A STUDY OF VARMA TRANSFORM

By

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1. INTRODUCTORY. The transform defined by the integral equation

$$\Phi(p) = p\int_0^\infty (pt)^{m-\frac{1}{2}} e^{-\frac{1}{2}pt} W_{k,m}(pt) f(t) \, dt, \tag{1.1}$$

which is a generalization of the classical LAPLACE transform

$$\Phi(p) = p\int_0^\infty e^{-pt} f(t) \, dt \tag{1.2}$$

has attracted the attention of many workers in the field of Pure Mathematics during the last decade. (1.1) was introduced into Mathematical Analysis by VARMA [**14**] and it reduces to (1.2) when $k + m = \frac{1}{2}$ by virtue of the identity

$$W_{\frac{1}{2}-m,m}(x) = x^{\frac{1}{2}-m} e^{-\frac{1}{2}x}.$$

The object of the present paper is to establish certain new theorems on the VARMA transform defined by (1.1). The theorems are quite general and their importance lies in the fact they give rise, as their particular cases, to many important results given earlier by GUPTA [**5**], NARAIN [**6**], SAKSENA [**7**] and SAXENA [**8, 10, 11** and **12**]. The well known TRICOMI's theorem of the Operational calculus also comes out as a special case of one of our results.

Throughout this paper the conventional notations $\Phi(p) \underset{k,m}{\overset{v}{=}} f(t)$ and $\Phi(p) \doteqdot f(t)$ will be used to denote (1.1) and (1.2) respectively. In what follows, n and s are positive integers and the symbol $\Delta(n\,;\,\alpha)$ denotes the set of parameters

$$\frac{\alpha}{n}, \frac{\alpha+1}{n}, \frac{\alpha+2}{n}, \ldots\ldots, \frac{\alpha+n-1}{n}.$$

2. THEOREM 1. If

$$\psi(p) \doteqdot f(t)$$

and

$$\Phi(p:\ a,\ \sigma) \underset{k,\,m}{\overset{v}{=}} t^{\sigma}\, e^{-a t^{s/n}}\, f(t^{s/n}),$$

then

$$\Phi(p: a,\ \sigma) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{\sigma+k+m}\, n^{\frac{1}{2}}\, p^{-\sigma}$$

$$\times \int_{\xi}^{\infty} (t+a)^{-1}\, G^{\,2s,\,0}_{\,s+n,\,2s}\left[\frac{n^n p^s}{s^s\ t^n}\,\middle|\,\begin{matrix}\Delta(s;\ \frac{3}{2}+\sigma-k+m),\ \Delta(n;\ 1)\\ \Delta(s;\ \sigma+1),\ \Delta(s;\ \sigma+2m+1)\end{matrix}\right] \psi(t+a)\, dt, \quad (2.1)$$

where $\xi = 0$ when $s > n$ but $\xi = p$ when $s \to n$,
provided that the integral is convergent, the LAPLACE transform of $|f(t)|$ and the VARMA transform of $|t^{\sigma}\, e^{-a t^{s/n}}\, f(t^{s/n})|$ exist, $R(a) > 0$, $R(p) > 0$ and $2m$ is not an integer.

PROOF. Since $\psi(p) \doteqdot f(t)$, we have

$$\frac{p\,\psi(p+a)}{p+a} \doteqdot e^{-at}\, f(t),\ R(p+a) > 0,$$

by virtue of a well known property of Operational calculus.
We also have [10]

$$p^{\varrho}\, e^{-\frac{1}{2}\alpha p^{n/s}}\, W_{k,\,m}(\alpha p^{n/s}) \doteqdot (2\pi)^{\frac{1}{2}(n-s)}\, s^{k+\frac{1}{2}}\, n^{\varrho-\frac{1}{2}}$$

$$\times\, t^{-\varrho}\, G^{\,2s,\,0}_{\,s+n,\,2s}\left[\frac{n^n \alpha^s}{s^s\ t^n}\,\middle|\,\begin{matrix}\Delta(s;\ 1-k),\ \Delta(n;\ 1-\varrho)\\ \Delta(s;\ \frac{1}{2}+m),\ \Delta(s;\ \frac{1}{2}-m)\end{matrix}\right],$$

where $R(p) > 0$, $|\arg \alpha^s| < \frac{1}{2}(s-n)\pi$ and $s > n$; when $s \to n$, $t > \alpha$.

Using the last two operational pairs in the Goldstein's form of the PARSEVAL'S FORMULA of the Operational calculus [4], we get

$$\int_{0}^{\infty} e^{-\frac{1}{2}\alpha t^{n/s}}\, W_{k,\,m}(\alpha\, t^{n/s})\, t^{\varrho-1}\, e^{-at}\, f(t)\, dt$$

$$= (2\pi)^{\frac{1}{2}(n-s)}\, s^{k+\frac{1}{2}}\, n^{\varrho-\frac{1}{2}} \int_{0}^{\infty} t^{-\varrho}(t+a)^{-1}\, G^{\,2s,\,0}_{\,s+n,\,2s}\left[\frac{n^n \alpha^s}{s^s\, t^n}\,\middle|\,\begin{matrix}\Delta(s;\ 1-k),\ \Delta(n;\ 1-\varrho)\\ \Delta(s;\frac{1}{2}+m),\ \Delta(s;\frac{1}{2}-m)\end{matrix}\right]$$

$$\psi(t+a)\, dt$$

Putting now $t = x^{s/n}$ in the integral on the left, replacing ϱ by $\frac{n}{s}(\sigma + m + \frac{1}{2})$, α by p and using [**2**, p. 209, eq. 8], we arrive at the result (2.1).

CORROLARY 1.ª Putting $k = \frac{1}{2} - m$ in the theorem, we obtain the result:

If

$$\psi(p) \doteqdot f(t)$$

and

$$\Phi(p:\ a,\ \sigma) \doteqdot t^{\sigma}\ e^{-at^{s/n}}\ f(t^{s/n}),$$

then

$$\Phi(p:\ a,\ \sigma) = (2\pi)^{\frac{1}{2}(n-s)}\ s^{\sigma+\frac{1}{2}}\ n^{\frac{1}{2}}\ p^{-\sigma}$$

$$\times \int_{\xi}^{\infty} (t+a)^{-1}\ G^{s,\ 0}_{n,\ s}\left[\frac{n^n p^s}{s^s t^n}\middle| \begin{matrix} \Delta\ (n;\ 1) \\ \Delta\ (s;\ \sigma+1) \end{matrix}\right] \psi\ (t+a)\ \mathrm{dt}, \quad (2.2)$$

where $\xi = 0$ when $s > n$ but $\xi = p$ when $s \to n$, provided that the integral is convergent, the LAPLACE transforms of $|\ f(t)\ |$ and $|\ t^{\sigma}\ e^{-at^{s/n}}\ f(t^{s/n})\ |$ exist, $R(a) > 0$ and $R(p) > 0$.

Particular cases of the cor. 1.ª (*i*) When $n = 1$, (2.2) yields the following result:

If

$$\psi(p) \doteqdot f(t)$$

and

$$\Phi(p:\ a,\ \sigma) \doteqdot t^{\sigma}\ e^{-at^{s}}\ f(t^{s}),$$

then

$$\Phi(p:\ a,\ \sigma) = (2\pi)^{-\frac{1}{2}(1+s)}\ s^{\sigma+\frac{1}{2}}\ p^{-\sigma}$$

$$\times \int_{\xi}^{\infty} (t+a)^{-1} \sum_{i,\,-i} \frac{1}{i} E\left\{\Delta\ (s;\ \sigma+1) :: \frac{p^s\ e^{i\pi}}{s^s\ t}\right\} \psi\ (t+a)\ \mathrm{dt}, \quad (2.3)$$

where $\xi = 0$ when $s > 1$ but $\xi = p$ when $s = 1$, provided that the integral is convergent, the LAPLACE transforms of $|\ f(t)\ |$ and $|\ t^{\sigma}\ e^{-at^{s}}\ f(t^{s})\ |$ exist, $R(a) > 0$ and $R(p) > 0$.

As a special case, when $a \to 0$, (2.2) and (2.3) reduce to results given earlier by SAXENA [**10, 8**].

(*ii*) On taking $s = 2$ also, we find that :

If

$$\psi(p) \doteqdot f(t)$$

and

$$\Phi(p : a, \sigma) \doteqdot t^{\sigma} e^{-at^2} f(t^2),$$

then

$$\Phi(p : a, \sigma) = 2^{-\frac{1}{2}(\sigma+2)} \pi^{-\frac{1}{2}} p$$

$$\times \int_0^{\infty} t^{-\frac{1}{2}(\sigma+1)} (t+a)^{-1} e^{-p^2/8t} D_{\sigma}\left(\frac{p}{\sqrt{2t}}\right) \psi(t+a) \, dt \qquad (2.4)$$

provided that the integral is convergent, the LAPLACE transforms of $|f(t)|$ and $|t^{\sigma} e^{-at^2} f(t^2)|$ exist, $R(a) > 0$ and $R(p) > 0$.

In obtaining (2.3), (2.4) we have used the results [**10**]

$$2\pi \, G^{s,0}_{1,s}\left[x \,\middle|\, \begin{matrix} 1 \\ \Delta(s; \varrho s) \end{matrix}\right] = \sum_{i,-i} \frac{1}{i} \, E\left\{\Delta(s; \varrho s) :: \varkappa e^{i\pi}\right\},$$

where $s = 2, 3, 4, \ldots$

and

$$G^{2\,0}_{1\,2}\left(\varkappa \,\middle|\, \begin{matrix} a \\ b, \, b+\frac{1}{2} \end{matrix}\right) = 2^{-\nu} x^b e^{-\frac{1}{2}x} D_{2\nu}(\sqrt{2x}) \text{ where } \nu = b - a + \frac{1}{2}.$$

We shall now illustrate the theorem by evaluating a few infinite integrals with ist help.

Example 1.[a] Taking [**3**]

$$f(t) = t^{-\varrho} e^{\frac{1}{2}at} W_{\lambda,\mu}(at)$$

$$\doteqdot \frac{\Gamma(\frac{3}{2} - \varrho + \mu)\,\Gamma(\frac{3}{2} - \varrho - \mu)}{\Gamma(2 - \varrho - \lambda)} a^{\mu+\frac{1}{2}} p^{\varrho-\mu-\frac{1}{2}}$$

$$\times \, {}_2F_1\left(\frac{3}{2} - \varrho + \mu, \frac{1}{2} + \mu - \lambda; \, 2 - \varrho - \lambda; \frac{p-a}{p}\right) = \psi(p)$$

where $R(\frac{3}{2} - \varrho \pm \mu) > 0$ and $R(p) > 0$, we have [**9**]

$$t^{\sigma} e^{-at^{s/n}} f(t^{s/n}) = t^{\sigma - \frac{s}{n}\varrho} e^{-\frac{1}{2}at^{s/n}} W_{\lambda,\mu}(at^{s/n})$$

$$\frac{\nu}{k, m} (2\pi)^{\frac{1}{2}(2-n-s)} s^{k+m+\sigma} n^{\lambda-\varrho+\frac{1}{2}} a^{\varrho} p^{-\sigma}$$

$$\times G^{2s,\,2n}_{s+2n,\,2s+n}\left[\frac{n^n p^s}{a^n s^s}\middle| \begin{matrix}\Delta(n;\tfrac{1}{2}+\varrho+\mu),\Delta(n;\tfrac{1}{2}+\varrho-\mu),\Delta(s;\tfrac{3}{2}+\sigma-k+m)\\ \Delta(s;\sigma+1),\Delta(s;\sigma+2m+1),\Delta(n;\lambda+\varrho)\end{matrix}\right]=\Phi(p{:}a,\sigma),$$

where $R(\sigma + m \pm m - \frac{s}{n}\varrho \pm \frac{s}{n}\mu + \frac{s}{2n} + 1) > 0$, $R(a) > 0$ and $R(p) > 0$.

Applying (2.1) and changing the parameters slightly, we get

$$\int_{\xi}^{\infty} {}_2F_1\left(\alpha,\beta;\gamma;-\frac{t}{a}\right) G^{2s,\,0}_{s+n,\,2s}\left[\frac{b^n}{t^n}\middle| \begin{matrix}\Delta(s;\tfrac{1}{2}+\sigma-k+m),\ \Delta(n;1)\\ \Delta(s;\sigma),\ \Delta(s;\sigma+2m)\end{matrix}\right]\mathrm{dt}$$

$$= (2\pi)^{1-n}\, n^{\alpha+\beta-\gamma-1}\, a.\,\{\Gamma(\gamma)/\Gamma(\alpha)\,\Gamma(\beta)\}$$

$$\times G^{2s,\,2n}_{s+2n,\,2s+n}\left[\frac{b^n}{a^n}\middle| \begin{matrix}\Delta(n;2-\alpha),\Delta(n;2-\beta),\ \Delta(s;\tfrac{1}{2}+\sigma-k+m)\\ \Delta(s;\sigma),\ \Delta(s;\sigma+2m),\Delta(n;2-\gamma)\end{matrix}\right], \tag{2.5}$$

where $\xi = 0$ when $s > n$ but $\xi = b$ when $s \to n$; $R[\alpha + \frac{n}{s}(\sigma + m \pm m)] > 1$, $R[\beta + \frac{n}{s}(\sigma + m \pm m)] > 1$, $R(a) > 0$, $|\arg b^n| < \frac{1}{2}(s-n)\pi$ and $2m$ is not an integer.

An interesting particular case of (2.5) is obtained on putting $k = \frac{1}{2} - m$, $n = 1$ and is the result

$$\int_{\xi}^{\infty} {}_2F_1\left((\alpha,\beta;\gamma;-\frac{t}{a}\right)\cdot \sum_{i,-i}\frac{1}{i}\ E\left\{\Delta(s;\sigma)::\frac{be^{i\pi}}{t}\right\}\mathrm{dt}$$

$$= 2\pi\, a.\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\ G^{s,\,2}_{2,\,s+1}\left[\frac{b}{a}\middle| \begin{matrix}2-\alpha,\ 2-\beta\\ \Delta(s;\sigma),\ 2-\gamma\end{matrix}\right], \tag{2.6}$$

where $\xi = 0$ when $s > 1$ and $\xi = b$ when $s = 1$, $R(\alpha + \frac{\sigma}{s}) > 1$, $R(\beta + \frac{\sigma}{s}) > 1$, $R(a) > 0$ and $|\arg b| < \frac{1}{2}(s-1)\pi$.

On taking $s = 2$, this gives

$$\int_0^{\infty} t^{-\frac{1}{2}(\nu+1)}\, e^{-p^2/8t}\, D_\nu\left(\frac{p}{\sqrt{2t}}\right) {}_2F_1\left(\alpha,\beta;\gamma;-\frac{t}{a}\right)\mathrm{dt}$$

$$= 2^{\frac{1}{2}(3\nu+2)}\, a\, p^{-(\nu+1)}\ \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\ G^{2\,2}_{2\,3}\left(\frac{p^2}{4a}\middle| \begin{matrix}2-\alpha,\ 2-\beta\\ \frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}\nu+1,\ 2-\gamma\end{matrix}\right), \tag{2.7}$$

where $R(\nu + 2\alpha) > 1$, $R(\nu + 2\beta) > 1$ and $R(p^2) > 0$.

When $\beta = \gamma$ and $\nu = 0$ (2.7) reduces to a known result [**3**, p. 139].

Example 1b. Starting with [3]

$$f(t) = t^{-(\mu+\frac{1}{2})}\, e^{at}\, k_{\nu+\frac{1}{2}}\,(zt)$$

$$\doteq \tfrac{1}{2}(2\pi)^{\frac{1}{2}} z^{-\frac{1}{2}}\, \Gamma(\nu-\mu+1)\, \Gamma(-\nu-\mu)\, p\,\{(p-a)^2-z^2\}^{\frac{1}{2}\mu}\, P_\nu^\mu\left(\frac{p-a}{z}\right) = \psi(p)$$

where $R(\mu-1) < R(\nu) < -R(\mu)$ and $R(p-a+z) > 0$, we have

$$t^\sigma\, e^{-ct^{s/n}}\, f(t^{s/n}) = t^{\sigma-s/n(\mu+\frac{1}{2})}\, k_{\nu+\frac{1}{2}}\,(zt^{s/n})$$

$$\doteq \sum_{k,\,m}^{v} (2\pi)^{\frac{1}{2}(3-2n-2s)}\, 2^{k+m+\sigma-\mu-\frac{3}{2}}\, s^{k+m+\sigma}\, n^{-\mu-\frac{1}{2}}\, z^{\mu+\frac{1}{2}}\, p^{-\sigma}$$

$$\times G^{4s,\,2n}_{2n+2s,\,4s}\left[\frac{4^{n-s}\, n^{2n}\, p^{2s}}{z^{2n}\, s^{2s}}\,\middle|\, \begin{matrix}\Delta(n;\, \frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu),\, \Delta(n;\, 1+\frac{1}{2}\mu-\frac{1}{2}\nu), \Delta(2s;\, \frac{3}{2}+\sigma-k+m)\\ \Delta(2s;\, \sigma+1),\ \Delta(2s;\, \sigma+2m+1)\end{matrix}\right],$$

$$= \Phi\,(p\colon a;\, \sigma)$$

where $R\,[\sigma+m\pm m-\frac{s}{n}(\mu+\frac{1}{2})\pm\frac{s}{n}(\nu+\frac{1}{2})+1] > 0$ and $R\,(p+z) > 0$.

Applying (2.1) and making slight changes in the parameters, we obtain

$$\int_\xi^\infty (t^2-a^2)^{\frac{1}{2}\mu}\, P_\nu^\mu\left(\frac{t}{a}\right) G^{2s,\,0}_{s+n,\,2s}\left[\frac{b^n}{t^n}\,\middle|\, \begin{matrix}\Delta\,(s;\, \frac{1}{2}+\sigma-k+m),\ \Delta\,(n;\, 1)\\ \Delta\,(s;\, \sigma),\ \Delta\,(s;\, \sigma+2m)\end{matrix}\right] \mathrm{dt}$$

$$= (2\pi)^{\frac{1}{2}(2-s-3n)}\, 2^{\sigma+k+m-\mu-\frac{3}{2}}\, n^{-\mu-1}\, a^{\mu+1}\, \frac{1}{\Gamma\,(\nu-\mu+1)\ \Gamma\,(-\nu-\mu)}$$

$$\times G^{4s,\,2n}_{2n+2s,\,4s}\left[\frac{4^{n-s}\, b^{2n}}{a^{2n}}\,\middle|\, \begin{matrix}\Delta(n;\, \frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu),\ \Delta(n;\, 1+\frac{1}{2}\mu-\frac{1}{2}\nu),\ \Delta(2s;\, \frac{1}{2}+6-k+m)\\ \Delta(2s;\, \sigma),\ \Delta\,(2s;\, \sigma+2m)\end{matrix}\right], \tag{2.8}$$

where $\xi=0$ when $s>n$ but $\xi=b$ when $s\to n$, $R[\frac{n}{s}(\sigma+m\pm m)-\mu-\nu] > 1$, $R[\frac{n}{s}(\sigma+m\pm m)-\mu+\nu] > 0$, $R\,(a) > 0$, $|\arg b^n| < \frac{1}{2}(s-n)\,\pi$ and $2m$ is not an integer.

When $k = \frac{1}{2}-m$ and $n = 1$, (2.8) yields the integral:

$$\int_\xi^\infty (t^2-a^2)^{\frac{1}{2}\mu}\, P_\nu^\mu\,(\tfrac{t}{a}).\sum_{i,-i} \tfrac{1}{i}\, E\left\{\Delta\,(s;\, \sigma)::\frac{be^{i\pi}}{t}\right\} \mathrm{dt}$$

$$= (2\pi)^{\frac{1}{2}(1-s)}\, 2^{\sigma-\mu-1}\, a^{\mu+1}\, \frac{1}{\Gamma\,(\nu-\mu+1)\ \Gamma\,(-\nu-\mu)}$$

$$\times\, G^{2s,\,2}_{2,\,2s}\left[\frac{4^{1-s}\, b^2}{a^2}\,\middle|\,\begin{matrix} 3/2 + \frac{1}{2}\,\mu + \frac{1}{2}\,\nu,\ \ 1 + \frac{1}{2}\,\mu - \frac{1}{2}\,\nu \\ \Delta\,(2s;\ \sigma)\end{matrix}\right], \tag{2.9}$$

where $\xi = 0$ when $s > 1$ but $\xi = b$ when $s = 1$, $R\,(\mu + \nu - \frac{\sigma}{s} + 1) < 0$,

$R\,(\mu - \nu - \frac{\sigma}{s}) < 0$, $R\,(a) > 0$ and $|\arg b| < \frac{1}{2}\,(s - 1)\,\pi$.

As a particular case, when we take $s = 2$, $\sigma = 1$ and $\nu = 0$, (2.9) reduces to a known result [**3**, p. 139].

3. THEOREM 2. If

$$\psi\,(p) \doteqdot f\,(t)$$

and

$$\Phi\,(p\colon a,\,\sigma)\ \underset{k,\,m}{\overset{V}{=\!=}}\ t^{\sigma}\ e^{-at^{-s/n}}\ f\,(t^{-s/n}),$$

then

$$\Phi\,(p\colon\ a,\ \sigma) = (2\pi)^{\frac{1}{2}(n-s)}\ s^{\sigma+k+m}\ n^{\frac{1}{2}}\ p^{-\sigma}$$

$$\times \int_0^\infty (t+a)^{-1}\, G^{2s,\,0}_{s,\,2s+n}\left[\frac{p^s\, t^n}{n^n\, s^s}\,\middle|\,\begin{matrix}\Delta(s;\ \frac{3}{2}+\sigma-k+m) \\ \Delta(s;\sigma+1),\, \Delta(s;\sigma+2m+1),\, \Delta(n;\,0)\end{matrix}\right] \psi\,(t + a)\ \mathrm{dt}, \tag{3.1}$$

provided that the integral is convergent, the LAPLACE transform of $|\,f\,(t)\,|$ and the VARMA transform of $|\,t^{\sigma}\ e^{-at^{-s/n}}\ s\,(t^{-s/n})\,|$ exist, $R\ (a) > 0$, $s \geq n$ and $R\ (p) > 0$ when $s > n$ but p is real and positive when $s = n$.

The proof is similar to that of Theorem 1 and we make use of the result [**10**]

$$p^{\varrho}\ e^{-\frac{1}{2}\alpha p^{-n/s}}\ W_{k,\,m}\,(\alpha\, p^{-n/s}) \doteqdot (2\pi)^{\frac{1}{2}(n-s)}\ s^{k+\frac{1}{2}}\ n^{\varrho-\frac{1}{2}}$$

$$\times\, t^{-\varrho}\ G^{2s,\,0}_{s,\,2s+n}\left[\frac{\alpha^s\, t^n}{n^n\, s^s}\,\middle|\,\begin{matrix}\Delta\,(s;\ 1 - k) \\ \Delta\,(s;\ \frac{1}{2} + m),\ \Delta\,(s;\ \frac{1}{2} - m),\ \Delta\,(n;\ \varrho)\end{matrix}\right],$$

where $R\,(n + 2s - 2s\varrho \pm 2nm) > 0$, $|\arg \alpha^s| < \frac{1}{2}\,(s - n)\,\pi$, $R\,(p) > 0$, and $s > n$; when $s = n$, α is real and positive.

COROLLARY 2.a When $k = \frac{1}{2} - m$, (3.1) reduces to the following result:

If

$$\psi(p) \doteqdot f(t)$$

and

$$\Phi(p\colon a, \sigma) \doteqdot t^{\sigma}\, e^{-at^{-s/n}}\, f(t^{-s/n}),$$

then

$$\Phi(p\colon a, \sigma) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{\sigma+\frac{1}{2}}\, n^{\frac{1}{2}}\, p^{-\sigma}$$

$$\times \int_0^\infty (t+a)^{-1}\, G^{s,\,0}_{0,\,s+n}\left[\frac{p^s\, t^n}{n^n\, s^s}\,\middle|\, \Delta(s;\sigma+1),\, \Delta(n;0)\right] \psi(t+a)\, \mathrm{dt}, \tag{3.2}$$

provided that the integral is convergent, the LAPLACE transforms of $|f(t)|$ and $|t^{\sigma}\, e^{-at^{-s/n}}\, f(t^{-s/n})|$ exist, $R(a) > 0$, $s \geq n$ and $R(p) > 0$ when $s > n$ but p is real and positive when $s = n$.

4. THEOREM 3. If

$$\Phi(p) \underset{k,\,m}{\overset{v}{=\!=}} h(t)$$

and

$$\psi(p\colon a, \sigma) \underset{\lambda,\,\mu}{\overset{v}{=\!=}} t^{\sigma}\, e^{-(at)^{n/s}}\, h(t^{n/s}), \tag{4.1}$$

then

$$\Phi(p^{n/s}) = (2\pi)^{\frac{1}{2}(n-3s+2)}\, s^{-(k+\frac{1}{2})}\, n^{\varrho-\lambda-\mu+1}\, \frac{p^{\frac{n}{s}(m+\frac{1}{2})}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}$$

$$\times \int_0^\infty t^{-(\varrho+1)}\, G^{2s+n,\,s}_{s+2n,\,2s+n}\left[\frac{n^n p^n}{s^s\, t^n}\,\middle|\, \begin{matrix}\Delta(s;\,k+1),\ \Delta(n;\,1-\varrho),\ \Delta(n;\,2\mu-\varrho+1)\\ \Delta(s;\,\frac{1}{2}+m),\ \Delta(s;\,\frac{1}{2}-m),\ \Delta(n;\,\frac{3}{2}-\varrho-\lambda+\mu)\end{matrix}\right]$$

$$\times\, \psi\left[t\colon p,\, \tfrac{n}{s}(m+\tfrac{1}{2})-\varrho\right] \mathrm{dt}, \tag{4.2}$$

provided that the integral is convergent, the VARMA transforms of $|h(t)|$ and $|t^{\sigma}\, e^{-(at)^{n/s}}\, h(t^{n/s})|$ exist, $R(s - s\varrho - nk + s\mu \pm s\mu) > 0$, $R(p) > 0$ and $3s > n$.

PROOF. By definition

$$\Phi(p^{n/s}) = p^{n/s} \int_0^\infty (p^{n/s}\, x)^{m-\frac{1}{2}}\, e^{-\frac{1}{2}xp^{n/s}}\, W_{k,\,m}(p^{n/s}\, x)\; h(x)\; dx.$$

Substituting for $W_{k,m}(xp^{n/s})$ from [**12**]

$$W_{k,m}(xp^{n/s}) = \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\, s^{\frac{1}{2}-k}\, n^{\varrho-\lambda-\mu}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, e^{-\frac{1}{2}xp^{n/s}}\, x^{\frac{s}{n}(\mu-\varrho+\frac{1}{2})}$$

$$\times \int_0^\infty t^{\mu-\varrho-\frac{1}{2}}\, e^{-\frac{1}{2}tx^{s/n}}\, W_{\lambda,\mu}(tx^{s/n})$$

$$\times\, G^{2s+n,\, s}_{s+2n,\, 2s+n}\left[\frac{n^n p^n}{s^s\, t^n}\middle|\begin{matrix}\Delta(s;\, k+1),\ \Delta(n;\, 1-\varrho),\ \Delta(n;\, 2\mu-\varrho+1)\\ \Delta(s;\, \frac{1}{2}+m),\ \Delta(s;\, \frac{1}{2}-m),\ \Delta(n;\, \frac{3}{2}-\varrho-\lambda+\mu)\end{matrix}\right] dt,$$

where $R(s - s\varrho - nk + s\mu \pm s\mu) > 0$, $|\arg x^s| < \frac{1}{2}(3s-n)\pi$, $3s > n$ and $R(p) > 0$, we get

$$\Phi(p^{n/s}) = \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\, s^{\frac{1}{2}-k}\, n^{\varrho-\lambda-\mu}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, p^{\frac{n}{s}(m+\frac{1}{2})}$$

$$\times \int_0^\infty x^{\frac{s}{n}(\mu-\varrho+\frac{1}{2})+m-\frac{1}{2}}\, e^{-xp^{n/s}}\, h(x) \left\{ \int_0^\infty t^{\mu-\varrho-\frac{1}{2}}\, e^{-\frac{1}{2}tx^{s/n}}\, W_{\lambda,\mu}(tx^{s/n})\right.$$

$$\left. G^{2s+n,\, s}_{s+2n,\, 2s+n}\left[\frac{n^n p^n}{s^s\, t^n}\middle|\begin{matrix}\Delta(s;\, k+1),\ \Delta(n;\, 1-\varrho),\ \Delta(n;\, 2\mu-\varrho+1)\\ \Delta(s;\, \frac{1}{2}+m),\ \Delta(s;\, \frac{1}{2}-m),\ \Delta(n;\, \frac{3}{2}-\varrho-\lambda+\mu)\end{matrix}\right] dt\right\} dx.$$

$$= \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\, s^{\frac{1}{2}-k}\, n^{\varrho-\lambda-\mu}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, p^{\frac{n}{s}(m+\frac{1}{2})}$$

$$\times \int_0^\infty t^{\mu-\varrho-\frac{1}{2}}\, G^{2s+n,\, s}_{s+2n,\, 2s+n}\left[\frac{n^n p^n}{s^s\, t^n}\middle|\begin{matrix}\Delta(s;\, k+1),\ \Delta(n;\, 1-\varrho),\ \Delta(n;\, 2\mu-\varrho+1)\\ \Delta(s;\, \frac{1}{2}+m),\ \Delta(s;\, \frac{1}{2}-m),\ \Delta(n;\, \frac{3}{2}-\varrho-\lambda+\mu)\end{matrix}\right]$$

$$\left\{ \int_0^\infty x^{\frac{s}{n}(\mu-\varrho-\frac{1}{2})+m-\frac{1}{2}}\, e^{-xp^{n/s}}\, e^{-\frac{1}{2}tx^{s/n}}\, W_{\lambda,\mu}(tx^{s/n})\, h(x)\, dx \right\} dt,$$

on changing the order of integration which is permissible under the given conditions by virtue of DE LA VALLEE POUSSIN'S theorem [**1**].

The theorem now follows on putting $x = u^{n/s}$ in the inner integral and then interpreting it by (4.1).

CORROLARY 3.ª When $\lambda = \frac{1}{2} - \mu$, (4.2) reduces to the following theorem:

If

$$\Phi(p) \underset{k,\,m}{\overset{v}{=\!=}} h(t)$$

and

$$\psi\,(p\colon\ a.\ \sigma) \doteqdot t^{\sigma}\ e^{-(at)^{n/s}}\ h\,(t^{n/s}),$$

then

$$\Phi\,(p^{n/s}) = \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\ s^{-(k+\frac{1}{2})}\ n^{\varrho+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, p^{\frac{n}{s}(m+\frac{1}{2})}$$

$$\times \int_0^\infty t^{-(\varrho+1)}\, G^{\,2s,\ s}_{s+n,\ 2s}\left[\frac{n^n p^n}{s^s\, t^n}\,\middle|\,\begin{matrix}\Delta\,(s;\,k+1),\ \Delta\,(n;\ 1-\varrho)\\ \Delta(s;\,\frac{1}{2}+m),\ \Delta\,(s;\,\frac{1}{2}-m)\end{matrix}\right]\psi\,[t\colon p,\,\tfrac{n}{s}\,(m+\tfrac{1}{2})-\varrho]\ dt, \tag{4.3}$$

provided that the integral is convergent and the VARMA transform of $|\,h\,(t)\,|$ and the LAPLACE transform of $|\,t^{\sigma}\ e^{-(at)^{n/s}}\ h\,(t^{n/s})\,|$ exist, $R\,(s - s\varrho - nk + \mu s \pm \mu s) > 0$, $R\,(p) > 0$ and $3s > n$.

Two particular cases of the theorem namely when (i) $n = 1$, $s = 1$ and (ii) $n = 2$, $s = 1$, have been obtained earlier by SAXENA [**11**].

5. THEOREM 4. If

$$\Phi\,(p)\ \underset{k,\,m}{\overset{v}{=\!=}}\ h\,(t)$$

and

$$\psi\,(p\colon\ a,\ \sigma)\ \underset{\lambda,\ \mu}{\overset{v}{=\!=}}\ t^{\sigma}\ e^{-(at)^{-n/s}}\ h\ (t^{-n/s}),$$

then

$$\Phi\,(p^{-n/s}) = \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\ s^{-(k+\frac{1}{2})}\ n^{-\varrho-\lambda-\mu+1}}{\Gamma\,(\frac{1}{2}-k+m)\,\Gamma\,(\frac{1}{2}-k-m)}\, p^{-\frac{n}{s}(m+\frac{1}{2})}$$

$$\times \int_0^\infty t^{-(\varrho+1)}\, G^{\,2s,\ s+n}_{s+n,\ 2s+2n}\left[\frac{p^{-n}\, t^n}{n^n\ s^s}\,\middle|\,\begin{matrix}\Delta\,(s;\,k+1),\ \Delta\,(n;\ \varrho+\lambda-\mu-\frac{1}{2})\\ \Delta\,(s;\,\frac{1}{2}+m),\ \Delta\,(s;\,\frac{1}{2}-m),\ \Delta\,(n;\varrho),\ \Delta\,(n;\ \varrho-2\mu)\end{matrix}\right]$$

$$\times\ \psi\,[t\colon p,\ -\tfrac{n}{s}\,(m+\tfrac{1}{2})-\varrho]\ dt, \tag{5.1}$$

provided that the integral is convergent, the VARMA transforms of $|\ h\ (t)\ |$ and $|\ t^{\sigma}\ e^{-(at)^{-n/s}}\ h\ (t^{-n/s})\ |$ exist, $R\,(n + 2s - 2s\varrho + {}$ ${}+ 2s\mu \pm 2s\mu \pm 2nm) > 0$, $R\,(p) > 0$ and $3s > n$.

PROOF. Proceeding as in Theorem 3 and using the following integral representation for WHITTAKER function

$$W_{k,m}(\varkappa p^{-n/s}) = \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\, s^{\frac{1}{2}-k}\, n^{\varrho-\lambda-\mu}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}$$

$$e^{-\frac{1}{2}\varkappa p^{-n/s}}\, \varkappa^{-\frac{s}{n}(\mu-\varrho+\frac{1}{2})} \int_0^\infty t^{\mu-\varrho-\frac{1}{2}}\, e^{-\frac{1}{2}t\varkappa^{-s/n}}\, W_{\lambda,\mu}(t\varkappa^{-s/n})$$

$$G^{2s,\, s+n}_{s+n,\, 2s+2n}\left[\frac{p^{-n}\, t^n}{n^n\, s^s}\,\middle|\, \begin{matrix}\Delta(s;\, k+1),\ \Delta(n;\, \varrho+\lambda-\mu-\frac{1}{2})\\ \Delta(s;\, \frac{1}{2}+m),\ \Delta(s;\, \frac{1}{2}-m),\ \Delta(n;\, \varrho),\ \Delta(n;\, \varrho-2\mu)\end{matrix}\right] dt,$$

where $R(n+2s-2s\varrho+2s\mu\pm 2s\mu\pm 2nm)>0$, $|\arg \varkappa^s| < \frac{1}{2}(3s-n)\pi$, $3s>n$ and $R(p)>0$, we arrive at the result (5.1).

We shall now give some very important corrolaries of this theorem.

CORROLARY 4.a. Putting $\lambda = \frac{1}{2}-\mu$ in (5.1), we obtain the following theorem:

If

$$\Phi(p) \underset{k,\, m}{\overset{v}{=\!=}} h(t)$$

and

$$\psi(p:\ a,\ \sigma) \doteqdot t^\sigma\, e^{-(at)^{-n/s}}\, h(t^{-n/s}),$$

then

$$\Phi(p^{-n/s}) = \frac{(2\pi)^{\frac{1}{2}(n-3s+2)}\, s^{-(k+\frac{1}{2})}\, n^{\varrho+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, p^{-n(m+\frac{1}{2})/s}$$

$$\times \int_0^\infty t^{-(\varrho+1)}\, G^{2s,\, s}_{s,\, 2s+n}\left[\frac{p^{-n}\, t^n}{n^n\, s^s}\,\middle|\, \begin{matrix}\Delta(s;\ k+1)\\ \Delta(s;\ \frac{1}{2}+m),\ \Delta(s;\ \frac{1}{2}-m),\ \Delta(n;\ \varrho)\end{matrix}\right]$$

$$\psi\left[t:\, p,\, -\tfrac{n}{s}(m+\tfrac{1}{2})-\varrho\right] dt, \tag{5.2}$$

provided that the integral is convergent, the VARMA transform of $|h(t)|$ and the LAPLACE transform of $|t^\sigma\, e^{-(at)^{-n/s}}\, h(t^{-n/s})|$ exist, $R(n+2s-2s\varrho+2s\mu\pm 2s\mu\pm 2nm)>0$, $R(p)>0$ and $3s>n$.

CORROLARY 4b. Again putting $\lambda = \frac{1}{2} - k$, $\mu = m$, $n = 1$, $s = 1$ and $\varrho = 2k + m + \frac{1}{2}$ in the theorem, we find that:

If

$$\Phi(p) \underset{k,\, m}{\overset{v}{=\!=}} h(t)$$

and

$$\psi(p\colon a, \sigma) \underset{\frac{1}{2} - k,\, m}{\overset{v}{=\!=\!=}} t^{\sigma} e^{-1/at} h\left(\tfrac{1}{t}\right),$$

then

$$\Phi\left(\tfrac{1}{p}\right) = \frac{2\sqrt{\pi}\, p^{-(k+m+1)}}{\Gamma(\frac{1}{2} - k + m)\, \Gamma(\frac{1}{2} - k - m)}$$

$$\times \int_0^{\infty} t^{-(k+m+1)} I_{-2k}\left(\sqrt{\tfrac{t}{p}}\right) K_{2m}\left(\sqrt{\tfrac{t}{p}}\right) \cdot \psi\left[t\colon p, -(2k + 2m + 1)\right] \mathrm{dt}, \tag{5.3}$$

provided that the integral is convergent, the VARMA transforms of $|h(t)|$ and $|t^{\sigma} e^{-1/at} h(\frac{1}{t})|$ exist, $R(\frac{1}{2} - k \pm m) > 0$ and $R(p) > 0$.

As a particular case of cor. 4.[a], when we put $n = 1$, $s = 1$ and $\varrho = k + 1$ in (5.2), we obtain the result:

If

$$\Phi(p) \underset{k,\, m}{\overset{v}{=\!=}} h(t)$$

and

$$\psi(p\colon a, \sigma) \doteqdot t^{\sigma} e^{-1/at} h\left(\tfrac{1}{t}\right),$$

then

$$\Phi\left(\tfrac{1}{p}\right) = \frac{2.\, p^{-(m+1)}}{\Gamma(\frac{1}{2} - k + m)\, \Gamma(\frac{1}{2} - k - m)} \int_0^{\infty} t^{-k-\frac{3}{2}} K_{2m}\left(2\sqrt{\tfrac{t}{p}}\right) \psi\left[t\colon p, -(\tfrac{3}{2} + k + m)\right] \mathrm{dt} \tag{5.4}$$

provided that the integral is convergent, the VARMA transform of $|h(t)|$ and the LAPLACE transform of $|t^{\sigma} e^{-1/at} h(\frac{1}{t})|$ exist, $R(1 + 2\mu \pm 2\mu \pm 2m - 2k) > 0$ and $R(p) > 0$.

6. Theorem 5. If

$$\Phi(p) \xlongequal[k,\, m]{v} h(t),$$

then

$$p^{\frac{n}{s}(1-\nu)}\ \Phi(p^{-n/s}) \xlongequal[\lambda,\, \mu]{v} f(t),$$

where

$$f(t) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{k+m+\nu}\, n^{-\lambda-\mu}$$

$$\times \int_0^\infty \varkappa^{-\nu}\, G^{2s,\, n}_{s+n,\, 2s+2n}\left[\frac{t^n\, \varkappa^s}{n^n\, s^s}\,\middle|\, \begin{matrix} \Delta(n; \lambda-\mu-\frac{1}{2}),\ \Delta(s; \nu-k+m+\frac{1}{2}) \\ \Delta(s;\nu), \Delta(s;\nu+2m), \Delta(n;0), \Delta(n;-2\mu) \end{matrix}\right] h(\varkappa)d\varkappa, \quad (6.1)$$

provided that the integral is convergent, the Varma transforms of $|h(t)|$ and $|f(t)|$ exist, $R(s+n\nu+nm\pm nm+s\mu\pm s\mu)>0$, $R(p)>0$ and $s \geq n$.

Proof. Interpreting the integrand in

$$p^{\frac{n}{s}(1-\nu)}\ \Phi(p^{-n/s}) = p^{n(\frac{1}{2}-m-\nu)/s}\int_0^\infty \varkappa^{m-\frac{1}{2}}\, e^{-\frac{1}{2}\varkappa p^{-n/s}}\, W_{k,m}(\varkappa p^{-n/s})\, h(\varkappa)\, d\varkappa$$

by [12]

$$p^{\varrho}\, e^{-\frac{1}{2}\varkappa p^{-n/s}}\, W_{k,m}(\varkappa p^{-n/s}) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{k+\frac{1}{2}}\, n^{\varrho-\lambda-\mu}$$

$$\times\, p\int_0^\infty (pt)^{\mu-\frac{1}{2}}\, e^{-\frac{1}{2}pt}\, W_{\lambda,\mu}(pt).\, t^{-\varrho}$$

$$G^{2s,\, n}_{s+n,\, 2s+2n}\left[\frac{\varkappa^s\, t^n}{n^n\, s^s}\,\middle|\, \begin{matrix} \Delta(n;\ \varrho+\lambda-\mu-\frac{1}{2}),\ \Delta(s;\ 1-k) \\ \Delta(s;\ \frac{1}{2}+m),\ \Delta(s;\ \frac{1}{2}-m),\ \Delta(n;\ \varrho),\ \Delta(n;\ \varrho-2\mu) \end{matrix}\right] \mathrm{dt}$$

where $R(n+2s-2s\varrho+2s\mu\pm 2s\mu\pm 2nm)>0$, $R(p)>0$, $|\arg \varkappa^s| < \frac{1}{2}(s-n)\pi$ and $s>n$, with $\varrho = \frac{n}{s}(\frac{1}{2}-m-\nu)$ and using [2, p. 209 eq. 8] we arrive at the result (6.1).

Below we give a few important particular cases of (6.1).

When we put $n = s = 1$, we obtain the following theorem due to Narain [6]:

CORROLARY 5.[a] If

$$\Phi(p) \frac{v}{\overline{\overline{k, m}}} h(t)$$

then

$$p^{1-\nu} \Phi\left(\tfrac{1}{p}\right) \frac{v}{\overline{\overline{\lambda, \mu}}} f(t)$$

where

$$f(t) = \int_0^\infty \varkappa^{-\nu} G_{24}^{21}\left(t\varkappa \,\middle|\, \begin{matrix} \lambda - \mu - \frac{1}{2}, v - k + m + \frac{1}{2} \\ \nu, \nu + 2m, 0, -2\mu \end{matrix}\right) h(\varkappa)\, d\varkappa \quad (6.2)$$

provided that the integral is convergent, the VARMA transforms of $|h(t)|$ and $|f(t)|$ exist, $R(\nu + \mu \pm \mu + m \pm m + 1) > 0$ and $R(p) > 0$.

When $\lambda = \frac{1}{2} - \mu$, then on replicing ν by $\frac{s}{n}\nu$, we obtain a theorem due to SAKSENA [7] :

CORROLARY 5b. If

$$\Phi(p) \frac{v}{\overline{\overline{k, m}}} h(t)$$

then

$$p^{\frac{n}{s} - \nu} \Phi(p^{-n/s}) \doteqdot f(t)$$

where

$$f(t) = (2\pi)^{\frac{1}{2}(n-s)} s^{k+m} n^{-\nu-\frac{1}{2}}$$

$$\times t^\nu \int_0^\infty G_{s, 2s+n}^{2s, 0}\left[\frac{t^n \varkappa^s}{n^n s^s} \,\middle|\, \begin{matrix} \Delta(s; \frac{1}{2} - k + m) \\ \Delta(s; 0), \Delta(s; 2m), \Delta(n; -\nu) \end{matrix}\right] h(\varkappa)\, d\varkappa, \quad (6.3)$$

provided that the integral is convergent, the VARMA transform of $|h(t)|$ and the LAPLACE transform of $|f(t)|$ exist, $R(s + s\nu + nm \pm \pm nm + s\mu \pm s\mu) > 0$ $R(p) > 0$ and $s \geq n$.

If both $\lambda = \frac{1}{2} - \mu$, $k = \frac{1}{2} - m$, we have a theorem due to GUPTA [5] :

CORROLARY 5c. If

$$\Phi(p) \doteqdot h(t)$$

then

$$p^{n/s-\nu}\ \Phi\,(p^{-n/s}) \doteqdot f\,(t),$$

where

$$f(t) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{\frac{1}{2}}\, n^{-\nu-\frac{1}{2}}\, t^{\nu} \int_0^{\infty} G^{\,s,\,0}_{\,0,\,s+n}\left[\frac{t^n\, x^s}{n^n\, s^s}\,\middle|\, \Delta(s;0),\, \Delta(n;-\nu)\right] h(x)\,dx, \qquad (6.4)$$

provided that the integral is convergent, the LAPLACE transforms of $|\,h\,(t)\,|$ and $|\,f\,(t)\,|$ exist, $R\,(s + s\nu + nm \pm nm + s\,\mu \pm s\mu) > 0$, $R\,(p) > 0$ and $s \geq n$.

On putting $n = 1$, $s = 1$ in the last corrolary we obtain the well known TRICOMI's theorem which states that if $\Phi\,(p) \doteqdot h\,(t)$, then

$$p^{1-\nu}\ \Phi\,(\tfrac{1}{p}) \doteqdot t^{\frac{1}{2}\nu} \int_0^{\infty} J_{\nu}\,(2\sqrt{tx})\; x^{-\frac{1}{2}\nu}\; h\,(x)\,dx = f\,(t) \qquad (6.5)$$

provided that the integral is convergent, the LAPLACE transforms of $|\,h\,(t)\,|$ and $|\,f\,(t)\,|$ exist and $R\,(p) > 0$.

7. THEOREM 6. If

$$\Phi\,(p) \underset{k,\;m}{\overset{v}{=\!=}} h(t),$$

then

$$p^{\frac{n}{s}(2\nu-1)}\ \Phi\,(p^{\frac{n}{s}}) \underset{\lambda,\;\mu}{\overset{v}{=\!=}} f\,(t),$$

where

$$f\,(t) = (2\pi)^{\frac{1}{2}(n-s)}\ s^{2\nu+k+m}\ n^{-\lambda-\mu}$$

$$\times \int_0^{\infty} x^{-2\nu}\, G^{\,2s+n,0}_{\,s+2n,\,2s+n}\left[\frac{n^n\, x^s}{s^s\ \ t^n}\,\middle|\,\begin{matrix}\Delta\,(s;\, 2\nu - k + m + \frac{1}{2}),\, \Delta\,(n;\, 1),\, \Delta(n;\, 2\mu + 1)\\ \Delta\,(s;\ 2\nu),\ \Delta\,(s;\, 2\nu + 2m),\ \Delta\,(n;\, \frac{3}{2} - \lambda + \mu)\end{matrix}\right] h\,(x)\,dx, \qquad (7.1)$$

provided that the integral is convergent, the VARMA transforms of $|\,h(t)\,|$ and $|\,f(t)\,|$ exist, $R(p) > 0$ and $s > n$.

PROOF. Proceeding as in Theorem 5 and using the following result due to SAXENA [**12**].

$$p^{\varrho}\, e^{-\frac{1}{2}\varkappa p^{n/s}}\, W_{k,m}\,(\varkappa p^{n/s}) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{k+\frac{1}{2}}\, n^{\varrho-\lambda-\mu}$$

$$\times\, p\int_0^{\infty} (pt)^{\mu-\frac{1}{2}}\, e^{-\frac{1}{2}pt}\, W_{\lambda,\mu}\,(pt).\, t^{-\varrho}$$

$$G^{\,2s+n,\,0}_{\,s+2n,\,2s+n}\left[\frac{n^n \varkappa^s}{s^s t^n}\,\middle|\,\begin{matrix}\Delta(s;\;1-k),\ \Delta(n;\;1-\varrho),\ \Delta(n;\;2\mu-\varrho+1)\\ \Delta(s;\;\frac{1}{2}+m),\ \Delta(s;\;\frac{1}{2}-m),\ \Delta(n;\;\frac{3}{2}-\varrho-\lambda+\mu)\end{matrix}\right]\mathrm{dt},$$

where $R(p) > 0$, $|\arg \varkappa^s| < \frac{1}{2}(s-n)\pi$ and $s > n$, we arrive at the result (7.1).

CORROLARY 6a. Putting $k = \frac{1}{2} - m$, $\lambda = \frac{1}{2} - \mu$ in (7.1), we obtain the result:

If

$$\Phi(p) \doteqdot h(t),$$

then

$$p^{\frac{n}{s}(2\nu-1)}\,\Phi(p^{n/s}) \doteqdot (2\pi)^{\frac{1}{2}(n-s)}\, s^{2\nu+\frac{1}{2}}\, n^{-\frac{1}{2}}\int_0^{\infty} \varkappa^{-2\nu}\, G^{\,s0}_{\,ns}\left[\frac{n^n x^s}{s^s t^n}\,\middle|\,\begin{matrix}\Delta(n;\;1)\\ \Delta(s;\;2\nu)\end{matrix}\right] h(\varkappa)\,d\varkappa = f(t), \tag{7.2}$$

provided that the integral is convergent, the LAPLACE transforms of $|h(t)|$ and $|f(t)|$ exist, $R(p) > 0$ and $s > n$.

We now give two interesting particular cases of this corrolary.

(*i*) When $n = 1$, we find that:

If

$$\Phi(p) \doteqdot h(t),$$

then

$$p^{\frac{1}{s}(2\nu-1)}\,\Phi(p^{\frac{1}{s}}) \doteqdot f(t),$$

where

$$f(t) = (2\pi)^{-\frac{1}{s}(1+s)}\, s^{2\nu+\frac{1}{2}}$$

$$\times \int_0^{\infty} x^{-2\nu} \sum_{i,-i} \frac{1}{i}\, E\left\{\Delta(s;\;2\nu) : : \frac{\varkappa^s e^{i\pi}}{s^s t}\right\} h(\varkappa)\,d\varkappa, \tag{7.3}$$

provided that the integral is convergent, the LAPLACE transforms of $|h(t)|$ and $|f(t)|$ exist, $R(p) > 0$ and $s = 2, 3, 4, \ldots$

(*ii*) If also $s = 2$, we arrive at a known result [**3**, p. 133]:

If

$$\Phi(p) \doteqdot h(t)$$

then

$$\pi^{\frac{1}{2}} 2^{\nu-\frac{1}{2}} p^{\nu-\frac{1}{2}} \Phi(p^{\frac{1}{2}}) \doteqdot t^{-\nu} \int_0^\infty e^{-x^2/8t} D_{2\nu-1}\left(\frac{x}{\sqrt{2t}}\right) h(x)\, dx = f(t) \quad (7.4)$$

provided that the integral is convergent, the LAPLACE transformations of $|h(t)|$ and $|f(t)|$ exist and $R(p) > 0$.

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REFERENCES

1. BROMWICH, T. J. I'A., *An introduction to the theory of Infinite series*, 1931.

2. ERDELYI, A., *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New-York, 1953.

3. ERDELYI, A., *Tables of Integral Transforms*, vol. 1, McGraw-Hill, New-York, 1954.

4. GOLDSTEIN, S., *Operational representations of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function*, Proc. London Math. Soc., 34 (1932), 103-125.

5. GUPTA, H. C., *On operational calculus*, Proc. Nat. Inst. Sc. (India), 14 (1948), 131-156.

6. NARAIN, R., *Certain properties of the generalized Laplace transform*, Rend. del Semin. Matematico, 15 (1955-56), 311-328.

7. SAKSENA, K. M., *Some theorems concerning a generalized Laplace transform*, Memoria publicada en Collectanea Mathematica, vol. X-Fasc. 1.º. Año 1958.

8. SAXENA, R. K., *Some integrals involving E-functions*, Proc. Glasg. Math. assoc., 4 (1960), 178-185.

9. SAXENA, R. K., *Some theorems on generalized Laplace transform 1*, Proc. Nat. Inst. Sc., India, 26A (1960), 400-413.

10. SAXENA, R. K., *Some theorems in operational calculus and infinite integrals involving Bessel function and G-functions*, Proc. Nat. Inst. Sc. (India), 27A (1961), 38-61.

11. SAXENA, R. K., *Some theorems on generalized Laplace transform-2*, Riv. Mat. Univ. Parma (2) 2 (1961), 287-300.

12. SAXENA, R. K., *Some theorems on generalized Laplace transform-3*, Riv. Mat. Univ. Parma (in press).

13. TRICOMI, F., *Sulla transformazione e il teorema di reciprocita di Hankel*, Rend. Acc. dei Lincei, 22 (1935), 564-571.

14. VARMA, R. S., *On a generalization of Laplace integral*, Proc. Nat. Acad. Sc. (India), 20A (1951), 209-216.