DEGREES OF POLYNOMIAL SOLUTIONS OF A CLASS OF RICCATI - TYPE DIFFERENTIAL EQUATIONS*

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1. INTRODUCTION

We consider the equation

$$(E) \qquad Ay' = B_0 + B_1 y + B_2 y^2 + \dots + B_n y^n \left(' = \frac{d}{dx}\right)$$

which comprises a wide range of familiar first order equations such as the linear equation, the RICCATI equation, ABEL's equation of the first kind and BERNOULLI's equation. Our aim here is to determine the possible degrees of polynomial solutions of (E), in the case where A, B_0, B_1, B_2, ..., B_n are polynomials in x, the independent variable. The results obtained for this equation will directly apply to polynomial RICCATI equations of order higher than the first, which arise in connection with factorization of polynomial differential operators of order higher than two. This application will be treated in a later paper. They are also applicable to a large class of higher order differential equations as we shall indicate. We shall also present a method to obtain the coefficients of polynomial solutions if the latter exist, and an algorithm for obtaining all polynomial solutions of (E) if it is known to have more than one polynomial solution. The problem of the *existence* of polynomial solutions will be treated subsequently.

2. CLASSIFICATION OF DEGREES

Let

$$y = c_m x^m + c_{m-1} x^{m-1} + \dots + c_0$$

be a solution of equation (E), where the c_i's are complex constants. By substituting in equation (E), we get

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$$A\,(mc_m x^{m-1} + \ldots) = B_0 + B_1\,(c_m x^m + \ldots) + B_2\,(c_m x^m + \ldots)^2 + \ldots + B_n\,(c_m x^m + \ldots)^n.$$

Let a, b_0, b_1, ..., b_n be the degrees and α, β_0, β_1, ..., β_n be the leading coefficients of A, B_0, B_1, ..., B_n respectively. Then this equation becomes

$$\alpha m c_m x^{m+a-1} + \ldots = \beta_0 x^{b_0} + \ldots + \beta_1 c_m x^{m+b_1} + \ldots + \beta_2 c_m{}^2 x^{2m+b_2} + \ldots + \beta_n\ c_m^n\ x^{nm+b_n} \tag{1}$$

In order that the right and left hand sides be equal, the coefficients of the highest degree terms must be equal. The term of highest degree on the left side is $\alpha m c_m x^{m+a-1}$; on the right hand side any one of the leading terms of the polynomials B_0, $B_1 y$, ..., $B_n y^n$ could be highest, depending on the value of m, which is precisely what we shall determine. Thus our procedure will be to assume any one of these $(n + 2)$ terms on the left and right hand sides as the highest degree term and by equating it to another term among the remaining $(n + 1)$ terms obtain values for m. We shall first consider the term $\alpha m c_m x^{m+a-1}$ on the left. This automatically leads us to consider the term $\beta_1 c_m x^{m+b_1}$ on the right, for if $a - 1 > b_1$, the term $\alpha m c_m x^{m+a-1}$ can be assumed to be the highest degree term and values of m can be obtained by equating it to the remaining admissible terms, except the term $\beta_1 c_m x^{m+b_1}$. But if $a - 1 < b_1$, the term $\beta_1 c_m x^{m+a-1}$ has higher degree than the term $\alpha m c_m x^{m+a-1}$ and this can no longer be considered as highest degree term. If $a - 1 = b_1$ then the terms $\alpha m c_m x^{m+a-1}$ and $\beta_1 c_m x^{m+b_1}$ have equal degrees and can both be assumed to be of highest degree and values of m obtained either by equating them to remaining terms or in the manner we shall further indicate. The remaining terms are independent of the values of $a - 1$ and b_1 and will be considered after the above two terms. Thus according as $a - 1 > b_1$, $a - 1 < b_1$, and $a - 1 = b_1$, we will have three different classes of values of m. They are as follows:

CLASS I. If $a - 1 > b_1$ and $\alpha m c_m x^{m+a-1}$ is assumed to be the highest degree term, then by equating it to the admissible terms on the right hand side (except the term $\beta_1 c_m x^{m+b_1}$) we have the following possible values for m:

(*i*) $m_1 = b_0 - (a - 1)$ if $b_0 \geq a - 1$, in which case B_0 is the polynomial of highest degree among A, B_0, B_1, ..., B_n.

(*ii*) $m_2 = (a - 1 - b_2)$ if $a - 1 \geq b_2$

(*iii*) $m_3 = (a - 1 - b_3)/2$ if $a - 1 \geq b_3$ and $a - 1 - b_3$ is a multiple of 2.

(*iv*) $m_4 = (a - 1 - b_4)/3$ if $a - 1 \geq b_4$ and $a - 1 - b_4$ is a multiple of 3.

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(*n*) $m_n = (a - 1 - b_n)/(n - 1)$ if $a - 1 \geq b_n$ and $a - 1 - b_n$ is a multiple of $n - 1$.

In addition to the above inequalities, we claim that for any m_i $(i = 2, 3, ..., n - 1)$ to be a possibility for a degree of a polynomial solution we must have $b_i \geq b_j$ for all $j > i$. For suppose $b_i < b_j$ for some $j > i$. Then if a solution of degree m_i is to exist, we are supposing that $x^{m_i + a - 1}$ and $x^{im_i + b_i}$ have the highest degree. But if $b_i < b_j$ for $j > i$, the term with $x^{jm_i + b_j}$ is clearly of a degree higher than the above two terms, which is a contradiction. Thus by the above statement and by (*i*) and (*ii*) we can say that in order that polynomial solutions of degrees $m_1, m_2, ..., m_n$ may exist simultaneously, it is necessary that $b_0 \geq a - 1 \geq b_2 \geq b_3 \geq ... \geq b_n$. If $b_0 < a - 1 < b_2 < b_3 < ... < b_n$ there can be no polynomial solutions of the degrees given above.

It is to be noted that if the degree m of a polynomial solution is equal to one of the values of m given above, say $m = m_j$ $(j > 1)$, then it is not necessary that $x^{m + a - 1}$ and $x^{jm + b_j}$ have the highest degree, although the converse is true. In such a case m will be placed into another class and we shall not say that $m = m_j$, from Class I. We shall say $m = m_j$ $(j > 1)$ from Class I if and only if $x^{m + a - 1}$ and $x^{jm + b_j}$ are among the highest degree terms. Similarly we shall say $m = m_1$ from Class I if and only if $x^{m + a - 1}$ and x^{b_0} are among the highest degree terms. In all further discussion similar identification of degrees and highest degree terms is made. Thus any value of m used for a degree of polynomial solution will automatically indicate which terms are among the highest degree terms, and no confusion should arise from such identification.

Class II. If $a-1 < b_1$ and $\beta_1 c_m x^{m+b_1}$ is assumed to be the highest degree term, by equating this to the admissible terms on the right hand side only (since $\alpha m c_m x^{m+a-1}$ is of degree less than that of the above), we have the following possible values of m :

(*i*) $m_1 = b_0 - b_1$ if $b_0 \geq b_1$

(*ii*) $m_2 = b_1 - b_2$ if $b_1 \geq b_2$

(*iii*) $m_3 = (b_1 - b_3)/2$ if $b_1 \geq b_3$ and $b_1 - b_3$ is even

(*iv*) $m_4 = (b_1 - b_4)/3$ if $b_1 \geq b_4$ and $b_1 - b_4$ is a multiple of 3

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(*n*) $m_n = (b_1 - b_n)/(n-1)$ if $b_1 \geq b_n$ and $b_1 - b_n$ is a multiple of $n-1$.

As before, polynomial solutions of degrees $m_1, m_2, \ldots, m_n$ from this class can exist simultaneously only if $b_0 \geq b_1 \geq b_2 \geq \ldots \geq b_n$. This is a necessary condition for $m_1, m_2, \ldots, m_n$ to be possibilities of degrees simultaneously. (Clearly this is not sufficient.) Thus if $b_0 < b_1 < b_2 < \ldots < b_n$ there are no polynomial solutions of degrees from this class.

Class III. If $a-1 = b_1$ and $\alpha m c_m x^{m+a-1}$ and $\beta_1 c_1 x^{m+b_1}$ are assumed to be of highest degree, then, if β_1/α is an integer, a value of m can be obtained by equating the coefficients of the above terms, provided that with this value of m, the remaining terms are of degree less than the degree of the above, i.e.

(*o*) $m_0 = \beta_1/\alpha$, if β_1/α is an integer such that

$$b_0 < (\beta_1/\alpha) + b_1$$

$$b_2 < b_1 - (\beta_1/\alpha)$$

$$b_3 < b_1 - 2(\beta_1/\alpha)$$

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$$b_n < b_1 - (n-1)(\beta_1/\alpha).$$

We call this value of m the *singular exponent*. The remaining values of m are found by equating $m + b_1$ to the exponents of other terms and are as follows:

(*i*) $m_1 = b_0 - b_1 = b_0 - (a-1)$ if $b_0 \geq b_1$

(*ii*) $m_2 = b_1 - b_2 = a - 1 - b_2$ if $b_1 \geq b_2$

(*iii*) $m_3 = (b_1 - b_3)/2 = (a - 1 - b_3)/2$ if $b_1 \geq b_3$

(*iv*) $m_4 = (b_1 - b_4)/3 = (a - 1 - b_4)/3$ if $b_1 \geq b_4$

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(*n*) $m_n = (b_1 - b_n)/(n-1) = (a - 1 - b_n)/(n-1)$ if $b_1 \geq b_n$.

Thus in this case the set of values given by Classes I and II coincide, and as before, in order that polynomial solutions of degrees $m_1, m_2, \ldots, m_n$ may exist simultaneously it is necessary that $b_0 \geq b_1 \geq b_2 \geq \ldots \geq b_n$.

The above classes are clearly mutually exclusive, since they are based on the fact that either $a - 1 > b_1$ or $a - 1 < b_1$ or $a - 1 = b_1$. But these do not exhaust all possibilities of degrees for polynomial solutions. If none of the above cases occur then m can have values from exactly one of the classes considered below.

CLASS IV$_1$. If x^{b_0} is the highest degree term then equating b_0 to the exponents of the terms to which it has not been equated before, we have the following values for m :

(*i*) $m_1 = (b_0 - b_2)/2$

(*ii*) $m_2 = (b_0 - b_3)/3$

(*iii*) $m_3 = (b_0 - b_4)/4$

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$(n-1)$ $m_{n-1} = (b_0 - b_n)/n$

provided $b_0 > b_i$ and $b_0 - b_i$ is a multiple for i for $i = 2, 3, \ldots, n$. As before the condition that polynomial solutions of degrees $m_1, \ldots,$

m_{n-1} may exist simultaneously is that $b_0 \geq b_2 \geq b_3 \geq \ldots \geq b_n$; no polynomial solutions of degrees from this class can exist if $b_0 < b_2 < b_3 < \ldots < b_n$.

CLASS IV_2. Similarly if $2m + b_2$ is assumed to be the highest exponent, then equating it to those exponents to which it has not been equated before we obtain the following possible values for m:

(*i*) $m_1 = b_2 - b_3$

(*ii*) $m_2 = (b_2 - b_4)/2$

(*iii*) $m_3 = (b_2 - b_5)/3$

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$(n-2)$ $m_{n-2} = (b_2 - b_n)/(n-2)$

provided $b_2 > b_i$ and $b_2 - b_i$ is a multiple of $i-2$, $i = 3, \ldots, n$. Polynomial solutions of degrees $m_1, \ldots, m_{n-2}$ can exist simultaneously only if $b_2 \geq b_3 \geq b_4 \geq \ldots \geq b_n$, and no polynomial solutions with degrees from the above class exist if $b_2 < b_3 < b_4 < \ldots < b_n$.

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CLASS IV_{n-1}. Equating the last two terms we have

(*i*) $m_1 = b_{n-1} - b_n$.

None of the above classes contains more than $n+1$ values for m. Hence there cannot exist more than $n+1$ polynomial solutions of different degrees simultaneously. The above results can be summarized in the following theorem :

THEOREM 1. *The polynomial solutions of equation* (E) *have degrees from exactly one of the following* $n+2$ *classes:*

(*i*) Class I $\quad n$ possible values for m

(*ii*) Class II $\quad n$ possible values for m

(*iii*) Class III $n+1$ possible values for m

(*iv*) Class IV_1 $n-1$ possible values for m

(*v*) Class IV_2 $n-2$ possible values for m

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$(n+2)$ Class IV_{n-1} 1 possible value for m.

There is no polynomial solution for equation (E) if $b_0 < a-1 < b_2 < b_3 < \ldots < b_n$ and $b_0 < b_1 < b_2$, except when $n=1$.

Information about all possible degrees of polynomial solutions of the linear, Riccati's, Abel's and some equations of higher orders can be obtained from above.

Corollary 1. For $n=1$ we have the linear equation

$$Ay' = B_0 + B_1 y$$

The polynomial solutions have degrees from exactly one of the following classes:

Class I (*i*) $m_1 = b_0 - (a-1)$ if $b_0 \geq a-1$

Class II (*i*) $m_1 = b_0 - b_1$ if $b_0 \geq b_1$

Class III (0) $m_0 = \beta_1/\alpha$ if β_1/α is a positive integer and if $b_0 < (\beta_1/\alpha) + b_1$

(*i*) $m_1 = b_0 - (a-1) = b_0 - b_1$ if $a-1 = b_1$, and $b_0 \geq b_1$.

From this we conclude that the linear equation can have at most two polynomial solutions of distinct degrees satisfying the equation simultaneously. There are no polynomial solutions if $b_0 < a-1$ and $b_0 < b_1$ and $a-1 \neq b_1$. If $a-1 = b_1$ and $b_0 < a-1$ then the only possible degree for a solution is the singular exponent.

Corollary 2. For $n=2$ we have the Riccati equation

$$Ay' = B_0 + B_1 y + B_2 y^2 \tag{a}$$

n = 1 | n = 2 | n = 3

Classes	m_0	m_1	m_2	m_3	. . .	m_i
I		$b_0-(a-1)$	$a-1-b_2$	$\frac{1}{2}(a-1-b_3)$	. . .	$\frac{1}{i-1}(a-1-b_i$
II		b_0-b_1	b_1-b_2	$\frac{1}{2}(b_1-b_3)$	. . .	$\frac{1}{i-1}(b_1-b_i)$
III	$\frac{\beta_1}{\alpha}$	$b_0-(a-1) = b_0-b_1$	$a-1-b_2 = b_1-b_2$	$\frac{1}{2}(a-1-b_3) = \frac{1}{2}(b_1-b_3)$	. . .	$\frac{1}{i-1}(a-1-b_i) = \frac{1}{i-1}(b_1-b_i)$
IV_1		$\frac{1}{2}(b_0-b_2)$	$\frac{1}{3}(b_0-b_3)$	$\frac{1}{4}(b_0-b_4)$	. . .	$\frac{1}{i+1}(b_0-b_{i+1})$
IV_2		b_2-b_3	$\frac{1}{2}(b_2-b_4)$	$\frac{1}{3}(b_2-b_5)$	. . .	$\frac{1}{i}(b_2-b_{i+2})$
IV_3		b_3-b_4	$\frac{1}{2}(b_3-b_5)$	$\frac{1}{3}(b_3-b_6)$	. . .	$\frac{1}{i}(b_3-b_{i+3})$
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IV_j		b_j-b_{j+1}	$\frac{1}{2}(b_j-b_{j+2})$	$\frac{1}{3}(b_j-b_{j+3})$		$\frac{1}{i}(b_j-b_{i+j})$
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IV_{n-3}		$b_{n-3}-b_{n-2}$	$\frac{1}{2}(b_{n-3}-b_{n-1})$	$\frac{1}{3}(b_{n-3}-b_n)$		
IV_{n-2}		$b_{n-2}-b_{n-1}$	$\frac{1}{2}(b_{n-2}-b_n)$			
IV_{n-1}		$b_{n-1}-b_n$				

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. . .	m_{n-3}	m_{n-2}	m_{n-1}	m_n
. . .	$\frac{1}{n-4}(a-1-b_{n-3})$	$\frac{1}{n-3}(a-1-b_{n-2})$	$\frac{1}{n-2}(a-1-b_{n-1})$	$\frac{1}{n-1}(a-1-b_n)$
. . .	$\frac{1}{n-4}(b_1-b_{n-3})$	$\frac{1}{n-3}(b_1-b_{n-2})$	$\frac{1}{n-2}(b_1-b_{n-1})$	$\frac{1}{n-1}(b_1-b_n)$
. . .	$\frac{1}{n-4}(a-1-b_{n-3})$ $=\frac{1}{n-4}(b_1-b_{n-3})$	$\frac{1}{n-3}(a-1-b_{n-2})$ $=\frac{1}{n-3}(b_1-b_{n-2})$	$\frac{1}{n-2}(a-1-b_{n-1})$ $=\frac{1}{n-2}(b_1-b_{n-1})$	$\frac{1}{n-1}(a-1-b_n)$ $=\frac{1}{n-1}(b_1-b_n)$
. . .	$\frac{1}{n-2}(b_0-b_{n-2})$	$\frac{1}{n-1}(b_0-b_{n-1})$	$\frac{1}{n-2}(b_0-b_n)$	
. . .	$\frac{1}{n-3}(b_2-b_{n-1})$	$\frac{1}{n-2}(b_2-b_n)$		
. . .	$\frac{1}{n-3}(b_3-b_n)$			

tions of equation (E)

This case was first treated by CAMPBELL [1] and then considered in more detail by CAMPBELL and Golomb [2]. We now have a systematic way of obtaining their results by putting $n = 2$ in Theorem 1.

COROLLARY 3. The equation

$$A_k\, y^{(k)} + A_{k-1}\, y^{(k-1)} + \dots + A_2\, y'' + Ay' = B_0 + B_1\, y + \dots + B_n\, y^n$$

where A_i's ($i = 2, 3, \dots, k$) are polynomials of degrees $a_2, a_3, \dots, a_k$ respectively such that $a_i \leq a$ ($i = 2, \dots, k$) has degrees, for polynomial solutions, from exactly one of the classes given by Theorem 1.

The table on the next page will make this classification clearer. To get all possible values of m for any value of n (say $n = i$), draw a line in the following manner: Starting from the vertical line immediately to the right of the column labeled m_i go down as far as the horizontal line below Class III and turn left going down one step at a time. Then the values given to the left and above this line are all the possible values for m in the case where $n = i$. The values for $n = 1, 2, 3$ have been marked in the table.

3. *Method for obtaining coefficients of the polynomial solutions*

If a polynomial solution exists, its degree m is one of the numbers given in Theorem 1. Its coefficients can be determined by the method we shall now discuss. We distinguish the following two cases :

CASE I. If m is not the singular exponent, then equating the coefficients of highest degree terms in equation (1), gives an equation in c_m, which can be at most of degree n. Hence c_m can have at most n possible values. To find c_{m-1} put $y = c_m x^m + y_1$ in (E). Then the equation becomes

$$A\,(mc_m\, x^{m-1} + y_1') = B_0 + B_1\,(c_m\, x^m + y_1) + B_2\,(c_m\, x^m + y_1)^2 + \dots + B_n\,(c_m\, x^m + y_1)^n$$

which is of the form

$$Ay'_1 = \bar{B}_0 + \bar{B}_1\, y_1 + \bar{B}_2\, y_1^2 + \dots + \bar{B}_n\, y_1^n$$

where

$$\bar{B}_0 = B_0 - Ac_m\, x^{m-1} + c_m\, x^m\, B_1 + c_m{}^2 x^{2m} B_2 + \dots + c_m{}^n x^{nm}\, B_n$$

$$\bar{B}_1 = B_1 + 2c_m\, x^m B_2 + \dots + nc_m{}^{n-1} x^{nm-m}\, B_n$$

etc.

Hence we get another equation of type (E) with which we can proceed in a similar manner and determine the coefficient c_{m-1}. Continuing in this way all coefficients can be determined consistently if a solution exists.

We may remark here that by this procedure we can also arrive at the conclusion that a polynomial solution of a certain degree does not exist. For example, take the simple RICCATI equation:

$$y' = 1 + x^2 y + x y^2.$$

One of the possible degrees is $m = 1$. Let

$$y = c_1 x + c_0$$

be a solution. Then by substitution in above we get

$$c_1 = 1 + x^2 (c_1 x + c_0) + x (c_1 x + c_0)^2$$

or $c_1 = 1 + c_1 x^3 + c_0 x^2 + c_1{}^2 x^3 + 2 c_0 c_1 x^2 + c_0{}^2 x.$

Equating highest degree terms we get

$$c_1 = -1.$$

Equating the second highest degree terms and putting $c_1 = -1$ we get

$$-2c_0 + c_0 = 0 \quad \therefore \quad c_0 = 0.$$

But that leaves us with

$$-1 = 1$$

when we equate the constant terms which is a contradiction. Hence no polynomial solution of degree 1 exists.

CASE II. If m is the singular exponent; i.e. if $m = \beta_1/\alpha$, by equating highest degree terms in x we get $m \alpha c_m = \beta_1 c_m$, which leaves c_m undetermined. Further, equating the coefficient of x^{m+b_1-1} to zero gives

$$(m-1)\alpha c_{m-1} + \alpha_{11} c_m m = \beta_1 c_{m-1} + \beta_{11} c_m + \ldots$$

where the terms not written out contain c_m and possibly $c_m^2, \ldots, c_m^n$ but not c_{m-i}, $i = 1, 2, \ldots, m$. (Here α_{11} and β_{11} are the coefficients following α in A and β_1 in B_1 respectively.) Thus c_{m-1} is determined uniquely as a function of $c_m, c_m^2, \ldots, c_m^n$. Similarly we can determine $c_{m-2}, \ldots, c_0$, as a function of $c_m, c_m^2, \ldots, c_m^n$. Finally comparing coefficients of $x^{b_1-1}, \ldots, x$ gives equations in c_m alone from which the value of c_m can be determined and then we can go back and find $c_{m-1}, c_{m-2}, \ldots, c_0$.

4. *The algorithm for polynomial solutions.* If it is known that equation (E) has more than one polynomial solution, then the problem of finding all polynomial solutions can be reduced to the problem of finding solutions of an algebraic equation of degree $n-1$. Thus if y and y_1 are polynomial solutions of equation (E), putting $u = y - y_1$, we get

$$Au' = u\left[\sum_{j=0}^{n-2}\sum_{i=j+1}^{n}\binom{i}{j} B_i\, y_1{}^j u^{i-(j+1)} + nB_n\, y_1{}^{n-1}\right] \tag{2}$$

Now if r is a zero of $u(x)$ of multiplicity p then r is a zero of the right hand side of multiplicity greater than or equal to p. Since r is a zero of $u'(x)$ of multiplicity $p-1$, r must be a zero of $A(x)$. Therefore the polynomial solutions of the differential equation (2) are of the form

$$u(x) = k(x-r_1)^{p_1}(x-r_2)^{p_2}\ldots(x-r_k)^{p_k} \tag{3}$$

where the r_i's are the zeros of $A(x)$ and p_i's are non-negative integers, and k is a constant. Since for a given u equation (2) is also an algebraic equation of degree $n-1$ in y_1, we have the following:

THEOREM 2. If equation (E) is known to have more than one polynomial solution then the class of functions y_1 defined by the equation

$$nB_n\, y_1{}^{n-1} + \sum_{j=0}^{n-2}\sum_{i=j+1}^{n}\binom{i}{j} B_i\, y_1{}^j\, u^{i-(j+1)} - \frac{Au'}{u} = 0 \tag{4}$$

where u is given by (3) includes all polynomial solutions of (E).

This algorithm is not applicable if (E) has only one polynomial solution. In that case equation (2) has only the trivial solution $u = 0$ and y_1 cannot be determined from equation (2).

Theorem 2 is not applicable to the linear equation since y_1 does not even appear in the equation. Equation (2) for the linear equation is

$$Au' = B_1 u$$

and all polynomial solutions of the equation

$$Ay' = B_0 + B_1 y$$

can be obtained by obtaining all polynomial solutions of $Au' = B_1 u$, if at least one polynomial solution of the former is known.

The case $n = 2$ of equation (E) has been treated by CAMPBELL and Golomb [2].

REFERENCES

1. J. G. CAMPBELL. — *A criterion for the polynomial solutions of a certain Riccati equation*, Amer. Math. Monthly, Vol. 59, 1952, pp. 388-389.

2. J. G. CAMPBELL and M. GOLOMB. — *On the polynomial solutions of a Riccati equation*, Amer. Math. Monthly, Vol. 61, 1954, pp.402-404.

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