AN ENTIRE FUNCTION ASSOCIATED WITH THE BESSEL FUNCTIONS

by

H. M. SRIVASTAVA

1. In a recent paper AL-SALAM and CARLITZ gave the expansion [1, p. 914]

$$\begin{aligned}&{}_pF_{q+1}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \mu+1, \nu+1, \beta_1, \ldots, \beta_{q-1};\end{matrix} -4xy\right] \\ (1.1)\quad &= \frac{\Gamma(\mu+1)\ \Gamma(\nu+1)}{\Gamma(\lambda+1)} \sum_{n=0}^{\infty} \frac{(\lambda+2n)\ \Gamma(\lambda+n)}{n!} x^n R(\lambda+2n, \mu+n, \nu+n, x) \\ &{}_{p+2}F_{q+1}\left[\begin{matrix}-n, \lambda+n, \alpha_1, \ldots, \alpha_p; \\ \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1, \beta_1, \ldots, \beta_{q-1};\end{matrix} y\right],\end{aligned}$$

where the entire function

$$(1.2)\qquad R(\lambda, \mu, \nu, z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda+m+1)_m z^m}{m!\ \Gamma(\mu+m+1)\ \Gamma(\nu+m+1)}$$

satisfies the relation [6, p. 147]

$$(1.3)\qquad J_\mu(2z)\, J_\nu(2z) = z^{\mu+\nu} R(\mu+\nu, \mu, \nu, z^2)$$

and is such that

$$(1.4)\qquad R\left(2\mu, \mu, \mu-\frac{1}{2}, z^2\right) = \frac{2}{\pi^{1/2}\left(\frac{1}{2}z\right)^\mu \Gamma(\mu)} \int_0^1 t^{\mu+1}(1-t^2)^{\mu-1} J_\mu(4zt)\, dt.$$

The purposes of the present paper are to exhibit (1.1) as a special case of a more general expansion which *I* have recently obtained and to give several other interesting results involving the function $R(\lambda, \mu, \nu, x)$. Some of the formulae hitherto known are shown as necessary consequences of the results of this paper.

2. The expansions given by me [4] express products of generalised hypergeometric functions as series of the NEUMANN type and have the advantage of being reducible to many well-known results.

To quote one we have[1] [4, expansion (3.1)]

$$\left(\frac{1}{2}z\right)^{\mu+\nu} {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix} - a^2 z^2\right] {}_PF_Q\left[\begin{matrix}\varrho_1, \ldots, \varrho_P; \\ \sigma_1, \ldots, \sigma_Q;\end{matrix} - b^2 z^2\right]$$

$$= \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{(\mu+\nu)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\,(\mu+\nu)_n}{n!} J_{\mu+n}(z)\, J_{\nu+n}(z) \tag{2.1}$$

$$\cdot\, F\left[\begin{matrix}-n,\ \mu+\nu+n,\ \mu+1,\ \nu+1 : \alpha_1, \ldots, \alpha_p;\ \varrho_1, \ldots, \varrho_P; \\ \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2) : \beta_1, \ldots, \beta_q;\ \sigma_1, \ldots, \sigma_Q;\end{matrix}\ a^2, b^2\right],$$

where the notation for the double hypergeometric function is due to BURCHNALL and CHAUNDY [2, p. 112] in preference to the one introduced by KAMPÉ DE FÉRIET.

Write $4x$ for z^2, y for a^2, Y for b^2 and make use of the relation (1.3). We thus find that

$${}_pF_{q+1}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \mu+1,\ \nu+1,\ \beta_1, \ldots, \beta_{q-1};\end{matrix} - 4xy\right] {}_PF_Q\left[\begin{matrix}\varrho_1, \ldots, \varrho_P; \\ \sigma_1, \ldots, \sigma_Q;\end{matrix} - 4xY\right]$$

$$= \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\lambda+1)} \sum_{n=0}^{\infty} \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{n!} x^n R(\lambda+2n,\ \mu+n,\ \nu+n,\ x) \tag{2.2}$$

$$\cdot\, F\left[\begin{matrix}-n,\ \lambda+n,\ \mu+1,\ \nu+1 : \alpha_1, \ldots, \alpha_p;\ \varrho_1, \ldots, \varrho_P; \\ \frac{1}{2}(\lambda+1), \frac{1}{2}(\lambda+2) : \mu+1,\ \nu+1,\ \beta_1, \ldots, \beta_{q-1};\ \sigma_1, \ldots, \sigma_Q;\end{matrix}\ y,\ Y,\right],$$

where $\lambda = \mu + \nu$.

Now the hypergeometric function on the right of (2.2) is equal to

$$\sum_{r=0}^{n} \frac{(-n)_r\,(\lambda+n)_r\,(\alpha_1)_r \ldots (\alpha_p)_r}{r!\left(\frac{\lambda+1}{2}\right)_r \left(\frac{\lambda+2}{2}\right)_r (\beta_1)_r \ldots (\beta_{q-1})_r} y^r$$

$$\cdot\, {}_{P+4}F_{Q+2}\left[\begin{matrix}-n+r,\ \lambda+n+r,\ \mu+1+r,\ \nu+1+r,\ \varrho_1, \ldots, \varrho_P; \\ \frac{1}{2}\lambda+\frac{1}{2}+r, \frac{1}{2}\lambda+1+r,\ \sigma_1, \ldots, \sigma_Q;\end{matrix}\ Y\right]$$

---

(1) In the actual formula a contracted notation is employed for the hypergeometric functions.

and this reduces to

$$ {}_{p+2}F_{q+1}\left[\begin{matrix} -n,\ \lambda+n,\ \alpha_1,\ \ldots,\ \alpha_p\,; \\ \frac{1}{2}\lambda+\frac{1}{2},\ \frac{1}{2}\lambda+1,\ \beta_1,\ \ldots,\ \beta_{q-1}\,; \end{matrix}\ y\right] $$

when $\boldsymbol{Y}=0$, and we have (1.1).

3. I now give some expansions of the NEUMANN type in $R(\lambda, \mu, \nu, z)$. Proceeding as in §2, from the formula [4, (1.2)]

$$ \left(\frac{1}{2}z\right)^{\lambda} {}_{p}F_{q}\left[\begin{matrix} \alpha_1,\ \ldots,\ \alpha_p\,; \\ \varrho_1,\ \ldots,\ \varrho_q\,; \end{matrix}\ a^2 z^2\right] {}_{P}F_{Q}\left[\begin{matrix} \beta_1,\ \ldots,\ \beta_P\,; \\ \sigma_1,\ \ldots,\ \sigma_Q\,; \end{matrix}\ b^2 z^2\right] $$

$$ (3.1) \qquad = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} \exp(z) \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda+n)\,\Gamma(2\lambda+n)}{n!}\, I_{\lambda+n}(z) $$

$$ \cdot\, F\left[\begin{matrix} -n,\ 2\lambda+n : \alpha_1,\ \ldots,\ \alpha_p\,;\ \beta_1,\ \ldots,\ \beta_P\,; \\ \lambda+\frac{1}{2} : \varrho_1,\ \ldots,\ \varrho_q\,;\ \sigma_1,\ \ldots,\ \sigma_Q\,; \end{matrix}\ \frac{1}{2}a^2 z,\ \frac{1}{2}b^2 z\right] $$

we have

$$ \left(\frac{1}{2}z\right)^{\lambda} R(\alpha, \beta, \gamma, z^2) $$

$$ (3.2) \qquad = \frac{\Gamma(\lambda)}{\Gamma(\beta+1)\,\Gamma(\gamma+1)} \exp(z) \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda+n)\,(2\lambda)_n}{n!}\, I_{\lambda+n}(z) $$

$$ \cdot\, {}_4F_4\left[\begin{matrix} -n,\ 2\lambda+n,\ \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1\,; \\ \lambda+\frac{1}{2},\ \alpha+1,\ \beta+1,\ \gamma+1\,; \end{matrix}\ -2z\right]. $$

This formula follows also from [4, (1.3)] if we employ the relation (1.3).

When $\beta=\gamma+\frac{1}{2}=\frac{1}{2}\alpha$ the ${}_4F_4$ reduces to a ${}_2F_2$, and if we further set $\lambda=\frac{1}{2}$, the formula is

$$ (3.3) \quad R\left(\alpha,\ \frac{1}{2}\alpha,\ \frac{1}{2}\alpha-\frac{1}{2},\ z^2\right) = \frac{2^{\alpha+\frac{1}{2}}}{z^{1/2}\,\Gamma(\alpha+1)} \exp(z) \sum_{n=0}^{\infty}(-1)^n \left(n+\frac{1}{2}\right) I_{n+\frac{1}{2}}(z) $$

$$ \cdot\, {}_2F_2\left[\begin{matrix} -n,\ n+1\,; \\ 1,\ \alpha+1\,; \end{matrix}\ -2z\right]. $$

From the expansions [4, (2.1) and (2.2)] we similarly have [1, (2.14)]

$$(2z)^{\lambda} R(\alpha, \beta, \gamma, a^2 z^2)$$

$$(3.4) \qquad = \frac{1}{\Gamma(\beta+1)\,\Gamma(\gamma+1)} \sum_{n=0}^{\infty} \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{n\,!} J_{\lambda+2n}(4z)$$

$$\cdot\ {}_4F_3\left[\begin{matrix} -n,\ \lambda+n, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1\,; \\ \alpha+1.\ \beta+1,\ \gamma+1\,; \end{matrix}\ a^2\right]$$

and

$$(2z)^{\lambda} R(\alpha, \beta, \gamma, a^2 z^2)$$

$$(3.5) \qquad = \frac{1}{\Gamma(\beta+1)\,\Gamma(\gamma+1)} \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} J_{\lambda+n}(4z)$$

$$\cdot\ {}_4F_3\left[\begin{matrix} -n, \lambda+1, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1\,; \\ \alpha+1, \beta+1, \gamma+1\,; \end{matrix}\ a^2\right].$$

When $a = 1$ and $\beta = \gamma + \frac{1}{2} = \frac{1}{2}\lambda$, the ${}_4F_3$ in (3.4) can be summed by making use of the identity [1, *p.* 915]

$$(3.6) \qquad \sum_{k=0}^{n} \frac{(-n)_k\,(\lambda+n)_k\,(\alpha+1)_{2k}}{k\,!\,(\lambda+1)_{2k}\,(\alpha+1)_k} = \frac{\lambda(\lambda-\alpha)_n}{(\lambda+2n)\,(\lambda)_n},$$

which is easy to verify, and we get [1, (2.16)].

If in (3.4) and (3.5) we set $\beta = \gamma + \frac{1}{2} = \frac{1}{2}\alpha$ and use VANDERMONDE's theorem to sum the resulting ${}_2F_1$ when $a = 1$, we obtain

$$(3.7) \quad R(\alpha, \tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha - \tfrac{1}{2}, z^2) = \frac{2^{\alpha-\lambda}}{\pi^{1/2}\, z^{\lambda}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} \frac{(\lambda+2n)\,\Gamma(\lambda+n)\,(\lambda-\alpha)_n}{\Gamma(\alpha+n+1)} J_{\lambda+2n}(4z)$$

and

$$(3.8) \qquad R(\alpha, \tfrac{1}{2}\alpha, \tfrac{1}{2}\alpha - \tfrac{1}{2}, z^2) = \frac{2}{\pi^{1/2}\, z^{\lambda}} \sum_{n=0}^{\infty} \frac{(2z)^n}{n\,!} \frac{(\alpha-\lambda)_n}{\Gamma(\alpha+n+1)} J_{\lambda+n}(4z)$$

respectively.

If we combine (3.3), (3.7) or (3.8) with (1.4), we get three interesting results. To quote one we have

$$\int_0^1 t^{\mu+1} (1 - t^2)^{\mu-1} J_\mu(4zt)\, dt$$

$$(3.9) \qquad = 2^{\mu-\lambda-1} z^{\mu-\lambda} \Gamma(\mu) \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} \frac{(2\mu-\lambda)_n}{\Gamma(2\mu+n+1)} J_{\lambda+n}(4z).$$

Next, from (1.1) we can deduce an expansion formula involving the Bessel polynomials

$$Y_n^{(\alpha)}(z) = {}_2F_0(-n, n+\alpha+1; -; \frac{1}{2} z).$$

We have

$$Y_n^{(\alpha)}(z)\, Y_n^{(\alpha)}(-z)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (\alpha+n+1)_k}{k!} \left(\frac{1}{2} z\right)^k {}_3F_2 \left[\begin{matrix} -k, -n, \alpha+n+1; \\ 1+n-k, -\alpha-n-k; \end{matrix} 1 \right],$$

and on summing the well-poised terminating ${}_3F_2$ (1) by DIXON's theorem, we find that

$$Y_n^{(\alpha)}(z)\, Y_n^{(\alpha)}(-z) = {}_4F_1 \left[\begin{matrix} -n, \alpha+n+1, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1; \\ \alpha+1; \end{matrix} z^2 \right].$$

The special case $p = 2q = 4$, $\lambda = \alpha + 1$ of (1.1), therefore, yields

$$\frac{\Gamma(\alpha+2)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\, {}_4F_3 \left[\begin{matrix} \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1, \frac{1}{2}\alpha+1, \frac{1}{2}\alpha+\frac{3}{2}; \\ \alpha+1, \mu+1, \nu+1; \end{matrix} -4x^2 z^2 \right]$$

$$(3.10) \qquad = \sum_{n=0}^{\infty} \frac{(\alpha+2n+1)\,\Gamma(\alpha+n+1)}{n!} x^{2n}\, Y_n^{(\alpha)}(z)\, Y_n^{(\alpha)}(-z)$$

$$\cdot\, R(\alpha+2n+1, \mu+n, \nu+n, x^2).$$

Al-Salam and Carlitz [1, p. 926] proved this formula in a different way.

Also, since the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n,\ n+\alpha+\beta+1\ ;\ \alpha+1\ ;\ \frac{1-z}{2}\right)$$

$$P_n^{(\alpha,\beta)}(-z) = (-1)^n\, P_n^{(\beta,\alpha)}(z)\ ;$$

it is easy to show that

$$P_n^{(\alpha,\beta)}(z)\ P_n^{(\alpha,\beta)}(-z)$$

$$= (-1)^n \frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\ {}_4F_3\left[\begin{matrix} -n, \alpha+\beta+n+1, \frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2); \\ \alpha+1, \beta+1, \alpha+\beta+1\ ; \end{matrix}\ (1-z^2)\right],$$

and therefore (1.1) gives

$$\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\ {}_4F_5\left[\begin{matrix} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1\ ; \\ \lambda,\ \alpha+1,\ \beta+1,\ \mu+1,\ \nu+1\ ; \end{matrix}\ 4x\,(z^2-1)\right]$$

$$(3.11) \qquad = \sum_{n=0}^{\infty} \frac{n!\ (\lambda+2n)\ \Gamma(\lambda+n)}{(\alpha+1)_n\,(\beta+1)_n}\, x^n\, P_n^{(\alpha,\beta)}(z)\ P_n^{(\beta,\alpha)}(z)$$

$$.\ R(\lambda+2n,\ \mu+n,\ \nu+n,\ x),$$

where $\lambda = \alpha + \beta + 1$.

The last formula appears in [1] with several misprints.

An expansion involving product of two ${}_1F_1$'s can also be derived from (1.1), and we have

$$\frac{\Gamma(\lambda+1).}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\ {}_1F_4\left[\begin{matrix} \frac{1}{2}\lambda+1\ ; \\ \lambda, \frac{1}{2}\lambda,\ \mu+1,\ \nu+1\ ; \end{matrix}\ -x^2 z^2\right]$$

$$(3.12) \qquad = \sum_{n=0}^{\infty} \frac{(\lambda+2n)\ \Gamma(\lambda+n)}{n!}\, x^{2n}\ {}_1F_1(-n\,;\lambda\,;z)\ {}_1F_1(-n\,;\lambda\,;-z)$$

$$.\ R(\lambda+2n,\ \mu+n,\ \nu+n,\ x^2),$$

and since the LAGUERRE polynomial

$$L_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;z),$$

from (3.12) we obtain

$$\frac{(\alpha+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\,{}_1F_4\left[\begin{matrix}\frac{1}{2}\alpha+\frac{3}{2}\,; \\ \alpha+1,\,\frac{1}{2}\alpha+\frac{1}{2},\,\mu+1,\,\nu+1\,;\end{matrix}\; -x^2z^2\right]$$

$$(3.13)\quad = \sum_{n=0}^{\infty} n!\,\frac{(\alpha+1+2n)}{(\alpha+1)_n}\,x^{2n}\,L_n^{(\alpha)}(z)\,L_n^{(\alpha)}(-z)\,R(\alpha+2n+1,\,\mu+n,\,\nu+n,\,x^2).$$

We remark in passing that expansions involving products of two GEGENBAUER polynomials or two LEGENDRE polynomials are particular cases of (3.11) and that from (3.10) we can deduce expansions involving product of two polynomials of the type

$$Y_n(z) = Y_n^{(0)}(z).$$

4. In this section *I* give various types of integral representations for the function $R(\lambda,\,\mu,\,\nu,\,z)$. We start with the relation

$$(4.1)\quad R(\lambda,\mu,\nu,z) = \frac{1}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2},\,\frac{1}{2}\lambda+1\,; \\ \lambda+1,\,\mu+1,\,\nu+1\,;\end{matrix}\; -4z\right],$$

which readily follows from (1.2), and replace the ${}_2F_3$ by its double integral of the EULERIAN type. We thus find that

$$\int_0^1\int_0^1 u^{2\lambda}\,v^{2\lambda+1}\,(1-u^2)^{\mu-\frac{1}{2}\lambda-\frac{1}{2}}\,(1-v^2)^{\nu-\frac{1}{2}\lambda-1}\,J_\lambda(4uvx)\,du\,dv$$

$$(4.2)\quad = \frac{\pi^{1/2}\,\Gamma(\mu-\frac{1}{2}\lambda+\frac{1}{2})\,\Gamma(\nu-\frac{1}{2}\lambda)}{2^{2\lambda+2}\,x^\lambda}\,R(\lambda,\,\mu,\,\nu,\,x^2),$$

provided that $Re(\lambda) > -\frac{1}{3}$, $Re(2\mu-\lambda) > -1$ and $Re(2\nu-\lambda) > 0$.

Next we make use of the LAPLACE's type integral [5]

$$(4.3)\qquad {}_2F_3\left[\begin{matrix}\sigma,\ \sigma+\frac{1}{2}\ ;\ \frac{\lambda^2}{z^2}\\ \varrho_1,\ \varrho_2,\ \varrho_3\ ;\end{matrix}\right]=\frac{z^{2\sigma}}{\Gamma(2\sigma)}\int_0^\infty e^{-zt}\,t^{2\sigma-1}\,{}_0F_3(-\,;\varrho_1,\varrho_2,\varrho_3;\ \frac{1}{4}\lambda^2t^2)\,dt,$$

$$Re(\sigma)>0,\ Re(z)>0,$$

in (4.1), and we get

$$R(\lambda,\ \mu,\ \nu,\ x^2)$$

$$(4.4)\qquad =\frac{1}{\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\int_0^\infty e^{-t}\,t^{\lambda}\,{}_0F_3(-\,;\lambda+1,\mu+1,\nu+1;\,-x^2t^2)\,dt,$$

valid when $Re(\lambda)>-1$.

From (4.4) we also have

$$\int_0^\infty e^{-t}\,t^{\lambda}\,{}_0F_1(\lambda+1\,;2xt)\ {}_0F_1(\lambda+1\,;-2xt)\,dt$$

$$(4.5)\qquad =\frac{\pi^{1/2}\,\{\Gamma(\lambda+1)\}^2}{2^{\lambda}}\,R(\lambda,\frac{1}{2}\lambda-\frac{1}{2},\frac{1}{2}\lambda,x^2),$$

provided that $Re(\lambda)>-1$.

The formula

$$R(\lambda,\ \mu,\ \nu,\ x)$$

$$(4.6)\qquad =\frac{1}{\Gamma(\frac{1}{2}\lambda+\frac{1}{2})\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\int_0^\infty e^{-t}\,t^{\frac{1}{2}\lambda-\frac{1}{2}}\,{}_1F_3\left[\begin{matrix}\frac{1}{2}\lambda+1\,;\\ \lambda+1,\ \mu+1,\ \nu+1;\end{matrix}\ -4xt\right]dt,$$

where $Re(\lambda)>-1$, follows from (4.1) if we employ

$$(4.7)\qquad {}_2F_3\left[\begin{matrix}\alpha,\ \beta\ ;\ \frac{\lambda}{z}\\ \varrho_1,\ \varrho_2,\ \varrho_3\ ;\end{matrix}\right]=\frac{z^{\alpha}}{\Gamma(\alpha)}\int_0^\infty e^{-zt}\,t^{\alpha-1}\,{}_1F_3\left[\begin{matrix}\beta;\ \lambda t\\ \varrho_1\ \varrho_2,\ \varrho_3;\end{matrix}\right]dt,$$

$$Re(\alpha),\ Re(z)>0\,;$$

which is a particular case of [5, (1.1)] when $n=p=q-2=1$.

If in (1.2) we use HANKEL's well-known generalisation of the second EULERIAN integral, namely

$$(4.8)\qquad \frac{1}{\Gamma(\nu+n+1)}=\frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-\nu-n-1}\,e^{t}\,dt,$$

we obtain a contour integral representation of $R(\lambda, \mu, \nu, z)$.

The formula is

$$R(\lambda, \mu, \nu, z)$$

$$(4.9) \qquad = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\, e^{t}\, {}_2F_2\left(\frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1; \lambda+1, \mu+1; -\frac{4z}{t}\right) dt,$$

in which $|\arg(t)| \leqq \pi$.

A repeated application of (4.8) in (1.2) gives

$$R(\lambda, \mu, \nu, z)$$

$$(4.10) \qquad = -\frac{1}{4\pi^2}\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)} s^{-\mu-1}\, t^{-\nu-1}\, e^{s+t}\, {}_2F_1\left(\frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1; \lambda+1; -\frac{4z}{st}\right) ds\, dt,$$

provided that both $|\arg(s)|$ and $|\arg(t)|$ do not exceed π.

Next we employ a technique suggested by RICE [3], and from SCHLÄFLI-SONINE integral [6, p. 176]

$$(4.11) \qquad J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^{\nu}}{2\pi i}\int_{-\infty}^{(0+)} \frac{dt}{t^{\nu+1}} \exp\left(t-\frac{z^2}{4t}\right),$$

we obtain

$$(4.12) \quad R(\lambda, \mu, \nu, z) = \frac{2^{\lambda-2}\, e^{\pi i\lambda}}{\pi \sin \pi\lambda}\int_{P}^{(1+,0+,1-,0-)} t^{-\mu-1}(1-t)^{-\nu-1} J_\lambda(T)\frac{dt}{T^\lambda}$$

when λ is non-integral; and if $Re(\lambda) > -1$ the formula is

$$(4.13) \qquad R(\lambda, \mu, \nu, z) = \frac{2^{\lambda-1}}{\pi i}\int_{C} t^{-\mu-1}(1-t)^{-\nu-1} J_\lambda(T)\frac{dt}{T^\lambda},$$

where $T = \left\{\frac{4z}{t(1-t)}\right\}$ and $\lambda = \mu + \nu$.

The contour for (4.12) is of the POCHHAMMER type, while C is any contour, starting and terminating at infinity, which can be deformed into the straight line joining $\frac{1}{2} - i\infty$ and $\frac{1}{2} + i\infty$ without passing over the points $t = 0$ and $t = 1$. In (4.12) both arg (t) and arg $(1 - t)$ vanish at the starting point P and in (4.13) they are zero at the point where C crosses the real axis between the origin and $t = 1$.

From (4.12) and (4.13) it follows that a general solution of the differential equation

$$\{\theta(\theta+\lambda)(\theta+\mu)(\theta+\nu)+z(2\theta+\lambda+1)(2\theta+\lambda+2)\}\ W=0, \quad \left(\theta=z\frac{d}{dz}\right) \tag{4.14}$$

satisfied by $R(\lambda, \mu, \nu, z)$, is

$$W = \int_{\mathcal{L}} t^{-\mu-1}(1-t)^{-\nu-1} f\left\{\frac{z}{t(1-t)}\right\} dt, \tag{4.15}$$

where $\mathcal{L}$ is any closed contour in the t-plane and $f(u)$ satisfies the differential equation

$$[\Phi(\Phi+\lambda)+u]\, f(u) = 0 \quad \left(\Phi = u\frac{d}{du}\right), \text{ where } u = \frac{z}{t\,(1-t)}. \tag{4.16}$$

A contour integral of Barnes's type for $R(\lambda, \mu, \nu, z)$ readily follows in view of (4.1), and we have

$$\Gamma(\mu+1)\,\Gamma(\nu+1)\,R(\lambda, \mu, \nu, z)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma\left\{\frac{1}{2}(\lambda+2s+1)\right\}\Gamma\left\{\frac{1}{2}(\lambda+2s+2)\right\}}{\Gamma(\lambda+s+1)\,\Gamma(\mu+s+1)\,\Gamma(\nu+s+1)}\,\Gamma(-s)\,(4z)^s\, ds, \tag{4.17}$$

the contour being suitably indented in the usual manner.

5. Some infinite integrals involving the function $R(\lambda, \mu, \nu, z)$ can be evaluated by means of [5, (1.1)]. To quote a few we have

$$\int_0^\infty e^{-pt}\, t^\sigma\, R(\lambda, \mu, \nu, zt)\, dt$$

$$= \frac{\Gamma(\sigma+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\, p^{-\sigma-1}\, {}_3F_3\left[\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1, \sigma+1; \\ \lambda+1, \mu+1, \nu+1;\end{matrix}\ -\frac{4z}{p}\right] \tag{5.1}$$

$$\int_0^\infty e^{-pt}\, t^{2\sigma-1}\, R(\lambda. \mu, \nu, z^2 t^2)\, dt$$

$$(5.2) \quad = \frac{\Gamma(2\sigma)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\, p^{-2\sigma}\, {}_4F_3 \left[\begin{matrix} \sigma, \sigma + \frac{1}{2}, \frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1\,; \\ \lambda+1, \mu+1, \nu+1\,; \end{matrix} \; -\frac{16\,z^2}{p^2} \right]$$

Formula (5.1) is true when $Re(p) > 0$ and $Re(\sigma) > -1$;

(5.2) is valid if $Re(\sigma) > 0$ and $Re(p) > 4\,|Re(z)|$.

Next we discuss some infinite integrals involving products of $R(\lambda, \mu, \nu, x)$. For convenience let us assume initially that z is real, and large compared with $|x|$ and $|y|$. We then have

$$\int_0^\infty t^{\sigma-1}\, R(\alpha, \beta, \gamma, x^2 t^2)\, R(\lambda, \mu, \nu, y^2 t^2)\, K_\varrho(4zt)\, dt$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{r+s}\,(\alpha+r+1)_r\,(\lambda+s+1)_s\, x^{2r}\, y^{2s}}{r!\, s!\, \Gamma(\beta+r+1)\,\Gamma(\gamma+r+1)\,\Gamma(\mu+s+1)\,\Gamma(\nu+s+1)}$$

$$\cdot \int_0^\infty t^{\sigma+2r+2s-1}\, K_\varrho(4zt)\, dt$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{r+s}\,(\alpha+r+1)_r\,(\lambda+s+1)_s\, x^{2r}\, y^{2s}}{r!\, s!\, \Gamma(\beta+r+1)\,\Gamma(\gamma+r+1)\,\Gamma(\mu+s+1)\,\Gamma(\nu+s+1)}$$

$$\cdot \frac{\Gamma\left\{\frac{1}{2}(\sigma-\varrho+2r+2s)\right\}\,\Gamma\left\{\frac{1}{2}(\sigma+\varrho+2r+2s)\right\}}{4\,(2z)^{\sigma+2r+2s}},$$

and therefore

$$\int_0^\infty t^{\sigma-1}\, R(\alpha, \beta, \gamma, x^2 t^2)\, R(\lambda, \mu, \nu, y^2 t^2)\, K_\varrho(4zt)\, dt$$

$$(5.3) \quad = \frac{\Gamma\left\{\frac{1}{2}(\sigma-\varrho)\right\}\,\Gamma\left\{\frac{1}{2}(\sigma+\varrho)\right\}}{2^{\sigma+2}\, z^\sigma\, \Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$$\cdot F\left[\begin{matrix} \frac{1}{2}(\sigma-\varrho), \frac{1}{2}(\sigma+\varrho) : \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1\,; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1\,; \\ : \alpha+1, \beta+1, \gamma+1;\ \lambda+1, \mu+1, \nu+1\,; \end{matrix} \; -\frac{x^2}{z^2}, -\frac{y^2}{z^2} \right]$$

in the notation of BURCHNALL and CHAUNDY [2].

By an appeal to the theory of analytic continuation it can be shown that this formula is true if

$$Re(\sigma) > |Re(\varrho)|$$

and each of the four numbers

$$Re(z \pm ix \pm iy)$$

is positive.

When $x = y$ and $\mu = \nu + \frac{1}{2} = \frac{1}{2}\lambda$ the double hypergeometric function is equal to

$$\sum_{k=0}^{\infty} \frac{\left(\frac{\sigma-\varrho}{2}\right)_k \left(\frac{\sigma+\varrho}{2}\right)_k}{k! \quad (\lambda+1)_k} \left(-\frac{x^2}{z^2}\right)^k$$

$$\cdot\ {}_4F_3\left[\begin{matrix} -k, -\lambda-k, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1; \\ \alpha+1, \beta+1, \gamma+1: \end{matrix}\ 1\right]$$

and this reduces to

$${}_4F_3\left[\begin{matrix} \frac{1}{2}(\sigma-\varrho), \frac{1}{2}(\sigma+\varrho), \frac{1}{2}(\alpha+\lambda+1), \frac{1}{2}(\alpha+\lambda+2); \\ \alpha+1, \lambda+1, \alpha+\lambda+1; \end{matrix}\ -\frac{4x^2}{z^2}\right]$$

if we set $\beta = \gamma + \frac{1}{2} = \frac{1}{2}\alpha$. We thus have

$$\int_0^{\infty} t^{\varrho-1} R(\lambda, \frac{1}{2}\lambda, \frac{1}{2}\lambda-\frac{1}{2}, x^2 t^2)\, R(\mu, \frac{1}{2}\mu, \frac{1}{2}\mu-\frac{1}{2}, x^2 t^2)\, K_\nu(8zt)\, dt$$

$$(5.4) \qquad = \frac{2^{\lambda+\mu-2\varrho-2}\, \Gamma\left\{\frac{1}{2}(\varrho-\nu)\right\} \Gamma\left\{\frac{1}{2}(\varrho+\nu)\right\}}{\pi\, z^{\varrho}\, \Gamma(\lambda+1)\, \Gamma(\mu+1)}$$

$$\cdot\ {}_4F_3\left[\begin{matrix} \frac{1}{2}(\varrho-\nu), \frac{1}{2}(\varrho+\nu), \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2); \\ \lambda+1, \mu+1, \lambda+\mu+1; \end{matrix}\ -\frac{x^2}{z^2}\right]$$

valid when $Re(\varrho) > |Re(\nu)|$ and $Re(z \pm ix) > 0$.

Now consider the integral

$$\int_0^\infty t^{\sigma-1}\, R(\alpha,\ \beta,\ \gamma,\ x^2 t^2)\ R(\lambda,\ \mu,\ \nu,\ y^2 t^2)\ J_\varrho(4zt)\ dt.$$

We start with the formula (5.3) which is true when

$$Re(\sigma) > |Re(\varrho)|,\ Re(z) > 0,$$

it being assumed that the numbers x and y are real and positive.

If we replace z by $\delta + iz$, where both δ and z are real, and use the asymptotic expansion of $K_\varrho\{4(\delta + iz)t\}$, it is easily seen that the integral converges uniformly for $\delta \geqq 0$, provided that it is convergent when $\delta = 0$ and $z \neq 0$. It follows that the integral is continuous when $\delta = 0$ and therefore (5.3) remains valid when z is replaced by $\pm\, iz$, where the new z is real and positive. Making use of the formula

$$\pi i\ J_\nu(z) = e^{-\frac{1}{2}\nu\pi i}\ K_\nu(-\,iz) - e^{\frac{1}{2}\nu\pi i}\ K_\nu(iz)$$

we find that

$$\int_0^\infty t^{\sigma-1}\ R(\alpha, \beta, \gamma, x^2 t^2)\ R(\lambda,\ \mu,\ \nu, y^2 t^2)\ J_\varrho(4zt)\ dt$$

$$(5.5)\quad = \frac{\Gamma\left\{\frac{1}{2}(\sigma+\varrho)\right\}}{2^{\sigma+1} z^\sigma\, \Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma\left\{1-\frac{1}{2}(\sigma-\varrho)\right\}}$$

$$.\ F\left[\begin{matrix}\frac{1}{2}(\sigma-\varrho),\ \frac{1}{2}(\sigma+\varrho) : \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1\ ;\ \frac{1}{2}\lambda+\frac{1}{2},\ \frac{1}{2}\lambda+1\ ; \\ -\quad : \alpha+1, \beta+1, \gamma+1;\ \lambda+1,\ \mu+1,\ \nu+1\ ;\end{matrix}\ \frac{x^2}{z^2},\ \frac{y^2}{z^2}\right]$$

provided that $Re(\sigma + \varrho) > 0,\ x,\ y,\ z > 0,\ z > x + y.$

In particular if we put $x = y$ and choose the parameters suitably we get

$$\int_0^\infty t^{\varrho-1}\ R(\lambda, \frac{1}{2}\lambda, \frac{1}{2}\lambda - \frac{1}{2}, x^2 t^2)\ R(\mu, \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}, x^2 t^2)\ J_\nu(8zt)\ dt$$

$$(5.6)\qquad = \frac{2^{\lambda+\mu-2\varrho-1}\, \Gamma\left\{\frac{1}{2}(\varrho+\nu)\right\}}{\pi\, z^{\varrho}\, \Gamma(\lambda+1)\, \Gamma(\mu+1)\, \Gamma\left\{1-\frac{1}{2}(\varrho-\nu)\right\}}$$

$$\cdot\ {}_4F_3\left[\begin{matrix}\frac{1}{2}(\varrho-\nu), \frac{1}{2}(\varrho+\nu), \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2);\\ \lambda+1,\ \mu+1,\ \lambda+\mu+1;\end{matrix}\ \frac{x^2}{z^2}\right]$$

valid if $Re(\varrho+\nu) > 0$ and $z > x > 0$.

In the special case $y = 0$ (5.5) gives

$$\int_0^{\infty} t^{\lambda-1}\, R(\alpha, \beta, \gamma, x^2 t^2)\, J_{\mu}(4zt)\, dt$$

$$(5.7)\qquad = \frac{\Gamma\left\{\frac{1}{2}(\lambda+\mu)\right\}}{2^{\lambda+1}\, z^{\lambda}\, \Gamma(\beta+1)\, \Gamma(\gamma+1)\, \Gamma\left\{1-\frac{1}{2}(\lambda-\mu)\right\}}$$

$$\cdot\ {}_4F_3\left[\begin{matrix}\frac{1}{2}(\lambda-\mu), \frac{1}{2}(\lambda+\mu), \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1;\\ \alpha+1,\ \beta+1,\ \gamma+1;\end{matrix}\ \frac{x^2}{z^2}\right]$$

valid when $Re(\lambda+\mu) > 0$ and $z > x > 0$.

Choose $\beta = \gamma + \frac{1}{2} = \frac{1}{2}\alpha$ and change the notation slightly. We thus obtain WEBER-SCHAFHEITLIN integral [6, p. 401]

$$(5.8)\qquad \int_0^{\infty} \frac{J_{\mu}(at)\, J_{\nu}(bt)}{t^{\lambda}} = \frac{b^{\nu}\, \Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}\right)}{2^{\lambda}\, a^{\nu-\lambda+1}\, \Gamma(\nu+1)\, \Gamma\left(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}$$

$$\cdot\ {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right),$$

provided that $Re(\mu+\nu+1) > Re(\lambda) > -1$ and $a > b > 0$, the additional condition $Re(\lambda) > -1$ being necessary to secure convergence of the integral when $a \neq b$.

If $\lambda + \mu + 1 = 0$ we can sum the ${}_2F_1$ when $a = b\sqrt{2}$, and we have

$$(5.9)\qquad \int_0^\infty t^{2\nu+1} J_{4\mu}(at)\, J_{2\nu}\left(\frac{at}{\sqrt{2}}\right) dt = \frac{\pi^{1/2}\left(\frac{1}{2}a^2\right)^{-\nu-1} \Gamma(2\mu+2\nu+1)}{\Gamma(2\mu-2\nu)\,\Gamma(\mu+2\nu+1)\,\Gamma(2\nu-\mu+1)},$$

this formula being true whenever

$$Re\left(\mu+\nu+\frac{1}{2}\right) > 0 \text{ and } 0 > Re(2\nu) \neq -1, -2, \ldots$$

It is not difficult to give a direct proof of (5.9).

We now proceed to evaluate some discontinuous integrals by means of RAMANUJAN's integral [6, p. 449]

$$(5.10)\qquad \int_{-\infty}^{+\infty} \frac{e^{it\xi}\, d\xi}{\Gamma(\mu+\xi)\,\Gamma(\nu-\xi)} = \begin{cases} \dfrac{\left(2\cos\frac{1}{2}t\right)^{\mu+\nu-2}}{\Gamma(\mu+\nu-1)}\, e^{\frac{1}{2}it(\nu-\mu)}, & (|t| < \pi), \\ 0, & (|t| > \pi), \end{cases}$$

where t is any real number.

By expanding in ascending powers of x and y, and then employing this formula, we see that

$$\int_{-\infty}^{+\infty} R(\lambda,\ \mu+\xi,\ \nu,\ x)\ R(\delta,\ \varrho,\ \sigma-\xi,\ y)\ e^{it\xi}\, d\xi$$

$$(5.11)\qquad = \frac{\left(2\cos\frac{1}{2}t\right)^{\mu+\sigma}}{\Gamma(\nu+1)\,\Gamma(\varrho+1)\,\Gamma(\mu+\sigma+1)}\, e^{\frac{1}{2}it(\sigma-\mu)}$$

$$\cdot\ F\left[\begin{matrix} - : \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1; \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}\delta+1; \\ \mu+\sigma+1 : \lambda+1,\ \nu+1;\ \delta+1,\ \varrho+1; \end{matrix}\ 2xe^{-\frac{1}{2}it}\cos\frac{1}{2}t,\ 2ye^{\frac{1}{2}it}\cos\frac{1}{2}t\right],$$

if $-\pi < t < \pi$; for other values of t, the integral vanishes.

When $t = 0$ we get

$$\int_{-\infty}^{+\infty} R(\lambda,\ \mu+\xi,\nu,x)\ R(\delta,\ \varrho,\ \sigma-\xi,\ y)\ d\xi$$

(5.12)
$$= \frac{2^{\mu+\sigma}}{\Gamma(\nu+1)\Gamma(\varrho+1)\Gamma(\mu+\sigma+1)} F\left[\begin{matrix} - & : \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1; \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}\delta+1; \\ \mu+\sigma+1 : & \lambda+1, \nu+1 ; \delta+1, \varrho+1 ; \end{matrix} 2x, 2y\right],$$

and for $x = y$ this leads to

$$\int_{-\infty}^{+\infty} R(\lambda, \mu+\xi, \nu, x) \, R(0, -\frac{1}{2}, \nu-\xi, x) \, d\xi$$

$$(5.13) \quad = \frac{2^{\mu+\nu}}{\pi^{1/2}\, \Gamma(\nu+1)\, \Gamma(\mu+\nu+1)} \, {}_1F_2 \, (\nu+\frac{1}{2}; \lambda+1, \mu+\nu+1; 2x),$$

provided that $\lambda = 2\nu$.

For $t = k\pi$, where k is real, we have

$$(5.14) \quad \int_{-\infty}^{+\infty} R(\lambda, \mu+\xi, \nu, x) \, R(\delta, \varrho, \sigma-\xi, y) \, e^{ik\xi\pi} \, d\xi = 0,$$

valid when $|k| > 1$.

6. In this section I evaluate some integrals, with finite limits, involving product of the function $R(\lambda, \mu, v, z)$. When $Re(\sigma) > 0$ and $Re(\varrho) > 0$ we can easily show that

$$\int_0^1 R(\alpha, \beta, \gamma, \frac{1}{4}xt) \, R(\lambda, \mu, \nu, \frac{1}{4}yt) \, t^{\sigma-1} (1-t)^{\varrho-1} \, dt$$

(6.1)
$$= \frac{\Gamma(\sigma)\,\Gamma(\varrho)\,\{\Gamma(\sigma+\varrho)\}^{-1}}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\nu+1)} F\left[\begin{matrix} \sigma : & \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1; \\ \sigma+\varrho : & \alpha+1, \beta+1, \gamma+1; \lambda+1, \mu+1, \nu+1; \end{matrix} -x, -y\right],$$

in the notation of BURCHNALL and CHAUNDY [2]; since

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (Re\ \alpha,\ Re\ \beta > 0).$$

When $x = y$ the hypergeometric function on the right is equal to

$$\sum_{k=0}^{\infty} \frac{(\sigma)_k \left(\frac{1}{2}\lambda+\frac{1}{2}\right)_k \left(\frac{1}{2}\lambda+1\right)_k}{k!\, (\sigma+\varrho)_k\, (\lambda+1)_k\, (\mu+1)_k\, (\nu+1)_k} (-x)^k$$

$$\cdot\ {}_6F_5\left[\begin{matrix} -k,\ -\lambda-k,\ -\mu-k,\ -\nu-k,\frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1\,; \\ \alpha+1,\ \beta+1,\ \gamma+1,\ \frac{1}{2}-\frac{1}{2}\lambda-k,\ -\frac{1}{2}\lambda-k\,; \end{matrix}\ 1\right],$$

and this reduces to

$$\sum_{k=0}^{\infty}\frac{(\sigma)_k\ (\alpha+\lambda+k+1)_k\ (-x)^k}{k!\ (\sigma+\varrho)_k\ (\alpha+1)_k\ (\lambda+1)_k}$$

on setting $\beta=\gamma+\frac{1}{2}=\frac{1}{2}\ \alpha$ and $\mu=\nu+\frac{1}{2}=\frac{1}{2}\ \lambda$.

From (6.1) we then have

$$\int_0^1 R(\lambda,\frac{1}{2}\ \lambda,\frac{1}{2}\ \lambda-\frac{1}{2},\frac{1}{4}\ xt)\ R(\mu,\frac{1}{2}\ \mu,\frac{1}{2}\ \mu-\frac{1}{2},\frac{1}{4}\ xt)\ t^{\nu}(1-t)^{\varrho-1}\ dt=$$

(6.2)

$$\frac{\Gamma(\nu+1)\ \Gamma(\varrho)\ \{\Gamma(\nu+\varrho+1)\}^{-1}}{\Gamma\left(\frac{1}{2}\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\lambda+1\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\ \mu+1\right)}\ \sum_{k=0}^{\infty}\frac{(-1)^k(\lambda+\mu+k+1)_k(\nu+1)_k}{k!(\lambda+1)_k(\mu+1)_k(\nu+\varrho+1)_k}\ x^k,$$

provided that $Re(\nu)>-1$ and $Re(\varrho)>0$.

If we prefer, the series on the right can be exhibited as a ${}_3F_4$.
If in (6.2) we set $\nu=\mu$ and change the notation slightly, we get

$$\int_0^1 R(\lambda,\frac{1}{2}\ \lambda,\frac{1}{2}\ \lambda-\frac{1}{2},\frac{1}{4}\ xt)\ R(\mu,\frac{1}{2}\ \mu,\frac{1}{2}\ \mu-\frac{1}{2},\frac{1}{4}\ xt)\ t^{\mu}(1-t)^{\nu-1}\ dt$$

(6.3) $$=\frac{2^{\lambda+\mu}\ \Gamma(\nu)}{\pi}\ R(\lambda+\mu,\lambda,\mu+\nu,x),\ (Re\ \mu>-1,\ Re\nu>0).$$

The last formula when re-written gives [1, p. 930]

(6.4) $$\int_0^1 R(\lambda,\mu,\nu,xt)\ t^{\nu}(1-t)^{\varrho-1}\ dt=\Gamma(\varrho)\ R(\lambda,\mu,\nu+\varrho,x),$$

which is valid if $Re(\nu)>-1$ and $Re(\varrho)>0$.

Also if we let $y=0$ in (6.1) we have

$$\int_0^1 R(\alpha,\beta,\gamma,xt)\ t^{\sigma-1}\ (1-t)^{\varrho-1}\ dt$$

$$(6.5) \quad = \frac{\Gamma(\sigma)\,\Gamma(\varrho)}{\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\sigma+\varrho)}\;{}_3F_4\left[\begin{matrix}\sigma,\ \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1; \\ \sigma+\varrho,\ \alpha+1,\ \beta+1,\ \gamma+1;\end{matrix}\; -4x\right],$$

$$Re(\sigma),\ Re(\varrho) > 0.$$

Putting $\sigma = \gamma + 1$ and making use of (4.1) we again get (6.4).

If in (6.5) we put $\alpha = \beta + \gamma$, employ (1.3) and change the notation slightly we see that

$${}_3F_4\left[\begin{matrix}\sigma,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \sigma+\varrho,\ \mu+1,\ \nu+1,\ \mu+\nu+1;\end{matrix}\; -4x^2\right]$$

$$(6.6) \quad = \frac{2\,\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma(\sigma+\varrho)}{\Gamma(\sigma)\,\Gamma(\varrho)\,x^{\mu+\nu}}\int_0^1 J_\mu(2xt)\,J_\nu(2xt)\,t^{2\sigma-\mu-\nu-1}\,(1-t^2)^{\varrho-1}\,dt,$$

$$[Re(\sigma) > 0,\ Re(\varrho) > 0]\,;$$

and since [6, p. 150]

$$J_\mu(z)\,J_\nu(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\,d\theta,$$

(6.6) becomes

$${}_3F_4\left[\begin{matrix}\sigma,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \sigma+\varrho,\ \mu+1,\ \nu+1,\ \mu+\nu+1;\end{matrix}\; -4x^2\right]\frac{\Gamma(\sigma)\,\Gamma(\varrho)}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma(\sigma+\varrho)}$$

$$(6.7) \quad = \frac{4}{\pi\,x^{\mu+\nu}}\int_0^1\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(4xt\cos\theta)\,t^{2\sigma-\mu-\nu-1}\,(1-t^2)^{\varrho-1}\cos(\mu-\nu)\theta\,dt\,d\theta,$$

valid when $Re(\sigma) > 0$, $Re(\mu+\nu) > -1$ and $Re(\varrho) > 0$.

The particular case $\sigma = \nu + 1$ of this formula is [1, p. 930]

$$(6.8) \quad R(\mu+\nu,\ \mu,\ \nu+\varrho,\ x^2) = \frac{4\,x^{-\mu-\nu}}{\pi\,\Gamma(\varrho)}\int_0^1\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(4xt\cos\theta)\,t^{\nu-\mu+1}\,(1-t^2)^{\varrho-1}\,\cdot$$

$$\cdot\,\cos(\mu-\nu)\theta\,dt\,d\theta,$$

provided that each of the three numbers

$$Re(\nu+1),\ Re(\nu+\mu+1),\ Re(\varrho)$$

is positive.

Next we make use of CAUCHY's formula

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{(\mu-\nu+m)i\theta} \cos^{\mu+\nu+m}\theta \, d\theta = \left(\frac{1}{2}\right)^{\mu+\nu+m} \binom{\mu+\nu+n}{\nu} \pi, \ Re(\mu+\nu) > -1,$$

and we have

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} R(\alpha, \beta, \gamma, \frac{1}{2} x e^{i\theta} \cos\theta) R(\lambda, \mu, \nu, \frac{1}{2} y e^{i\theta} \cos\theta) e^{(p-q)i\theta} \cos^{p+q}\theta \, d\theta$$

$$(6.9) \quad = \frac{\pi \, \Gamma(p+q+1)}{2^{p+q}\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(p+1)\Gamma(q+1)}$$

$$. \, F\left[\begin{matrix} p+q+1 : \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1 ; \\ p+1 : \alpha+1, \beta+1, \gamma+1 ; \lambda+1, \mu+1, \nu+1 ; \end{matrix} \; -x, -y\right],$$

valid when $Re(p+q) > -1$.

We have just seen that the double hypergeometric function in (6.1) simplifies in many special cases. By applying the same technique, therefore, a number of interesting particular cases of (6.9) will follow.

The special case $y = 0$ is

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} R(\alpha, \beta, \gamma, \frac{1}{2} x e^{i\theta} \cos\theta) \, e^{(p-q)i\theta} \cos^{p+q}\theta \, d\theta$$

$$(6.10) \quad = \frac{\pi}{2^{p+q}\Gamma(\beta+1)\Gamma(\gamma+1)} \binom{p+q}{p} {}_3F_4\left[\begin{matrix} p+q+1, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \\ p+1, \alpha+1, \beta+1, \gamma+1 ; \end{matrix} \; -x\right].$$

If in this we write $p+q$ for β, $4x$ for x and make use of (4.1), we get the known formula [1]

$$\frac{\pi}{2^{\beta+\delta}\Gamma(\delta+1)} R(\alpha, \beta, \gamma, x)$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} R(\alpha, \beta+\delta, \gamma, 2xe^{i\theta}\cos\theta) \, e^{(\beta-\delta)i\theta} \cos^{\beta+\delta}\theta \, d\theta,$$

valid when $Re(\beta+\delta) > -1$.

7. Finally *I* give some NEUMANN type expansions in products of

$R(\lambda, \mu, \nu, x)$. One such formula follows from (2.1) if we set $p = P = 2$, $q = Q = 3$, and we have

$$R(\alpha, \beta, \gamma, az) \; R(\lambda, \mu, \nu, bz) \frac{\Gamma(\delta+1)}{\Gamma(\varrho+1)\;\Gamma(\sigma+1)}$$

(7.1)

$$= \frac{1}{\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\delta+2n)\,\Gamma(\delta+n)}{n!} z^n R(\delta+2n, \varrho+n, \sigma+n, z)$$

$$\cdot F\left[\begin{matrix} -n,\ \delta+n,\ \varrho+1,\ \sigma+1 : \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1 ; \\ \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}\delta+1 : \alpha+1, \beta+1, \gamma+1; \lambda+1, \mu+1, \nu+1 ; \end{matrix} \; a, b\right].$$

When $a = b$, the double hypergeometric function equals

$$\sum_{k=0}^{n} \frac{(-n)_k\,(\delta+n)_k\,(\varrho+1)_k\,(\sigma+1)_k\,(\alpha+1)_{2k}}{k!\,(\alpha+1)_k\,(\beta+1)_k\,(\gamma+1)_k\,(\delta+1)_{2k}} a^k$$

$$\cdot\, {}_6F_5\left[\begin{matrix} -k, -\alpha-k, -\beta-k, -\gamma-k, \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1 ; \\ \lambda+1,\ \mu+1,\ \nu+1, \frac{1}{2}-\frac{1}{2}\alpha-k, -\frac{1}{2}\alpha-k ; \end{matrix} \; 1\right],$$

and this can be further reduced by assigning suitable particular values to the parameters.

If in (7.1) we set $b = 0$, we get the known formula [1, (2.3)]

$$R(\alpha, \beta, \gamma, az)$$

$$(7.2) \quad = \frac{\Gamma(\varrho+1)\;\Gamma(\sigma+1)}{\Gamma(\beta+1)\,\Gamma(\gamma+1)\;\Gamma(\delta+1)} \sum_{n=0}^{\infty} \frac{(\delta+2n)\;\Gamma(\delta+n)}{n!} z^n\, R(\delta+2n, \varrho+n, \sigma+n, z)$$

$$\cdot\, {}_6F_5\left[\begin{matrix} -n,\ \delta+n,\ \varrho+1,\ \sigma+1, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \\ \alpha+1,\ \beta+1,\ \gamma+1, \quad \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}\delta+1 ; \end{matrix} \; a\right],$$

which has numerous special cases of interest.

When $\varrho = \sigma + \frac{1}{2} = \frac{1}{2}\,\delta$, (7.2) leads to the formula (3.4).

Further expansions in products of $R(\lambda, \mu, \nu, x)$ can be derived from (3.1) and the formulae [4, (2.1) and (2.2)]. We thus have

$$z^{\delta} R(\alpha, \beta, \gamma, az^2) R(\lambda, \mu, \nu, bz^2) \frac{\Gamma(2\delta)}{\Gamma(\delta)}$$

$$(7.3) \quad = \frac{\exp(2z)}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} (-1)^n \frac{(\delta+n)\Gamma(2\delta+n)}{n!} J_{\delta+n}(2z)$$

$$. F\left[\begin{matrix} -n, 2\delta+n : \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1 ; \\ \delta+\frac{1}{2} : \alpha+1, \beta+1, \gamma+1 : \lambda+1, \mu+1, \nu+1 ; \end{matrix} \; -az, -bz\right],$$

$$(2z)^{\delta} R(\alpha, \beta, \gamma, a^2 z^2) R(\lambda, \mu, \nu, b^2 z^2)$$

$$(7.4) \quad = \frac{1}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\delta+2n)\Gamma(\delta+n)}{n!} J_{\delta+2n}(4z)$$

$$. F\left[\begin{matrix} -n, \delta+n : \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1 ; \\ - \quad : \alpha+1, \beta+1, \gamma+1 ; \lambda+1, \mu+1, \nu+1 ; \end{matrix} \; a^2, b^2\right]$$

and

$$(2z)^{\delta} R(\alpha, \beta, \gamma, a^2 z^2) R(\lambda, \mu, \nu, b^2 z^2)$$

$$(7.5) \quad = \frac{1}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} J_{\delta+n}(4z)$$

$$. F\left[\begin{matrix} -n, \delta+1 : \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 ; \frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\lambda+1 ; \\ - \quad : \alpha+1, \beta+1, \gamma+1 ; \lambda+1, \mu+1, \nu+1 ; \end{matrix} \; a^2, b^2\right].$$

Formulae (3.2), (3.4) and (3.5) are clearly special cases of these expansions.

REFERENCES

1. W. A. AL-SALAM and L. CARLITZ. — *Some functions associated with the Bessel functions*, Journal of Mathematics and Mechanics (Indiana). Vol. 12 (1963), pp. 911-934.

2. J. L. BURCHNALL and T. W. CHAUNDY. — *Expansions of Appell's double hypergeometric functions II*, Quarterly Journal of Mathematics (Oxford) Vol. 12 (1941), pp. 112-128.

3. S. O. RICE. — *On contour integrals for the product of two Bessel functions*, Quarterly Journal of Mathematics (Oxford). Vol. 6 (1935), pp. 52-64.

4. H. M. SRIVASTAVA. — *Some expansions in products of hypergeometric functions*. in course of publication.

5. H. M. SRIVASTAVA. — *Some infinite integrals associated with hypergeometric functions*, read at the 34 th annual session (February 13, 1965) *of the National Academy of Sciences of India* (Section A).

6. G. N. WATSON. — *Theory of Bessel functions*, second edition. Cambridge and New York, 1944.

Department of Mathematics
The University
Jodhpur, India