CERTAIN OPERATORS IN THE SPACE ANALYTIC DIRICHLET TRANSFORMATIONS

By

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1. INTRODUCTION. Let $\mathbf{C}$ denote the field of complex numbers equipped with the usual topology. Denote by χ, the family of all transformations f: $\mathbf{C} \to \mathbf{C}$, such that

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}; \quad s = \sigma + it\varepsilon; \tag{1.1}$$

where $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to +\infty$ with n, and further,

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty; \tag{1.2}$$

$$\limsup_{n\to\infty} \frac{\log \lambda_n}{n} = D < \infty. \tag{1.3}$$

The topological aspect (in various directions) of this space has been studied in details by one of us and Husain (see [2], [3]). However, in an earlier paper of ours [4], we have considered two topologies on χ, namely:

(i) The topology τ_1 generated by the family of semi-norms (indeed, norms), $\{M(\sigma, f); \sigma \text{ real}\}$, where

$$M(\sigma, f) = \sup_{-\infty < t < \infty} |f(\sigma + it)|;$$

and

(ii) The topology τ_2 generated by the family of semi-norms (indeed, norms), $\{p(\sigma, \ldots), \sigma \text{ is real}\}$, where $p(\sigma, f) = \sum_{n=1}^{\infty} |a_n| e^{\sigma\lambda_n}$. We have carlier denoted $p(\sigma, f)$ by $||f; \sigma||$.

From the well-known Cauchy-Ritt inequality, namely

$$|a_n| \leq M(\sigma, f) \ \exp\ \{-\lambda_n \sigma\};$$

valid for all real σ, it follows that

$$M(\sigma, f) \leq p(\sigma, f) \leq C(k)\, M(\sigma + k, f),\ k > 0,$$

$C(k)$ being a constant depending on k only. Hence one finds that the topologies τ_1 and τ_2 are equivalent. To falicitate our work, we recall few things more from our earlier work [4]. A sequence $\{f_1 : n \geq 0\} \subset \chi$, is said to be a *base* if to each $f \varepsilon \chi$, their corresponds a unique sequence $\{a_n\}$ in $\mathbf{C}$ such that

$$f = \sum_{n=0}^{\infty} a_n f_n$$

A base $\{f_n : n \geq 0\}$ is called a *genuine base* if the corresponding coefficients in the expansion of an f satisfy (1.2). A sequence $\{f_n : n \geq 0\}$ is called an *absolute base* if it is a base in χ and the infinite series corresponding to each $f \ \varepsilon\ \chi$ is absolutely convergent with respect to τ_1 (or equivalently with respect to τ_2). A sequence $\{f_n\}$ in χ is called a *proper base* for χ if it is a genuine as well as an absolute base for χ. We have earlier shown that if $\{f_n\}$ is a proper base, then([4] Theorem (2.1))

$$\limsup_{n \to \infty} \frac{\log\ M(\sigma, f_n)}{\lambda_n} < +\infty,$$

for each real σ.

Our aim in this paper is to characterize certain continuous linear operators on χ and use them in the determination of praper bases in χ. Our two main results stated and proved below are in the form of theorems.

2. Characterisations: Let $\delta_n \ \varepsilon\ \chi$, where

$$\delta_n(z) = e^{z\lambda_n}\ (n \geq 1), \text{ then we have;}$$

THEOREM 2.1: Let $\{\alpha_n : n \geq 1\} \subset \chi$. Suppose T is a linear operator from χ into χ, such that $T(\delta_n) = \alpha_n$; $n \geq 1$. Then if T is continuous then,

$$\limsup_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} < +\infty; \tag{2.1}$$

for all real σ.

Conversely, if (2.1) holds good, then there exists a continuous linear operator $T : \chi \to \chi$, such that

$$T(\delta_n) = \alpha_n, \ n \geq 1.$$

PROOF. Suppose T is a continuous linear operator from χ into χ with $T(\delta_n) = \alpha_n$, $n \geq 1$. Then for a given σ, there exists a Δ (all reals) such that

$$\begin{aligned} M(\sigma, T\delta_n) = M(\sigma, \alpha_n) &\leq K\, M(\Delta, \delta_n) \\ &= K\, e^{\Delta\lambda_n} \end{aligned}$$

Hence

$$\limsup_{n\to\infty} \frac{\log M(\sigma, \alpha_n)}{\lambda_n} < +\infty$$

Thus (2.1) follows.

Conversely, assume that (2.1) is true. Let $\alpha \varepsilon \chi$, then, α is represented by

$$\alpha = \sum_{n=1}^{\infty} a_n \delta_d,$$

where the coefficients a_n's satisfy (1.2). Since (2.1) holds, therefore there exists a $M = M(\sigma)$, depending on σ, such that

$$\frac{\log M(\sigma, \alpha_n)}{\lambda_n} \leq M \quad \text{for all} \quad n \geq n_0,$$

or,

$$M(\sigma, \alpha_n) \leq e^{M\lambda_n} \quad \text{for all} \quad n \geq n_0$$

Therefore, noting that (1.2) is already valid for the coefficients a_n's, we find that

$$\sum_{n=1}^{\infty} a_n \alpha_n$$

is absolutely convergent in χ and as χ is complete, we find that the proceeding series converges in χ and so it represents an element of χ. Hence there is a natural transformation $T : \chi \to \chi$, such that

$$T(\alpha) = \sum_{n=1}^{\infty} \alpha_n a_n, \ \alpha = \sum_{n=1}^{\infty} a_n \delta_n.$$

Clearly $T(\delta_n) = \alpha_n$, $n = 1, 2, 3, \ldots$. We are now required to show that T is continuous on (χ, τ_1). To do this it is sufficient to prove that T is continuous on (χ, τ_2). The norms $||\ldots, \sigma||$ are continuous on χ, therefore given any real σ, we find

$$\begin{aligned} ||T(\alpha)\,;\ \sigma|| &= ||\lim_{N \to \infty} \sum_{n=1}^{N} a_n \alpha_n\,;\ \sigma|| \\ &\leq \lim_{N \to \infty} \sum_{n=1}^{N} |a| \cdot ||\alpha_n\,;\ \sigma_n|| \\ &\leq \lim_{N \to \infty} \sum_{n=1}^{N} |a_n|\, e^{\Delta \lambda_n},\ \Delta = \Delta(\sigma) \\ &= ||\alpha\,;\ \Delta|| \end{aligned}$$

Thus T: $(\chi, ||\ldots, \Delta||) \to (\chi, ||\ldots, \sigma||)$ is continuous and as σ is arbitrary, we find that $T : \chi \to \chi$ is continuous.

Next, we prove

THEOREM 2.2: If T is a linear operator on χ to itself, such that T and T^{-1} are continuous. Then $\{T(\delta_n) : n \geq 1\}$ is a proper base in the closed subspace $T(\chi)$ of χ. Conversely, if $\{\alpha_n : n \geq 1\}$ is a proper base in a closed subspace Y of χ, then there exists a continuous linear operator $T : \chi \to \chi$, such that $T(\delta_n) = \alpha_n$.

PROOF. Suppose first that T is the one as mentioned in the hypothesis. Then $T(\chi)$ is a closed subspace of χ. Let $T(\delta_n) = \alpha_n$, $n \geq 1$. Let $f \varepsilon T(\chi)$, tehen

$$T^{-1}(f) = \sum_{n=1}^{\infty} a_n \delta_n$$

where a_n's satisfy (1.2). Now

$$\sum_{n=1}^{M} a_n \delta_n \to T^{-1}(f) \quad \text{in} \quad \chi \text{ as } M \to \infty \tag{2.2}$$

But T is continuous and linear and so (2.2) implies,

$$f = \sum_{n=1}^{\infty} a_n \alpha_n \tag{2.3}$$

Since (2.1) holds, this implies that $\sum_{n=1}^{\infty} M(\sigma, a_n \alpha_n)$ converges for every real σ. Also the representation of f in (2.3) is unique (since T^{-1} is continuous), we conclude that $\{\alpha_n : n \geq 1\}$ is a proper base for $T(\chi)$. Conversely, let $\{\alpha_n\}$ be a proper base for a closed subspace Y of χ. Hence (2.1) holds. Therefore from Theorem 2.1, there exists a continuous linear operator T on χ into itself, such that $T(\delta_n) = \alpha_n$, $n \geq 1$. Let $f \,\varepsilon\, \chi$, $f \neq 0$. Then f is represented by (1.1) whose coefficient $a_n' s$ satisfy (1.2). Thus

$$T(f) = \sum_{n=1}^{\infty} a_n \alpha_n \neq 0$$

Therefore T is one-to-one. Hence T is a continuous algebraic isomorphism from χ onto Y. Now by applying Banach theorem ([1], p. 41) we see that T^{-1} exists and is continuous (observe that Y is complete). The proof of the theorem is now complete.

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REFERENCES

[1] BANACH S. *Théorie des opérations Lineaires*, Warsaw (1932).

[2] HUSAIN T., and KAMTHAN, P. K. *Space of entire functions represented by Dirichlet series*. Collectanea Math., 19, No. 3, (1968), 203-16.

[3] KAMTHAN, P. K. *F. K-spaces for entire Dirichlet functions; Collect.* Math., 20, (1969), 271-80.

[4] KAMTHAN, P. K. and GAUTAM, S. K. SINGH, *Bases in the space of analytic Dirichlet transformations*. (sent for publication).

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