$x^{-1/4}\ W_{1/4,\pm 1/4}(x) = e^{-x/2} \qquad \dots (1.3)$$

We shall call $\Phi(p)$ as VARMA TRANSFORM of FIRST KIND of $h(t)$. Equation (1.2) is symbolically denoted as

$$\Phi(p) \underset{k,\ m}{\overset{V}{=\!=}} h(t)$$

and equation (1.1) as usual shall be denoted as

$$\Psi(p) \doteqdot h(t)$$

The object of this paper is to prove some theorems in the transform (1.2) and to evaluate some infinite integrals and a recurrance formula with the help of those theorems.

2. THEOREM 1

If

$$\Phi(p) \frac{V}{\overline{k,\ m}} t^{-m-3/4} g(t^{1/2}).$$

and

$$\Psi(p,\ \alpha) \doteqdot e^{-\alpha t^2} g(t)$$

then

$$\Phi(p) = \frac{(2p)^{k+3/4}}{\Gamma(1-2k+2m)} \int_0^\infty 2m - 2k - 1\ e^{-t^2/8p}\, {}_1F_1 \left[\begin{matrix} 2m + \frac{1}{2} \\ 1-k+m \end{matrix} ;\ \frac{t^2}{8p} \right] \Psi(t,p)dt \qquad \ldots (2.1)$$

provided $R(^1/_2 - k + m) > 0,\ R(p) > 0$, VARMA Transform of first kind of $|t^{-m-3/4} g(t^{1/2})|$ and LAPLACE Transform of $|e^{-\alpha t^2} g(t)|$ both exist and the integral involved in (2.1) is absolutely convergent.

Proof

We have by hypothesis

$$\Psi(p,\ \alpha) \doteqdot e^{-\alpha t^2} g(t) \qquad \ldots (2.2)$$

and also (2., p. 294)

$$p^{-2m} e^{\alpha} p^2 W_{k,m}(2\alpha p^2) \doteqdot \frac{2^{1-k+m} (2\alpha)^{1/2(1+k+m)}}{\Gamma(1-2k+2m)} t^{m-k-1} e^{-t^2/(16\alpha)} M_{-\frac{k+3m}{2},\ \frac{m-k}{2}} (t^2/(8\alpha)) \qquad \ldots (2.3)$$

where $R(k-m) < {}^1/_2,\ R(\alpha) > 0,\quad R(p) > 0$.

Now using the relations (2.2) and (2.3) in PARSEVAL'S GOLDSTEIN theorem (7) we get

$$\int_0^\infty t^{-2m-1} W_{k,m}(2\alpha t^2)\ g(t)\ \mathrm{dt} = \frac{2^{1-k+m} (2\alpha)^{1/2(1+k+m)}}{\Gamma(1-2k+2m)}$$

$$\times \int_0^\infty t^{m-k-2} e^{-t^2/16\alpha} M_{-\frac{k+3m}{2},\ \frac{m-k}{2}} (t^2/8\alpha)\ \Psi(t,\ \alpha)\ dt$$

In the left integral replacing t^2 by t, multiplying both sides by $\alpha(2/d)^{-1/4}$ and finally on replacing α by p and using the relation (1., p. 264) we get the theorem.

$$M_{k,m}(x) = e^{-x/2} x^{m+1/2}\ {}_1F_1 [{}^1/_2 - k + m\ ;\ 2\ m + 1\ ;\ x]$$

COROLLARY

In the above theorem if we take $k = \pm m = {}^1/_4$, the theorem reduces to the form

If

$$\Phi(p) \doteqdot t^{-1}\, g(t^{1/2})$$

and

$$\Psi(p, \alpha) \doteqdot e^{-\alpha t^2}\, g(t)$$

then

$$\Phi(p) = 2p \int_0^\infty t^{-2}\, e^{-1/8t^2/p}\, \Psi(t, p)\, dt \qquad \ldots (2.4)$$

provided $R(p) > 0$, LAPLACE Transform of $|t^{-1}\, g(t^{1/2})|$ and $|e^{-\alpha t^2}$ $g(t)|$ exists and the integral involved in the equation (2.4) is absolutely convergent.

EXAMPLE

If we take

$$g(t) = t^{v-1}$$

then (2., p. 146)

$$\Psi(p, \alpha) = \frac{2^v\, p\, \Gamma(v)}{(8\alpha)^{v/2}}\, e^{p^2/(8\alpha)}\, D_{-v}(p/\sqrt{2\alpha})$$

Now by definition we have

$$\Phi(p) \underset{k,\, m}{\overset{V}{=\!=}} t^{-m+v/2-5/4}$$

that is

$$\Phi(p) = \frac{(2p)^{m-v/2+5/4}\, \Gamma({}^v/_2 - 2m)\, \Gamma({}^v/_2)}{2\, \Gamma({}^1/_2 - k - m + {}^v/_2)}\; {}_2F_1\left[\begin{matrix} {}^v/_2 - 2m,\ \frac{v}{2} \\ {}^1/_2 - k - m + {}^v/_2 \end{matrix};\ \frac{1}{2}\right] \qquad \ldots (2.5)$$

by virtue of the relation (5)

$$x^n\, e^{-qx} \underset{k,\, m}{\overset{V}{=\!=}} \frac{\Gamma(n \pm m + {}^5/_4)}{2\,(2p)^n\, \Gamma(n - k + {}^7/_4)}\; {}_2F_1\left[\begin{matrix} n \pm m + {}^5/_4 \\ n - k + {}^7/_4 \end{matrix};\ \frac{1}{2} - \frac{q}{2p}\right] \qquad \ldots (2.6)$$

where $R(n \pm m + {}^5/_4) > 0$, $R(p) > R(po) > 0$ and $|p| > |q|$.

Substituting the value of $\Psi(p, \alpha)$ in (2.1) and equating this value of $\Phi(p)$ with that of obtained in (2.5) we get

$$\int_0^\infty t^{(2m-2k)} D_{v-1}(t/\sqrt{2p})\ {}_1F_1\left[\begin{matrix} {}^1/_2 + 2m \\ 1 - k + m \end{matrix}; t^2/8p\right] dt$$

$$= \frac{(2p)^{m-k+1/2}\ \Gamma({}^1/_2 - {}^v/_2)\ \Gamma({}^1/_2 - {}^v/_2 - 2m)}{2v\ \Gamma(1 - k - {}^v/_2 - m)\ \Gamma_v}\ {}_2F_1\left[\begin{matrix} {}^1/_2 - {}^v/_2, {}^1/_2 - {}^v/_2 - 2m \\ 1 - k - {}^v/_2 - m \end{matrix}; \frac{1}{2}\right] \quad \ldots (2.7)$$

By adjusting the parameters we get

$$\int_0^\infty t^{a/2+2b-2} D_{v-1}(pt)\ {}_1F_1\left[a; b; p^2 t^2/4\right] dt$$

$$= \frac{p^{1-2b}\ \Gamma({}^1/_2 - {}^v/_2)\ \Gamma(1 - a - {}^v/_2)}{2^v\ \Gamma_v\ \Gamma({}^1/_2 - a - {}^v/_2 + b)}\ {}_2F_1\left[\begin{matrix} {}^1/_2 - {}^v/_2, 1 - a - {}^v/_2 \\ {}^1/_2 - a - {}^v/_2 + b \end{matrix}; \frac{1}{2}\right] \quad \ldots (2.8)$$

3. THEOREM 2

If

$$\Phi(p) \underset{k, m}{\overset{V}{=\!=}} t^{-k-7/4}\ g(1/t)$$

and

$$\Psi(p, \alpha) \doteqdot e^{-\alpha/t}\ g(t)$$

then

$$\Phi(p) = \frac{(2p)^{5/4}}{\Gamma({}^1/_2 - k \pm m)} \int_0^\infty t^{-k-3/2} K_{2m}(2\sqrt{2pt})\ \Psi(t, p)\ dt \quad \ldots (3.1)$$

provided $R({}^1/_2 - k - m) > 0$, $R(p) > 0$, VARMA Transform of First Kind of $|t^{-k-7/4}\ g(1/t)|$ and LAPLACE Transform of $|e^{-\alpha/t}\ g(t)|$ both exist and the integral involved in the equation (3.1) is absolutely convergent.

Proof

We have by definition

$$\Psi(p, \alpha) \doteqdot e^{-\alpha/t}\ g(t) \quad \ldots (3.2)$$

and also (2, p. 294)

$$e^{\alpha/p}\, p^{k+1}\, W_{k,m}\,(2\alpha/p) \doteqdot \frac{2^{3/2}\,\alpha^{1/2}\,t^{-k-1/2}}{\Gamma(1/2-k\pm m)} K_{2m}\,(2\sqrt{2\alpha t}) \quad \ldots (3.3)$$

where

Now using the relations (3.2) and (3.3) in PARSEVAL'S GOLDSTEIN theorem (7) weget

$$\int_0^\infty t^k\, W_{k,m}\,(2\alpha/t)\; g\,(t)\; dt$$

$$= \frac{2^{3/2}\,\alpha^{1/2}}{\Gamma(1/2-k\pm m)} \int_0^\infty t^{-k-3/2}\; K_{2m}\,(2\sqrt{2\alpha t})\; \Psi\,(t,\,\alpha)\; dt$$

In the left integral replacing t by $1/t$, multiplying both sides by $\alpha\,(2\alpha)^{-1/4}$ and finally on replacing α by p we obtain the theorem.

COROLLARY

In the above theorem taking $k=\pm m=1/4$ we get the theorem as If

$$\Phi\,(p) \doteqdot \frac{1}{t^3}\, g\,(1/t)$$

and

$$\Psi\,(p,\,\alpha) \doteqdot e^{-\alpha/t}\; g\,(t)$$

then

$$\Phi\,(p) = p\int_0^\infty t^{-2}\; e^{-2\sqrt{2pt}}\; \Psi\,(t,\,p)\; dt \quad \ldots (3.4)$$

provided provided $R\,(p) > 0$, LAPLACE Transform of $|\,e^{-\alpha/t}\; g\,(t)\,|$ and $|\,t^{-2}\; g\,(1/t|$ exists and the integral involved in the equation (3.4) is absolutely convergent.

EXAMPLE

If we take

$$g\,(t) = t^{v-1}$$

then (2., p. 146)

$$\Psi\,(p,\,\alpha) = 2\alpha^{v/4-1/4}\, p^{5/4-v/4}\, K_{v/2-1/2}\,(2\alpha^{1/2}\, p^{1/2})$$

Substituting the value of $\Psi(p, \alpha)$ in (3.1) we get

$$\Phi(p) = \frac{(2p)^{5/4}\, 2p^{\nu/2}}{\Gamma(1/2 - k \pm m)} \int_0^\infty t^{1/2-k-\nu/2}\, K_{\nu/2-1/2}(2p^{1/2} t^{1/2})\, K_{2m}(2\sqrt{2pt})\, dt$$

$$= \frac{2^{m-1/4}\, p^{k+1/4+\nu/2}\, \Gamma\, 1 - k - 3\,\nu/2 \pm m}{\Gamma(3/2 - k - \nu/2)} \times$$

$$\times\ {}_2F_1\left[\begin{matrix} 1/2 - k + m - \nu \pm v \\ 3/2 - k - \nu/2 \end{matrix}; -1\right] \quad \ldots (3.5)$$

by virtue of the relation (3., p. 145)

$$\int_0^\infty x^{\sigma-1} K_\mu(ax)\, K_\nu(yx)\, dx = \frac{2^{\sigma-3}\, a^{-\nu-\sigma}\, \Gamma\left(\frac{\sigma \pm \mu + \nu}{2}\right) \Gamma\left(\frac{\sigma \pm \mu - \nu}{2}\right)}{\Gamma(\sigma)}\, y^\nu$$

$$F_1\left[\begin{matrix} (\sigma \pm \mu + \nu)/2 \\ \sigma \end{matrix}; 1 - \frac{y^2}{a^2}\right]$$

Where $Re(\sigma) > |Re\ \mu| + |Re\ \nu|$ and $R(y + \alpha) > 0$.

Again since by hypothesis

$$\Phi(p) \underset{k,\, m}{\overset{V}{=\!=}} t^{-k-3/4-\nu}$$

that is

$$\Phi(p) = \frac{\Gamma(\pm\, m - k - \nu - 1/2)}{2(2p)^{-k-7/4-\nu}\, \Gamma(-\,\nu - 2k)}\, {}_2F_1\left[\begin{matrix} \pm m - k - \nu - 1/2 \\ -\,\nu - 2k \end{matrix}; \frac{1}{2}\right] \quad \ldots (3.6)$$

by virtue of the result (2.6).

Now equating the values of $\Phi(p)$ from the relations (3.5) and (3.6) we get the known result [I., p. 105]

4. Theorem 3

If

$$\Phi(p) \underset{k,\, m}{\overset{V}{=\!=}} t^{n-1}\, \Psi(t)$$

and

$$\Psi(p) \underset{\lambda,\, \mu}{\overset{V}{=\!=}} f(t)$$

then

$$\Phi(p) = \frac{p/2}{(2p)^n} \sum_{r=0}^{\infty} \frac{(2p)^{-r}}{r!} \int_0^{\infty} \frac{1}{x} (p+x)^r G_{33}^{22}\left(\frac{p}{x} \middle| \begin{matrix} -\frac{1}{4} \pm \mu, \frac{3}{4} - k + n + r \\ {}^1/_4 \pm m + n + r, \lambda - {}^3/_4 \end{matrix}\right) f(x)\, dx \quad \dots (4.1)$$

provided $R(p) > 0$, $R(\mu + \mu_1 + \frac{1}{4}) > 0$ where $f(x) = 0\ (x^{\mu 1})$ for small x, VARMA Transform of First Kind of $|f(t)|$ and $|t^{n-1}\, \Psi(t)|$ exists and the integral involved in the equation (4.1) is absolutely convergent.

Proof

We have by hypothesis

$$\Phi(p) = p \int_0^{\infty} (2pt)^{-1/4}\, W_{k,m}(2pt)\, t^{n-1}\, \Psi(t)\, dt \quad \dots (4.2)$$

and

$$\Psi(p) = p \int_0^{\infty} (2pt)^{-1/4}\, W_{\lambda,\mu}(2pt)\, f(t)\, dt \quad \dots (4.3)$$

Substituting the value of $\Psi(t)$ from the relation (4.3) in (4.2) weget

$$\Phi(p) = p \int_0^{\infty} (2pt)^{-1/4}\, W_{k,m}(2pt)\, t^n \int_0^{\infty} (2tx)^{-1/4}\, W_{\lambda,\mu}(2tx)\, f(x)\, dx,\, dt$$

$$= p \int_0^{\infty} f(x)\, dx \int_0^{\infty} t^n (2pt)^{-1/4} (2tx)^{-1/4}\, W_{k,m}(2pt)\, W_{\lambda,\mu}(2tx)\, dt$$

$$= \frac{p}{(2p)^n} \sum_{r=0}^{\infty} \frac{(2p)^{-r}}{r!} \int_0^{\infty} (p+x)^r f(x)\, dx\, G_{12}^{20}\left(2xt \middle| \begin{matrix} \frac{3}{4} - \lambda \\ {}^1/_4 \pm \mu \end{matrix}\right) G_{12}^{20}\left(2pt \middle| \begin{matrix} {}^3/_4 - k + n + r \\ {}^1/_4 \pm m + n + r \end{matrix}\right) dt$$

$$= \frac{p/2}{(2p)^n} \sum_{r=0}^{\infty} \frac{(2p)^{-r}}{r!} \int_0^{\infty} \frac{1}{x} (p+x)^r G \int_0^{\infty} {}_{33}^{22}\left(\frac{p}{x} \middle| \begin{matrix} -{}^1/_4 \pm \mu, {}^3/_4 - k + n + r \\ {}^1/_4 \pm m + n + r, \lambda - {}^3/_4 \end{matrix}\right) f(x)\, dx$$

by virtue of the relation (3., p. 422)

$$\int_0^{\infty} G_{pq}^{mn}\left(\alpha x \middle| \begin{matrix} a_1, \dots, ap \\ b_1, \dots, bq \end{matrix}\right) G_{rs}^{kl}\left(\beta x \middle| \begin{matrix} c_1, \dots, cr \\ d_1, \dots, ds \end{matrix}\right) dn$$

$$= \frac{1}{\alpha} G^{k+n,\, l+m}_{r+q,\, s+p}\left(\beta/\alpha \,\middle|\, \begin{matrix} -b_1, \ldots, -b_m, c_1 \ldots cr, -b_{m+1}, \ldots, -bq \\ -a_1, \ldots -a_n, d_1 \ldots ds, -a_{n+1}, \ldots, -ap \end{matrix}\right)$$

The change of the order of integration is permissible by virtue of de la vallee POUSSIN's theorem (8., p. 504) when the VARMA Transform of First Kind of $|t^{n-1}\, \Psi(t)|$ and $|f(t)|$ exists and the resulting integral is absolutely convergent.

COROLLARY 1

In the above theorem if we take $k = \pm\, m = {}^1\!/_4$ we get the theorem given by BOSE (6., p. 19) by using the result (1., p. 213)

$$G^{mn}_{pq}\left(\lambda x \,\middle|\, \begin{matrix} a_1 \ldots ap \\ b_1 \ldots bq \end{matrix}\right) = \lambda^{b_1} \sum_{r=0}^{\infty} \frac{1}{Lr} (1-\lambda)^r\, G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_1 \ldots ap \\ b_1 + r, b_2, \ldots bq \end{matrix}\right) \qquad \ldots (4.4)$$

COROLLARY 2

Again, in the above theorem if we take $\lambda = \pm\, \mu = {}^1\!/_4$ we get the theorem in the following form by using the relation (4.4).

If

$$\Phi(p) \underset{k,\, m}{\overset{V}{=\!=}} t^{n-1}\, \Psi(t)$$

and

$$\Psi(p) \doteqdot f(t)$$

then

$$\Phi(p) = \tfrac{1}{2}(2p)^{-n} \frac{\Gamma({}^3\!/_4 \pm m + n)}{\Gamma({}^1\!/_4 - k + n)} \int_0^{\infty} (p+x)^n\, {}_2F_1\left[\begin{matrix} {}^3\!/_4 \pm m + n \\ {}^1\!/_4 - k + n \end{matrix}; \frac{1}{2} - \frac{x}{2p}\right] f(x)\, dn \qquad \ldots (4.5)$$

provided $R(p) > 0$, $R({}^1\!/_2 + \mu_1) > 0$ where $f(x) \pm 0\,(x^{\mu 1})$ for small x, VARMA Transform of $|t^{n-1}\, \Psi(t)|$ and LAPLACE Transform of $|f(t)|$ both exist and the integral linvolved in (4.5) is absolutely convergent.

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