SOME THEOREMS CONCERNING MEIJER TRANSFORM

by

K. C. GUPTA*

1. INTRODUCTION. MEIJER [6] has given a generalization of LAPLACE transform

$$\phi(p) = p \int_0^\infty e^{-pt} h(t) \, dt \tag{1.1}$$

in the form

$$\phi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p \int_0^\infty (pt)^{\frac{1}{2}} K_\nu(pt) \, h(t) \, dt \tag{1.2}$$

VARMA [8] has introduced another generalization of (1.1) by the integral equation

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{m-\frac{1}{2}} W_{k,m}(pt) \, h(t) \, dt \tag{1.3}$$

on putting $\nu = \pm\frac{1}{2}$ in (1.2) and $k + m = \frac{1}{2}$ in (1.3) we observe that both of them reduce to (1.1).

As usual, in (1.2) $\phi(p)$ will be called the MEIJER transform of $h(t)$ and $h(t)$ the inverse MEIJER transform of $\phi(p)$. Similarly in (1.3) $\phi(p)$ will be termed as the VARMA transform of $h(t)$ and $h(t)$ the inverse VARMA transform of $\phi(p)$.

In a recent paper [4] in these memoirs I had obtained the inverse MEIJER transform of the G-function. Here we shall use the result proved there for obtaining two theorems involving the transform (1.2) and two other theorems connecting the transforms defined by (1.2) and (1.3). Several results given earlier by TRICOMI [7], GUPTA [5], etc., follow as particular cases of our findings. A few infinite integrals involving WHITTAKER function, hypergeometric

* Present address: Malaviya Regional Engineering College, Jaipur (India).

function and MACROBERT'S E-function have been evaluated by way of the application of the theorems.

Throughout this note (1.1), (1.2) and (1.3) will be denoted symbolically as

$$\phi(p) \doteqdot h(t), \quad \phi(p) \underset{\nu}{\overset{k}{=\!=}} h(t) \text{ and } \phi(p) \underset{k,\, m}{\overset{v}{=\!=}} h(t)$$

respectively.

The symbol $\Delta(l, a)$ has been used to represent the sequence of parameters

$$\frac{a}{l}, \frac{a+1}{l}, \ldots \frac{a+l-1}{l}$$

and $\Delta(l, a \pm b)$, $\Gamma(a \pm b)$ will be used to denote

$$\Delta(l, a+b), \Delta(l, a-b) \text{ and } \Gamma(a+b). \Gamma(a-b)$$

respectively.

In what follows n and s are positive integers.

2. THEOREM 1. If

$$\phi(p) \underset{\nu}{\overset{k}{=\!=}} h(x)$$

and

$$p^{\sigma-\frac{3}{2}\frac{n}{s}} \phi(p^{\frac{n}{s}}) \underset{\mu}{\overset{k}{=\!=}} \psi(x)$$

then

$$\psi(x) = (2n)^{\sigma-\frac{1}{2}} (2\pi)^{n-s} x^{-\sigma} \int_0^\infty t^{\frac{1}{2}} h(t)\, G^{2s,\, 0}_{2n,\, 2s}\left[\left(\frac{t}{2s}\right)^{2s} \left(\frac{2n}{x}\right)^{2n} \middle| \begin{matrix} \Delta\left(n, \frac{3-2\sigma\pm 2\mu}{4}\right) \\ \Delta\left(s, \pm\frac{\nu}{2}\right) \end{matrix}\right] dt \tag{2.1}$$

provided that the integral is convergent, the MEIJER transform of $|h(x)|$ and $|\psi(x)|$ exist, $s > n$ and $R(p) > 0$.

PROOF: We have by the definition of MEIJER transform

$$p^{\sigma-\frac{3}{2}\frac{n}{s}}\phi(p^{\frac{n}{s}}) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p^{\sigma}\int_0^{\infty} t^{\frac{1}{2}} K_{\nu}(t\, p^{\frac{n}{s}})\, h(t)\, dt \tag{2.2}$$

also [4] gives

$$p^{\sigma} K_{\nu}(t\, p^{\frac{n}{s}}) = (2n)^{\sigma-\frac{1}{2}}(2n)^{n-s}\, p\int_0^{\infty}(px)^{\frac{1}{2}} K_{\mu}(px)\, x^{-\sigma}$$

$$\times G^{2s,\,0}_{2n,\,2s}\left[\left(\frac{t}{2s}\right)^{2s}\left(\frac{2n}{x}\right)^{2n}\middle|\begin{matrix}\dfrac{\Delta(n,\, 3-2\sigma\pm 2\mu)}{4}\\ \Delta\left(s,\, \pm\dfrac{\nu}{2}\right)\end{matrix}\right]dx \tag{2.3}$$

where

$$s > n,\ |\arg t| < \tfrac{\pi}{2}\left(1-\tfrac{n}{s}\right) \text{ and } R(p) > 0.$$

Substituting the value of $p^{\sigma} K_{\nu}(tp^{\frac{n}{s}})$ from (2.3) in (2.2) we have

$$p^{\sigma-\frac{3}{2}\frac{n}{s}}\phi(p^{\frac{n}{s}}) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(2n)^{\sigma-\frac{1}{2}}(2\pi)^{n-s}\, p\int_0^{\infty} t^{\frac{1}{2}} h(t)$$

$$\times\left\{\int_0^{\infty}(px)^{\frac{1}{2}} K_{\mu}(px)\, x^{-\sigma} G^{2s,\,0}_{2n,\,2s}\left[\left(\frac{t}{2s}\right)^{2s}\left(\frac{2n}{x}\right)^{2n}\middle|\begin{matrix}\dfrac{\Delta(n,\, 3-2\sigma\pm 2\mu)}{4}\\ \Delta\left(s,\, \pm\dfrac{\nu}{2}\right)\end{matrix}\right]dx\right\}dt$$

On inverting the order of integration which is easily seen to be permissible by virtue of DE LA VALL'EE POUSSIN's theorem [1, p. 504] under the conditions stated with the theorem we get

$$p^{\sigma-\frac{3}{2}\frac{n}{s}}\phi(p^{\frac{n}{s}}) = (2n)^{\sigma-\frac{1}{2}}(2\pi)^{n-s}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} p\int_0^{\infty}(px)^{\frac{1}{2}} K_{\mu}(px)\, x^{-\sigma}$$

$$\times\left\{\int_0^{\infty} t^{\frac{1}{2}} h(t)\, G^{2s,\,0}_{2n,\,2s}\left[\left(\frac{t}{2s}\right)^{2s}\left(\frac{2n}{x}\right)^{2n}\middle|\begin{matrix}\dfrac{\Delta(n,\, 3-2\sigma\pm 2\mu)}{4}\\ \Delta\left(s,\, \pm\dfrac{\nu}{2}\right)\end{matrix}\right]dt\right\}dx \tag{2.4}$$

interpreting the R.H.S. of (2.4) with the help of (1.2) we get the theorem stated above.

COROLLARY 1. By putting $n = 1$; $s = 2$ and $\mu = \frac{1}{2}$ in (2.1) G-function in the integrand breaks up into a WHITTAKAR function and the theorem takes the following form:

If

$$\phi(p) \underset{\nu}{\overset{k}{=}} h(x)$$

and

$$p^{\lambda-\frac{1}{4}} \phi(p^{\frac{1}{2}}) \doteqdot \psi(x)$$

then

$$\psi(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (x)^{-\lambda} \int_0^\infty t^{-\frac{1}{2}} h(t)\, e^{-\frac{1}{8}\frac{t^2}{x}} W_{\lambda,\frac{\nu}{2}}\left(\frac{t^2}{4x}\right) dt \tag{2.5}$$

provided that the integral is convergent, the LAPLACE transform of $|\psi(x)|$ and the MEIJER transform of $|h(x)|$ exist and $R(p) > 0$.

The above corollary is equivalent to the following interesting sequence due to GUPTA [5]:

If

$$\phi(p) \doteqdot t^{\frac{1}{2}(\nu-\mu+1)} \psi\left(\frac{1}{t}\right)$$

and

$$\psi(p) \underset{\frac{\mu}{2},\frac{\nu}{2}}{\overset{v}{=}} f(\sqrt{t})$$

then

$$\pi^{-\frac{1}{2}} \left(\frac{p}{2}\right)^{\frac{1}{2}-\mu} \phi\left(\frac{p^2}{4}\right) \underset{\nu}{\overset{k}{=}} t^{\nu+\frac{1}{2}} f(t)$$

provided that LAPLACE transform of $\left| t^{\frac{1}{2}(\nu-\mu+1)} \psi\left(\frac{1}{t}\right)\right|$, VARMA transform of $|f(\sqrt{t})|$, and the MEIJER transform of $|t^{\nu+\frac{1}{2}} f(t)|$ exist and $R(p) > 0$.

COROLLARY 2. When $\mu = \nu = \frac{1}{2}$ and $n = 1$ we get

If

$$\phi(p) \doteqdot h(x)$$

and

$$p^{l-\frac{1}{s}}\phi(p^{\frac{1}{s}}) \doteqdot \psi(x)$$

then

$$\psi(x) = (2\pi)^{-\frac{1}{2}(1+s)}\, s^{sl+\frac{1}{2}} \int_0^\infty t^{-ls}\, h(t) \sum_{i,-i} \frac{1}{i} E\left(\Delta(s, ls) :: \frac{e^{i\pi}}{x}\left(\frac{t}{s}\right)^s \right) dt \tag{2.6}$$

where $\sum_{i,-i}$ indicates that to the expression following it i is to be replaced by $-i$ and the two expressions are to be added.

Provided that the integral is convergent, LAPLACE transforms of $|h(x)|$ and $|\psi(x)|$ exists, $s > 1$ and $R(p) > 0$.

(2.6) reduces to a well known result [3, p. 133] if $s = 2$.

Example 1.1. Taking [3, p. 294]

$$p^{\lambda-\frac{1}{4}}\phi(p^{\frac{1}{2}}) = p^{1-\sigma} e^{\frac{1}{2}\frac{p}{a}} W_{k,\mu}\left(\frac{p}{a}\right)$$

$$\doteqdot \frac{a^{-k}\, x^{\sigma-k-1}}{\Gamma(\sigma-k)}\, {}_2F_1\left(\begin{matrix}\frac{1}{2}-\mu-k, \frac{1}{2}+\mu-k\,; \\ \sigma-k\end{matrix} - ax\right) = \psi(x)$$

where $|\arg a| < \pi$ and $R(\sigma-k) > 0$.

We have

$$\phi(p) = p^{\frac{5}{2}-2\lambda-2\sigma} e^{\frac{1}{2}\frac{p^2}{a}} W_{k,\mu}\left(\frac{p^2}{a}\right).$$

Writing down the inverse MEIJER transform of the right hand side [4] we get

$$h(t) = \frac{2^{2-2\lambda-2\sigma}(2\pi)^{\frac{1}{2}}\, t^{2\lambda+2\sigma-\frac{5}{2}}}{\Gamma(\frac{1}{2}-k\pm\mu)}\, G^{1,2}_{2,3}\left[\frac{at^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2}-\mu, \frac{1}{2}+\mu \\ -k, \frac{3}{2}-\lambda-\sigma-\frac{\nu}{2}, \frac{3}{2}-\lambda-\sigma+\frac{\nu}{2}\end{matrix}\right]$$

Substituting the above values of $|h(t)|$ and $|\psi(x)|$ in (2.5) and using [2, p. 215] we have

$$\int_0^\infty e^{-\frac{1}{8}\frac{t^2}{x}} W_{\lambda,\frac{\nu}{2}}\left(\frac{t^2}{4x}\right) t^{2\lambda+2\sigma-2k-3}\, {}_2F_2\left[\begin{matrix}\frac{1}{2}-k-\mu, \frac{1}{2}-k+\mu \\ \lambda+\sigma-k-\frac{\nu}{2}-\frac{1}{2}, \lambda+\sigma-k+\frac{\nu}{2}-\frac{1}{2}\end{matrix}; -\frac{at^2}{4}\right] dt$$

$$= \frac{2^{2\lambda+2\sigma-2k-3}\, x^{\lambda+\sigma-k-1}\, \Gamma(\lambda+\sigma-k\pm\frac{\nu}{2}-\frac{1}{2})}{\Gamma(\sigma-k)}\, {}_2F_1\left(\begin{matrix}\frac{1}{2}-k-\mu, \frac{1}{2}-k+\mu \\ \sigma-k\end{matrix}; -ax\right) \tag{2.7}$$

where $R(\lambda + \sigma - k \pm \frac{\nu}{2} - \frac{1}{2}) > 0$, $R(\varkappa) > 0$, and $|\arg a| < \frac{\pi}{2}$.

Example 1.2. Starting with [3, p. 283] we have

$$\phi(p) = 2\, a^{\frac{1}{2}}\, p^{\mu}\, K_{\nu}(2\, a^{\frac{1}{2}}\, p^{\frac{1}{2}})$$

$$\doteqdot \varkappa^{\frac{1}{2}-\mu}\, e^{-\frac{1}{2}\frac{a}{x}}\, W_{\mu-\frac{1}{2},\, \frac{\nu}{2}}\left(\frac{a}{\varkappa}\right) = h(\varkappa)$$

where $R(a) > 0$.

Therefore

$$p^{\varrho-\frac{1}{s}}\, \phi(p^{\frac{1}{s}}) = 2\, a^{\frac{1}{2}}\, p^{\varrho-\frac{\mu-1}{s}}\, K_{\nu}(2\, a^{\frac{1}{2}}\, p^{\frac{1}{2s}})$$

Taking the inverse LAPLACE transform [4] of the right hand side and applying [2, p. 210] we obtain

$$\psi(\varkappa) = (2\pi)^{-s}\, a^{\frac{3}{2}-\mu-\varrho s}\, s^{2(\varrho s+\mu-1)} \sum_{i,-i} \frac{1}{i}\, E\left(\Delta(s,\, \mu + \varrho s \pm \tfrac{\nu}{2} - 1) : : \frac{a^s\, e^{i\pi}}{s^{2s}\, \varkappa}\right)$$

where $|\arg a| < \frac{\pi}{2}(1 - \frac{1}{2s})$ and $R(p) > 0$.

Substituting the above values of $h(t)$ and $\psi(\varkappa)$ in (2.6) we get

$$\int_0^{\infty} t^{\frac{1}{2}-\mu-\varrho s}\, e^{-\frac{1}{2}\frac{a}{t}}\, W_{\mu-\frac{1}{2},\, \frac{\nu}{2}}\left(\frac{a}{t}\right) \sum_{i,-i} \frac{1}{i}\, E\left(\Delta(s,\, \varrho\, s) : : (\tfrac{t}{s})^s\, \frac{e^{i\pi}}{\varkappa}\right) dt$$

$$= a^{\frac{3}{2}-\mu-\varrho s}\, (2\pi)^{\frac{1}{2}(1-s)}\, s^{\varrho s\, \cdot\, 2\mu-\frac{5}{2}} \sum_{i,-i} \frac{1}{i}\, E\left(\Delta(s,\, \mu+\varrho s \pm \tfrac{\nu}{2} - 1) : : \frac{a^s\, e^{i\pi}}{s^{2s}\, \varkappa}\right) \quad (2.8)$$

where $s > 1$, $R(a) > 0$ and $|\arg \varkappa| < (s-1)\frac{\pi}{2}$.

3. THEOREM 2. If

$$\phi(p) \underset{\nu}{\overset{k}{=\!=}} h(\varkappa)$$

and

$$p^{\sigma+\frac{3}{2}\frac{n}{s}}\, \phi(p^{-\frac{n}{s}}) \underset{\mu}{\overset{k}{=\!=}} \psi(\varkappa)$$

then

$$\psi(\varkappa) = (2n)^{\sigma-\frac{1}{2}} (2\pi)^{n-s} \varkappa^{-\sigma} \int_0^\infty t^{\frac{1}{2}} h(t)$$

$$\times G^{2s,\,0}_{0,\,2s+2n}\left[\left(\frac{t}{2s}\right)^{2s}\left(\frac{\varkappa}{2n}\right)^{2n}\middle|\,\Delta(s,\pm\tfrac{\nu}{2}),\,\Delta\left(n,\frac{2\sigma\pm2\mu+1}{4}\right)\right] dt \qquad (3.1)$$

provided that the integral is convergent, MEIJER transforms of $|h(x)|$ and $|\psi(\varkappa)|$ exist, $R(\frac{3}{2}s \pm n\nu \pm \mu s - \sigma s) > 0$ $s \geq n$ and $R(p) > 0$.

To prove the theorem we use the result [4].

$$p^{\sigma} K_{\nu}(t\,p^{-\frac{n}{s}}) \underset{\mu}{\overset{k}{=}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (2n)^{\sigma-\frac{1}{2}} (2\pi)^{n-s} \varkappa^{-\sigma}$$

$$\times\; G^{2s,\,0}_{0,\,2s+2n}\left[\left(\frac{t}{2s}\right)^{2s}\left(\frac{\varkappa}{2n}\right)^{2n}\middle|\,\Delta(s,\pm\tfrac{\nu}{2}),\,\Delta\left(n,\frac{2\sigma\pm2\mu+1)}{4}\right]$$

where $R(\pm n\nu + \frac{3}{2}s \pm \mu s - \sigma s) > 0$, $|\arg t| < (1-\frac{n}{s})\frac{\pi}{2}$, $s > n$ and $R(p) > 0$ and proceed in a manner similar to that of theorem 1. When $n = s$, t should be real and positive.

COROLLARY. Taking $n = s = 1$ in (3.1) we obtain

If

$$\phi(p) \underset{\nu}{\overset{k}{=}} h(\varkappa)$$

and

$$p^{\sigma+\frac{3}{2}} \phi\left(\frac{1}{p}\right) \underset{\mu}{\overset{k}{=}} \psi(\varkappa)$$

then

$$\psi(\varkappa) = 2^{\sigma+\frac{3}{2}} x^{-\sigma-1} \int_0^\infty t^{-\frac{1}{2}} h(t)\, S_2\left(-\frac{\nu}{2}, \frac{\nu}{2}, \frac{2\sigma-2\mu+1}{4}, \frac{2\sigma+2\mu+1}{4}; \frac{t\varkappa}{4}\right) dt, \quad (3.2)$$

provided that the integral is convergent, MEIJER transforms of $|h(\varkappa)|$ and $|\psi(\varkappa)|$ exist, $R(\pm\nu\pm\mu-\sigma+\frac{3}{2}) > 0$ and $R(p) > 0$.

If we put $\mu = \nu = \frac{1}{2}$ in the above corollary we get TRICOMI'S theorem [7].

Example 2.1. Let

$$h(x) = x^{\varrho} e^{-\frac{1}{2}ax^2} W_{k,m}(ax^2)$$

then [5] gives

$$\phi(p) = (2\pi)^{-\frac{1}{2}} 2^{\varrho+\frac{1}{2}} p^{-\varrho} G^{2,2}_{2,3}\left[\frac{p^2}{4a} \middle| \begin{matrix} \frac{1}{2}-m, \frac{1}{2}+m \\ \frac{3+2\varrho+2\nu}{4}, \frac{3+2\varrho-2\nu}{4}, k \end{matrix}\right]$$

Therefore

$$p^{\sigma+\frac{3}{2}} \phi\left(\tfrac{1}{p}\right) = (2\pi)^{-\frac{1}{2}} 2^{\varrho+\frac{1}{2}} p^{\sigma+\varrho+\frac{3}{2}} G^{2,2}_{2,3}\left[\frac{1}{4a p^2} \middle| \begin{matrix} \frac{1}{2}-m, \frac{1}{2}+m \\ \frac{3+2\varrho-2\nu}{4}, \frac{3+2\varrho+2\nu}{4}, k \end{matrix}\right]$$

Writing down the inverse MEIJER transform [4] of the R.H.S. we get

$$\psi(x) = 2^{2\varrho+\sigma+\frac{3}{2}} x^{-\varrho-\sigma-\frac{3}{2}} G^{2,2}_{2,5}\left[\frac{x^2}{16a} \middle| \begin{matrix} \frac{1}{2}-m, \frac{1}{2}+m \\ \frac{3+2\varrho-2\nu}{4}, \frac{3+2\varrho+2\nu}{4}, k, \frac{\varrho+\sigma-\mu+2}{2}, \frac{\varrho+\sigma+\mu+2}{2} \end{matrix}\right]$$

Substituting the values of $h(t)$ and $\psi(x)$ in (3.2) we obtain

$$\int_0^\infty t^{\varrho-\frac{1}{2}} e^{-\frac{1}{2}at^2} W_{k,m}(at^2)\, S_2\left(-\frac{\nu}{2}, \frac{\nu}{2}, \frac{2\sigma-2\mu+1}{4}, \frac{2\sigma+2\mu+1}{4}; \frac{tx}{4}\right) dt$$

$$= 2^{2\varrho} x^{-\varrho-\frac{1}{2}} G^{2,2}_{2,5}\left[\frac{x^2}{16a} \middle| \begin{matrix} \frac{1}{2}-m, \frac{1}{2}+m \\ \frac{3+2\varrho-2\nu}{4}, \frac{3+2\varrho+2\nu}{4}, k, \frac{\varrho+\sigma-\mu+2}{2}, \frac{\varrho+\sigma+\mu+2}{2} \end{matrix}\right] \quad (3.3)$$

where $R(\varrho \pm 2m \pm \nu + \frac{5}{2}) > 0$, $R(a) > 0$ and $R(x) > 0$.

4. We shall now give some theorems involving the transforms (1.2) and (1.3)

THEOREM 3. If

$$\phi(p) \underset{k,\, m}{\overset{\nu}{=\!=}} h(x)$$

and

$$p^{\sigma-(m+\frac{1}{2})\frac{2n}{s}}\phi\left(p^{\frac{2n}{s}}\right) \underset{\nu}{\overset{k}{=}} \psi(x)$$

then

$$\psi(x) = (2n)^{\sigma-\frac{1}{2}}(2\pi)^{\frac{1}{2}(2n-s)} s^{k+\frac{1}{2}} x^{-\sigma}$$

$$\times \int_0^\infty t^{m-\frac{1}{2}} h(t)\, G^{2s,\,0}_{s+2n,\,2s}\left[\left(\frac{t}{s}\right)^s \left(\frac{2n}{x}\right)^{2n} \middle| \begin{matrix} \Delta(s,\, 1-k),\, \Delta\left(n,\, \frac{3-2\sigma\pm 2\nu}{4}\right) \\ \Delta\left(s, \frac{1}{2}\pm m\right) \end{matrix}\right] dt, \quad (4.1)$$

provided that the integral is convergent, the VARMA transform of $|h(x)|$ and the MEIJER transform of $|\psi(x)|$ exist, $s > 2n$ and $R(p) > 0$.

PROOF : To prove (4.1) we use the result [4]

$$p^\sigma e^{-\frac{1}{2}t\, p^{\frac{2n}{s}}} W_{k,m}\left(t p^{\frac{2n}{s}}\right) \underset{\nu}{\overset{k}{=}} (2n)^{\sigma-\frac{1}{2}}(2\pi)^{\frac{1}{2}(2n-s)} s^{k+\frac{1}{2}}$$

$$\times x^{-\sigma} G^{2s,\,0}_{s+2n,\,2s}\left[\left(\frac{t}{s}\right)^s \left(\frac{2n}{x}\right)^{2n} \middle| \begin{matrix} \Delta(s,\, 1-k),\, \Delta\left(n,\, \frac{3-2\sigma\pm 2\nu}{4}\right) \\ \Delta\left(s, \frac{1}{2}\pm m\right) \end{matrix}\right]$$

where $|\arg t| < \left(1-\frac{2n}{s}\right)\frac{\pi}{2}$, $s > 2n$ and $R(p) > 0$

and proceed in a manner similar to that given during the proof of theorem 1.

COROLLARY. When $k + m = \frac{1}{2}$; $\sigma = \nu - \frac{1}{2}$ and $n = 1$ the theorem takes the following form

If

$$\phi(p) \doteqdot h(x)$$

and

$$p^{\nu-\frac{1}{2}-\left(\frac{2m+1}{2s}\right)}\phi\left(p^{\frac{1}{s}}\right) \underset{\nu}{\overset{k}{=}} \psi(x)$$

then

$$\psi(x) = 2^{\nu-m}(2\pi)^{-s} s^{1-m} x^{\frac{1}{2}-\nu}$$

$$\times \int_0^\infty t^{m-\frac{1}{2}} h(t) \sum_{i,-i} \frac{1}{i} E\left(\Delta\left(2s, \frac{1}{2}-m\right): 1-\nu :: \left(\frac{t}{2s}\right)^{2s} e^{i\pi}\frac{4}{x^2}\right) dt \quad (4.2)$$

Provided that the integral is convergent, the LAPLACE transform of $|h(\varkappa)|$ and the MEIJER transform of $|\psi(\varkappa)|$ exist, $s > 1$ and $R(p) > 0$. (4.2) can also be obtained from (2.1) if we take $\nu = \frac{1}{2}$ and $n = 1$ in it.

Example 3.1. We have [3, p. 294]

$$\phi(p) = p^{-2\mu}\, e^{\frac{1}{2}ap^2}\, W_{k,\mu}(ap^2)$$

$$\doteqdot \frac{2^{\mu-k+1}\, a^{\frac{1}{2}(\mu+k+1)}}{\Gamma(1-2k+2\mu)}\, \varkappa^{-k-1+\mu}\, e^{-\frac{\varkappa^2}{8a}}\, M_{-\frac{1}{2}(k+3\mu),\, \frac{1}{2}(\mu-k)}\left(\frac{\varkappa^2}{4a}\right) = h(\varkappa)$$

where $R(a) > 0$, $R(k-\mu) < \frac{1}{2}$.

Therefore

$$p^{\nu-\frac{1}{2}-\left(\frac{2m+1}{2s}\right)}\, \phi(p^{\frac{1}{s}}) = p^{\nu-\frac{1}{2}-\frac{1}{2s}(4\mu+2m+1)}\, e^{\frac{1}{2}ap^{\frac{2}{s}}}\, W_{k,\mu}(ap^{\frac{2}{s}})$$

Writing down the inverse MEIJER transform [4] of R.H.S. we have

$$\psi(\varkappa) = \frac{2^{\nu-1-\frac{1}{2s}(4\mu+2m+1)}\, (2\pi)^{\frac{1}{2}(4-3s)}\, s^{\frac{1}{2}(1-2k)}}{\Gamma(\frac{1}{2}-k\pm\mu)}\, \varkappa^{\frac{1}{2}-\nu+\frac{1}{2s}(4\mu+2m+1)}$$

$$\times\, G^{2s,\,s}_{s+2,\,2s}\left[\left(\frac{a}{s}\right)^s \frac{4}{\varkappa^2}\,\middle|\, \begin{matrix} \Delta(s,1+k),\ \frac{4s(1-\nu)+2m+4\mu+1}{4s},\ \frac{4s+2m+4\mu+1}{4s} \\ \Delta(s,\frac{1}{2}\pm\mu) \end{matrix}\right]$$

where $|\arg a| < (3-\frac{2}{s})\frac{\pi}{2}$ and $R(p) > 0$.

Putting these values of $h(t)$ and $\psi(\varkappa)$ in (4.2) we get

$$\int_0^\infty t^{\mu+m-k-\frac{3}{2}}\, e^{-\frac{t^2}{8a}}\, M_{-\frac{1}{2}(k+3\mu),\, \frac{1}{2}(\mu-k)}\left(\frac{t^2}{4a}\right)$$

$$\times \sum_{i,-i} \frac{1}{i}\, E\left(\Delta(2s, \tfrac{1}{2}-m) : 1-\nu :: \left(\frac{t}{2s}\right)^{2s} e^{i\pi}\frac{4}{\varkappa^2}\right) dt$$

$$= \frac{\Gamma(1-2k+2\mu)\ (2\pi)^{\frac{1}{2}(4-s)}}{\Gamma(\frac{1}{2}-k\pm\mu)\ a^{\frac{1}{2}(\mu+k+1)}}\, s^{m-k-\frac{1}{2}}\, 2^{k+m-\mu-2-\frac{1}{2s}(4\mu+2m+1)}$$

$$\times\, \varkappa^{\frac{1}{2s}(4\mu+2m+1)}$$

$$\times\, G^{2s,\,s}_{s+2,\,2s}\left[\left(\frac{a}{s}\right)^s \frac{4}{\varkappa^2}\,\middle|\, \begin{matrix} \Delta(s,1+k),\ \frac{4s(1-\nu)+4\mu+2m+1}{4s},\ \frac{4s+4\mu+2m+1}{4s} \\ \Delta(s,\frac{1}{2}\pm\mu) \end{matrix}\right] \quad (4.3)$$

where $R(2\mu - 2k + 1) > 0$, $|\arg \varkappa| < (s-1)\frac{\pi}{2}$, $s > 1$, and $R(a) > 0$.

Proceeding as in theorem 1 and applying the result [4]

$$p^{\sigma}\, e^{-\frac{1}{2}tp^{-\frac{2n}{s}}}\, W_{k,m}\,(tp^{-\frac{2n}{s}}) \underset{\nu}{\overset{k}{=\!=}} (2n)^{\sigma-\frac{1}{2}}\, (2\pi)^{\frac{1}{2}(2n-s)}\, s^{k+\frac{1}{2}}$$

$$\times\, \varkappa^{-\sigma}\, G^{2s,\,0}_{s,\,2s+2n}\left[\left(\frac{t}{s}\right)^{s}\left(\frac{\varkappa}{2n}\right)^{2n} \middle|\, \begin{matrix} \Delta(s,\,1-k) \\ \Delta(s,\,\tfrac{1}{2}\pm m),\ \Delta\!\left(n,\,\dfrac{2\sigma \pm 2\nu + 1}{4}\right)\end{matrix}\right]$$

where $R(n + \frac{3}{2}s \pm 2m\, n \pm \nu\, s - \sigma s) > 0$, $|\arg t| \leq (1 - \frac{2n}{s})\frac{\pi}{2}$, $s \geq 2n$ and $R(p) > 0$ we get the following theorem.

THEOREM 4. If

$$\phi\,(p) \underset{k,\,m}{\overset{v}{=\!=}} h\,(\varkappa)$$

and

$$p^{\sigma+(m+\frac{1}{2})\frac{2n}{s}}\,\phi\,(p^{-\frac{2n}{s}}) \underset{\nu}{\overset{k}{=\!=}} \psi\,(\varkappa)$$

then

$$\psi\,(\varkappa) = (2n)^{\sigma-\frac{1}{2}}\,(2\pi)^{\frac{1}{2}(2n-s)}\, s^{k+\frac{1}{2}}\,\varkappa^{-\sigma}$$

$$\times \int_0^{\infty} t^{m-\frac{1}{2}}\, h\,(t)\; G^{2s,\,0}_{s,2s+2n}\left[\left(\frac{t}{s}\right)^{s}\left(\frac{\varkappa}{2n}\right)^{2n} \middle|\, \begin{matrix} \Delta(s,\,1-k) \\ \Delta(s,\tfrac{1}{2}\pm m),\, \Delta\!\left(n,\,\dfrac{2\sigma\pm 2\nu+1}{4}\right)\end{matrix}\right] dt, \quad (4.4)$$

provided that the integral is convergent, VARMA transform of $|h(\varkappa)|$ and MEIJER transform of $|\psi(\varkappa)|$ exist, $s \geq 2n$ and $R(p) > 0$.

COROLLARY. On putting $k + m = \frac{1}{2}$; $\nu = \frac{1}{2}$, $n = 1$, and $s = 2$ in (4.4) we get TRICOMI's theorema [7]

ACKNOWLEDGEMENT

The author is thankful to Dr. K. C. SHARMA for his keen interest and help during the preparation of this paper.

Department of Mathematics,
University of Rajasthan,
JAIPUR, INDIA.

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