STUDIES ON OVERCONVERGENCE

by

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INTRODUCTION

The idea of overconvergence may be illustrated by considering the following example set out by M. B. PORTER [21] in 1906. This work of PORTER incidentally, marked the discovery of the phenomenon of overconvergence. Let the series

$$\sum a_{n_k} z^{n_k} \quad \text{with} \quad n_{k+1} > 2n_k \tag{1}$$

have radius of overgence 1. The series

$$F(w) = \sum a_{n_k} w^{n_k} (1 + w)^{n_k} \tag{2}$$

converges uniformly then in the interior of the lemniscate (CASSINIAN) defined by

$$| w (1 + w) | = 1 \tag{3}$$

and diverges at each point outside. The series (2) converges absolutely for $|w|\ (1 + |w|) < 1$, and (2) defines $F(w)$ as an analytic function inside the region L defined by (3). Rearrangement of (2) as a power series in w therefore yields

$$F(w) = \sum c_n\ w^n. \tag{4}$$

The radius of convergence of (4) is that of the largest circle contained in L, having as center the point $w = 0$. There is no overlapping of powers when one forms (4) from (2) ; consequently the sequence of partial sums of the series in (2) is a subsequence of the sequence of partial sums of the series in (4). Hence (4) has a subsequence of partial sums which converges inside L, and therefore at points exterior to the circle of convergence of (4). We note also that the convergence of this subsequence of (4) is uniform in any closed subregion of L. Whenever a power series has a subsequence of partial sums converging uniformly to a limit function in a neighborhood of a point on the circle of convergence, the series is said to be overconvergent at that point. As it will become apparent, a series which is overconvergent at a point has the same property at each regular point of the circle of convergence. The limit function to which such a subsequence converges is analytic in any region where the conververgence is uniform (WEIERSTRASS' theorem), and must therefore coincide with the fuction defined by the series. Overconvergence thus affords a method, though not a practical one, of effecting the analytic continuation of certain fuctions, defined by power series and regular at some point on the circle of convergence.

The convergence of a particular sequence of partial sums of a power series in domains exterior to the circle of convergence was rediscovered twice before PORTER received credit for making the first observation. It was ALEXANDER OSTROWSKI who conceived the main ideas of the theory of overconvergence. Before proceeding with his development, we must mention a result due to J. HADAMARD which contains the germ of OSTROWSKI's fundamental theorem and to which we shall have occasion to refer later.

(5) Let

$$f(z) = \sum c_n\ z^{p_n}.$$

Assume also that $p_{n+1} \geq (1 + \theta)p_n$ for some $\theta > 0$ and for all sufficiently large n.

Then the circle of convergence is a natural boundary of $f(z)$.[6].

This theorem is commonly called HADAMARD's gap theorem. Eventually the only remaining terms in the series (5) are at the ends of 'long' gaps. OSTROWSKI modified the severity of this condition and was able to arrive at his first overconvergence theorem.

(6) If

$$f(z) = \sum c_n z^n.$$

has radius of convergence 1, and $c_n = 0$ when $n_k < n \leq N_k (k = 1, 2, ...)$ where $\{n_k\}$ and $\{N_k\}$ are two sequences of positive integers such that $N_k \geq (1 + \theta) n_k$ for some positive θ, then the subsequence of partial sums

$$s_{n_k}(z) = \sum_{n=0}^{n_k} c_n z^n,$$

converges uniformly in the neighborhood of each regular point on the circle of convergence.

Section 1 of this thesis consits in a new proof for this theorem. It may be added here that we may speak of overconvergent DIRICHLET series also ; however, we will be concerned throughout with power series only.

Ranges of zero coefficients of the type hypothesized above are known as HADAMARD gaps. One need look no further than PORTER's example (1), with $n_{k+1} = 2n_k + 1$ to see that HADAMARD gaps are by no means necessary for a series to possess overconvergence properties. Indeed, one can always add a series with convergence radius greater than one to the series in (6) and get an overconvergent series having no gaps. In the literature, a power series which can be written as the sum of two power series, one with HADAMARD gaps, the other with radius of convergence greater than the given one is said to have a lacunary structure. The connection between this type of series and the overconvergence theorem :

(7) Every series $\sum c_n z^n$ which has an overconvergent subsequence of partial sums has a lacunary structure.

Other conditions on the coefficients of a power series are sufficient for producing overconvergence in the neighborhood of regular points on the circle of convergence. In 1934, MACINTYRE [14] showed that the theorem of VIVANTI-BOREL-DIENES could be generalized in a way involving overconvergence along a radius. In Section 2 it is shown that overconvergence in a neighborhood is implied by MACINTYRE'S hypothesis, which is essentially that $|\arg c_n| \leq \alpha < \pi/2$ for $n_k \leq n \leq (1 + \theta) n_k$. Evidentelly this condition is implied by $c_n = 0$ for the corresponding ranges of n. The transformation $z = w^p (1 + w)/2$ used in Section 2 was already employed by MORDELL to prove HADAMARD'S theorem and by ESTERMANN [5] in his demonstration of OSTROWSKI'S first theorem.

In Section 3, there is stated a group of overconvergence theorems which hypothesize certain restrictions on the distribution of sign changes occuring in the real parts of the coefficients of the series. Variations of these conditions have been used by a number of authors — POLYA, FABRY, and LANDAU among them — to establish the existence of a singularity on a specified arc of the circle of convergence. One could say that the overconvergence results in Section 3 and the FABRY type theorems are related in somewhat the same way as OSTROWSKI'S first theorem and HADAMARD'S gap theorem. Consider also other theorems of the type which conclude that such and such a point on the circle of convergence is a singular point of the function. It is possible that these theorems might establish overconvergence in the neighborhood of the point were it a regular one.

Without becoming too precise at the moment, we can see from the proof of OSTROWSKI'S second theorem (see Section 5) that if the gaps in the series (6) occur too frequently, then it is not possible to have overconvergence of the subsequence $\{s_{n_k}(z)\}$. Another way of looking at this is to consider the intervals between gaps; if these are not 'long enough' so to speak, then there is no overconvergence. Section 4 deals with this idea in what is throught to be a novel way, making use of a system of linear equations and an elementary distance property of points distributed on a circular arc. The results obtained generalize HADAMARD'S theorem and show also that in the series (6), $\overline{\lim}\, n_{k+1}/N_k$ must be greater than 1 if the sequence $\{s_{n_k}(z)\}$ overconverges.

Section 5 is concerned with a slight improvement in the proof of OSTROWSKI'S second theorem given by DIENES [4, p. 370]. The theorem is then thought of as applying to overconvergent series with HADAMARD gaps, enabling one to estimate numerically a lower bound for $\overline{\lim}\, n_{k+1}/N_k$, referred to in the last paragraph. This lower bound, as it turns out, is dependent on the inner mapping radius of the region in which the series is overconvergent.

Finally, it was thought worthwhile including in Section 6 a few of the other basic results on overconvergence. In the course of these observations some attempt is made as well to point out several possibilities for further investigation.

1. A NEW PROOF FOR OSTROWSKI'S FIRST THEOREM

Many proofs have been devised for this theorem since it was originally done in 1921. See, for example [2, p. 12], [11], and [7, p. 311] among others. The range of difficulty among these proofs is quite wide, some demanding considerable preparation. OSTROWSKI's initial method rests on HADAMARD's three-circles theorem, while what may be the simplest method is based on MORDELL's proof of HADAMARD's gap theorem and can be found in [2, p. 17]. The present effort is mainly computational and depends on nothing more formidable than a linear transformation.

THEOREM I. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

have radius of convergence 1. Let $\{n_k\}$ and $\{N_k\}$ be two sequences of positive integers such that

$$(1+\theta)\, n_k \leq N_k \text{ and } n_k < N_k < n_{k+1}.$$

Assume also that

$$c_n = 0 \quad \text{for} \quad n_k < n \leq N_k\,;\ k = 1,\ 2,\ \ldots,$$

where θ is a fixed real number greater than zero. Then the subsequence

$$s_{n_k}(z) = \sum_{n=0}^{n_k} c_n z^n \tag{2}$$

of partial sums of (1) converges uniformly in a neighborhood of each regular point on $|z| = 1$.

Proof: The linear transformation

$$z = w\left(\frac{\varrho^2 - 1}{2\varrho}\right)\Big/\left(1 - \frac{w}{\varrho}\right) \tag{3}$$

maps the circle $|w| = 1$ into the circle $\left|z - \frac{1}{2}\right| = r = \varrho/2$ in the

z-plane. The origins correspond under this transformation and the points $w = 1$ and $w = -1$ are mapped into $z = \frac{1}{2} + r$ and $z = \frac{1}{2} - r$ respectively. By hypothesis,

(1) has radius of convergence 1. Let us assume for the moment that $f(z)$ is regular at $z = 1$. Therefore $f(z)$ is regular in and on a circle C having center $z = \frac{1}{2}$ and radius $r > \frac{1}{2}$. Set $\varrho = 2r$ and $A = (\varrho^2 - 1)/2\varrho$. Then

$$F(w) = f\left(Aw/(1 - \frac{w}{\varrho})\right) \tag{4}$$

is regular in and on the unit circle. Hence $F(w)$ has a power series expansion about $w = 0$, namely

$$F(w) = \sum_{m=0}^{\infty} b_m w^m \tag{5}$$

having radius of convergence $R > 1$. We can obtain the b_m explicitly by substituting the right side of (3) in (1). That is,

$$b_m = \sum_{i=1}^{m} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}. \tag{6}$$

It is useful to note that (2) is also a regular function in and on C. So that

$$S_{n_k}(w) \equiv s_{n_k}\left(Aw/(1 - \frac{w}{\varrho})\right)$$

is a regular function of w in and on $|w| = 1$. Therefore

$$S_{n_k}(w) = \sum_{m=0}^{\infty} {}_{n_k}d_m w^m, \text{ where } {}_{n_k}d_m = \sum_{i=1}^{\min(n_k, m)} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \tag{7}$$

has radius of convergence $R' > 1$. Note that $S_{n_k}(w)$ is not the sum of the first n_k terms of (5). We would like to arrive now at an estimate for the absolute value of sums of the form

$$(a) \quad \sum_{1 \leq i \leq \lambda_1 m} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \quad \text{and} \tag{8}$$

$$(b) \quad \sum_{\lambda_2 m \leq i \leq m} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i},$$

where for a given m, λ_1 and λ_2 are such that $1/m \leq \lambda_1 < \lambda_2 \leq 1$. First of all, for any $\varepsilon > 0$ there is a number $K = K(\varepsilon)$ such that $|c_n| \leq K(1+\varepsilon)^n$. This means that the absolute value of (8a) is less than or equal to

$$K(\varepsilon)(1+\varepsilon)^m \sum_{1 \leq i \leq \lambda_1 m} A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}$$

while a similar expression may be written for (8b). If we form quotients of consecutive terms of the finite sequence $A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}$ for $i = 1, 2, \ldots, m$, we find that the maximum term occurs when $i = [m(\varrho^2 - 1)/(\varrho^2 + 1) + 1]$ in which the square brackets denote the greatest integer function. Choose some λ such that $1/m \leq \lambda \leq$ $\leq (\varrho^2 - 1)/\varrho^2 + 1)$. Certainly then

$$\sum_{1 \leq i \leq [\lambda m]} A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \leq m A^{[\lambda m]} \binom{m-1}{m-[\lambda m]} \Big/ \varrho^{m-[\lambda m]}$$

$$\leq m A^{[\lambda m]} \binom{m}{m-[\lambda m]} \Big/ \varrho^{m-[\lambda m]},$$

and upon making use of

$$e(k/e)^k < k! < ek(k/e)^k, \text{ (k an integer)}$$

we find, after some calculation — routine but somewhat lengthy, that

$$\text{(9)} \quad \overline{\lim_{m \to \infty}} \left\{ \sum_{1 \leq i \leq \lambda m} A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \right\}^{1/m} \leq A^\lambda / \varrho^{1-\lambda} (1-\lambda)^{(1-\lambda)} \lambda^\lambda.$$

On the other hand, if λ is chosen so that $(\varrho^2 - 1)/(\varrho^2 + 1) < \lambda \leq 1$ then

$$\sum_{[\lambda m] \leq i \leq m} A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \leq m A^{[\lambda m]} \binom{m}{m-[\lambda m]}$$

and further,

$$\text{(10)} \quad \overline{\lim_{m \to \infty}} \left\{ \sum_{[\lambda m] \leq i \leq m} A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \right\}^{1/m} \leq$$
$$A^\lambda / \varrho^{1-\lambda} (1-\lambda)^{(1-\lambda)} \lambda^\lambda.$$

Consider now the function

$$g(\lambda) = A^\lambda / \varrho^{1-\lambda} (1-\lambda)^{(1-\lambda)} \lambda^\lambda$$

which is continuous and differentiable on $(0,1)$. It is easy to see that its maximum is reached when $\lambda = (\varrho^2 - 1)/(\varrho^2 + 1)$ and that $g((\varrho^2 - 1)/(\varrho^2 + 1)) = (\varrho^2 + 1)/2\varrho$. Also $g(\lambda)$ increases in the interval $(0, (\varrho^2 - 1)/(\varrho^2 + 1)]$ and decreases in $[(\varrho^2 - 1)/(\varrho^2 + 1), 1)$.

If the sequences $\{n_k\}$ and $\{N_k\}$ and the real number θ satisfy the hypotheses given above, it is always possible to find for each sufficiently large k, an integer p_k and a real number $\mu = \mu(\theta)$ with $0 < \mu < 1$ such that

$$n_k < (1 - \mu)\, p_k < p_k \leq (1 + \mu)\, p_k \leq \theta n_k \leq N_k.$$

For example, $p_k = [n_k(1 + \theta)/2]$ and $\mu = (\theta - 1)/(\theta + 1)$ will do. Set $\lambda' = (1 - \mu)(\varrho^2 - 1)/(\varrho^2 + 1)$ and $\lambda'' = (1 + \mu)(\varrho^2 - 1)/(\varrho^2 + 1)$. Then $\max\{g(\lambda'), g(\lambda'')\} = \sigma < (\varrho^2 + 1)/2\varrho$. Put $\lambda_1 = (\varrho^2 - 1)/(\varrho^2 + 1)$ and define $G_{p_k}(w)$ as

$$G_{p_k}(w) = \sum_{0 \leq m \leq P_k/\lambda_1} b_m w^m \tag{11}$$

a partial sum of the series in (5). Form the difference

$$\begin{aligned} S_{p_k}(w) - G_{p_k}(w) &= \sum_{m=0}^{\infty} {}_{p_k}d_m w^m - \sum_{0 \leq m < P_k/\lambda_1} b_m w^m \\ &= \sum_{0 \leq m < P_k} {}_{p_k}d_m w^m + \sum_{P_k < m}^{\infty} {}_{p_k}d_m w^m \\ &\quad - \sum_{0 \leq m \leq P_k} b_m w^m - \sum_{P_k < m \leq P_k/\lambda_1} b_m w^m; \end{aligned} \tag{12}$$

which by (7) is valid at least for $|w| \leq 1$. We see also from (7) that the first and third sums on the right side of (12) are equal. Thus the left hand side of (12) becomes, on writing out the expression for b_m and breaking up the last sum

$$\sum_{P_k < m \leq P_k/\lambda_1} {}_{p_k}d_m w^m + \sum_{P_k/\lambda_1 < m}^{\infty} {}_{p_k}d_m w^m - \sum_{P_k < m \leq P_k/\lambda_1} w^m \left\{ \sum_{i=1}^{P_k} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i} \right\}$$
$$- \sum_{P_k < m \leq P_k/\lambda_1} w^m \sum_{i > P_k}^{m} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}.$$

Now the first and third sums are equal so that taking the definition of ${}_{p_k}d_m$ into account again we have

$$S_{p_k}(w) - G_{p_k}(w) = \sum_{m > P_k/\lambda_1}^{\infty} w^m \sum_{i=1}^{P_k} c_i A^i \binom{m-1}{m-i} \varrho^{m-i}$$

$$- \sum_{P_k < m \leq P_k/\lambda_1} w^m \sum_{i > P_k}^{m} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}.$$

Because of the zero coefficients these sums may be written in the form

$$S_{p_k}(w) - G_{p_k}(w) = \sum_{m > P_k/\lambda_1}^{\infty} w^m \sum_{i=1}^{P_k(1-\mu)} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}$$

$$- \sum_{P_k < m \leq P_k/\lambda_1} w^m \sum_{i > P_k(1+\mu)}^{m} c_i A^i \binom{m-1}{m-i} \Big/ \varrho^{m-i}$$

$$= \sum_{m > P_k/\lambda_1}^{\infty} P_m w^m - \sum_{P_k < m \leq P_k/\lambda_1} Q_m w^m,$$

the notation being evident. In the first sum on the right side of the last expression $i \leq p_k(1-\mu)$ and $m > p_k/\lambda_1$; therefore $i/m < \lambda_1(1-\mu) = \lambda'$. In the second, $i > p_k(1+\mu)$ and $m \leq p_k/\lambda_1$ so that $i/m > (1+\mu)\lambda_1 = \lambda''$. Hence by (9) and (10)

$$\overline{\lim_{m\to\infty}} \, | P_m |^{1/m} \leq (1+\varepsilon)\sigma \equiv \alpha \text{ and } \lim_{m\to\infty} | Q_m |^{1/m} \leq (1+\varepsilon)\sigma \equiv \alpha.$$

By taking ε small enough α can be less than $(\varrho^2+1)/2\varrho$. (See the definition of σ above.) Therefore

$$\sum_{m > P_k/\lambda_1}^{\infty} P_m w^m \text{ and } \sum_{P_k < m \leq P_k/\lambda_1} Q_m w^m$$

tend uniformly to zero as $k \to \infty$, in and on a circle of radius R_1 with center the origin such that $R_1 > 2\varrho/(\varrho^2+1)$. Consequently

$$| S_{p_k}(w) - G_{p_k}(w) |$$

tends uniformly to zero with increasing k in and on a circle C_2 with center the origin and radius $R_2 = \min(R_1, 1)$. Since

$$| S_{p_k}(w) - F(w) | \leq | S_{p_k}(w) - G_{p_k}(w) | + | G_{p_k}(w) - F(w) |$$

and the second term on the right side tends uniformly to zero in and on $|w| = 1$, the left hand side tends to zero uniformly in and on C_2. This means that $| s_{p_k}(z) - f(z) |$ tends to zero uniformly in and on the image circle of C_2 under (3). Now C_2 contains $w = 2\varrho/(\varrho^2+1)$ in its interior and 1 is the image of $w = 2\varrho/(\varrho+1)$

under (3). Hence 1 is contained in a neighborhood throughout which the partial sums $s_{p_k}(z)$ of the series (1) converge uniformly to $f(z)$. But because of the zero coefficients $s_{n_k}(z) = s_{p_k}(z)$ and the result is established for $z = 1$. If one replaces z by $z = e^{ie}$ in the series, the zero coefficients are not changed. That is, if $z = e^{ie}$ is a regular point of the given series, the new series will have $z = 1$ as a regular point, and the previous proof applies to it. This remark establishes the theorem.

2. Other coefficient conditions implying overconvergence

While commenting on the material in chapter 5 of [9], Dr. Macintyre shows that $\Sigma c_n z^n$ has a subsequence of partial sums which converges uniformly in some interval $[1, 1+\varepsilon]$ of the real axis if $f(z) = \Sigma c_n z^n$ is regular at $z = 1$ and $c_n \geq 0$ for $n_k \leq n \leq n_k(1+\theta)$ [14]. Here $\{n_k\}$ is an increasing sequence of integers and θ is a real number greater than zero. It is shown here that this result may be extended to convergence of the subsequence in a full neighborhood of $z = 1$. This turns out to be an immediate consequence of the following section, but it is of interest that the conclusion is available upon the use of considerably less demanding methods than those employed there.

Theorem I. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

have radius of convergence 1 and be regular at $z = 1$. Let $\{n_k\}$ be an increasing sequence of positive integers and θ be a real number greater than zero. If $c_n \geq 0$ for $n_k \leq n \leq n_k(1+\theta)$ for k sufficiently large, then the sequence

$$\sum_{n=0}^{n_k(1+\delta)} c_n z^n, \quad \delta = (1+\theta)^{1/2} - 1 \tag{2}$$

converges uniformly to $f(z)$ in a neighborhood of $z = 1$.

Proof: We observe the fact that for p a positive integer $f(w^p(1+w)/2)$ is a regular function of w for $|w| < 1 + \sigma\ (\sigma > 0)$ if $f(z)$ is regular for $|z| < 1$ and $z = 1$. We have

$$f\left(w^p(1+w)/2\right) = \sum_{m=0}^{\infty} b_m w^m \text{ where}$$

(3)

$$b_m = \sum_{\frac{m}{p+1} \leq n \leq \frac{m}{p}} c_n \binom{n}{m-pn} \Big/ 2^n$$

and the infinite series in (3) has radius of convergence $r \geq 1 + \sigma$. From the structure of the terms we can see that c_n enters into the sum for b_m if and only if

(4) $$m/(p+1) \leq n \leq m/p.$$

If we consider the partial sum (2) and let $z = w^p(1 + w)/2$, then

(5) $$\sum_{n=0}^{n_k(1+\delta)} c_n w^{np}(1+w)^n/2^n = \sum_{m=0}^{pn_k(1+\delta)} b_m w^m + \sum_{pn_k(1+\delta)}^{(p+1)n_k(1+\delta)} b_m{}^* w^m.$$

The coefficients b_m in the first sum of the right side of (5) are the same as in (3) above. Once m exceeds $pn_k(1+\delta)$ however, the b_m in (3) may involve n larger than $n_k(1+\delta)$, as shown by (4). The $b_m{}^*$ differ then from the b_m in the corresponding range in this, that the c_n used in the sum for b_m might not all occur in the sum for $b_m{}^*$. Those c_n which do affect $b_m{}^*$ appear with the same real coefficients as in the sum for b_m. For b_m, with m in the range $pn_k(1+\delta) < m \leq (p+1)n_k(1+\delta)$, (4) requires that n satisfy

$$\frac{pn_k(1+\delta)}{p+1} < n \leq \frac{(p+1)n_k\ (1+\delta)}{p}.$$

If this last range of n could be included in one for which $c_n \geq 0$, then leaving some of these c_n out when forming $b_m{}^*$ would serve to make $b_m{}^* \leq b_m$. That is, we want

$$n_k \leq \frac{p(1+\delta)\,n_k}{p+1} \leq \frac{(p+1)n_k\,(1+\delta)}{p} \leq n_k\,(1+\theta).$$

In other words we have the conditions

(6) (a) $\dfrac{1+\delta}{1+\frac{1}{p}} \geq 1$ and (b) $(1+\delta)\left(1+\dfrac{1}{p}\right) \leq 1+\theta$

to satisfy. Given $\theta > 0$, we select δ such that $(1 + \delta) = (1 + \theta)^{1/2}$. Then choose an integer p so large that $1 + \frac{1}{p} \leq 1 + \delta$. This ensures that (6a) and (6b) are true. Working only with the second sum on the right side of (5) we have

$$(7) \qquad | \Sigma\, b_m{}^* w^m | \leq \Sigma \,| b_m{}^* | \, | w^m | = \Sigma\, b_m{}^* \,| w^m | \leq \Sigma\, b_m \,| w^m |,$$

the index m running between $pn_k (1 + \delta)$ and $(p + 1)\, n_k (1 + \delta)$. By the uniform convergence of the series (3) in a circle $|w| = 1 + \sigma'$ for some $\sigma' > 0$, it follows that the left side of (5) tends to

$$\lim_{k \to \infty} \sum_{m=0}^{pn_k(1+\delta)} b_m\, w^m, \text{ in other words to } f(w^p (1 + w)/2)$$

uniformly in $| w | = 1 + \sigma'$. Or what is the same thing

$$\lim_{k \to \infty} \sum_{n=0}^{n_k(1+\delta)} c_n\, z^n = f(z)$$

uniformly with respect to z for $|z| < 1$, and z in some neighborhood of $z = 1$.

If we make the assumption that $| \arg c_n | \leq \alpha < \pi/2$ for n in $n_k \leq n \leq n_k(1 + \theta)$ for sufficiently large k, the same result holds. For as explained above

$$b_m{}^* = \sum_{n=n_1}^{n_2} c_n \binom{n}{m - pn} \Big/ 2^n \text{ and } b_m = \sum_{n=n_1}^{n_3} c_n \binom{n}{m - pn} \Big/ 2^n$$

for suitably chosen n_2 and n_3 with $n_2 \leq n_3$. Then $|\arg b_m{}^*| \leq \alpha < \pi/2$ and $| \arg b_m | \leq \alpha < \pi/2$. Also $R(b_m{}^*) \leq R(b_m)$. Therefore $| b_m{}^* | \leq R(b_m{}^*) \sec \alpha$, $R(b_m) \sec \alpha \leq | b_m | \sec \alpha$. So that as in (7), $| \Sigma\, b_m{}^*\, w^m | \leq \sec \alpha\, \Sigma\, | b_m | \, | w |^m$ and the concluding statements of the proof above may be repeated.

Going a little further, suppose that is is possible to find a real number β_k for each k so that

$$| \arg c_n - \beta_k | \leq \alpha < \pi/_2 \text{ for } n \text{ in } n_k \leq n \leq n_k (1 + \theta).$$

Then $| \arg (b_m{}^*\, e^{-i\beta_k}) | \leq \alpha < \pi/2$ and $| \arg (b_m\, e^{-i\beta_k} | \leq \alpha < \pi/2$. Similarly, $R (b_m{}^*\, e^{-i\beta_k}) \leq R (b_m\, e^{-i\beta_k})$ and it follows that $| b_m{}^* | = | b_m{}^*\, e^{-i\beta_k} | \leq | b_m\, e^{-i\beta_k} | \sec \alpha = | b_m | \sec \alpha$. These last remarks show that $c_n \geq 0$ for $n_k \leq n \leq n_k(1+\theta)$ may be replaced by $|\arg c_n| \leq \alpha < \pi/2$ or $|\arg c_n - \beta_k| \leq \alpha < \pi/2$ in Theorem I. The last condition evidently includes the other two.

3. Coefficient conditions of the Fabry-Polya type

There is a group of theorems associated with the name of Fabry and further developed by Faber, Polya, Landau and others. These results are concerned with certain of the coefficients of a power series and their relation to the singularities on the circle of convergence. In the present section, some of these theorems are recast so that they yield sufficient conditions for overconvergence of the series. The first theorem, included here for convenient reference, is a generalization of a basic result by Fabry and is due to Polya [1].

Theorem I. Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \ \overline{\lim_{n\to\infty}} |f_n|^{\frac{1}{n}} = 1 \tag{1}$$

and $\{n_k\}$ be an increasing sequence of positive integers with the following properties:

1. For each $n \varepsilon \{n_k\}$ there is a real number β_n such that

$$\overline{\lim_{n\to\infty}} |f'_n|^{\frac{1}{n}} = 1, \ n \varepsilon \{n_k\}$$

where $f'_n = R(f_n e^{i\beta_n})$.

2. The maximal density of those indices at which sign changes of $f'_{n+p} = R(f_{n+p} e^{i\beta_n})$ occur in the intervals $-\theta n \leq p \leq \theta n$, $n \varepsilon \{n_k\}$ $(k = 1, 2, ...)$ is Δ where $0 \leq \Delta < 1$ and θ is a real number independent of n_k such that $0 < \theta < 1$. (Note that β_n does not depend on p.)

Then on the arc $|z| = 1$, $|\arg z| \leq \Delta\pi$, there is at least one singular point of $f(z)$.

For the idea of maximal density and its use in this connection see Polya [19]. By modifying the theorem just quoted, we can arrive at a more general overconvergence theorem. First we prove.

Theorem II. Let $f(z)$ satisfy (1) above, θ be a real number such that $0 < \theta < 1$, and $\{n_k\}$ be an increasing sequence of positive integers with the following properties:

1. $|\arg f_{n+p}| \leq \alpha < \pi/2$ or $|\arg f_{n+p} - \pi| \leq \alpha < \pi/2$ for some fisxed α, for all $n + p$ satisfying $-\theta n \leq p \leq \theta n$ and $n \varepsilon \{n_k\}$.

2. $$\overline{\lim_{k\to\infty}} \left\{ \sup_{i \varepsilon I_k} |f_i|^{\frac{1}{i}} \right\} = 1$$

where I_k is the closed interval $[(1 - \theta)^{1/2} n_k, (1 + \theta)^{1/2} n_k]$.

3. The maximal density of those indices at which sign changes of $f'_{n+p} = R(f_{n+p})$ occur in the intervals $[(1 - \theta) n_k, (1 + \theta) n_k]$ $(k = 1, 2, \ldots)$ is Δ where $0 \leq \Delta < 1$. Then, on the arc $|z| = 1$, $|\arg z| \leq \Delta\pi$, there is at least one singular point of $f(z)$.

Proof: In each interval I_k, determine p'_k so that

$$|f_{p'_k}|^{1/p'_k} = \sup_{i \varepsilon I_k} |f_i|^{1/i}.$$

Now define intervals I'_k centered on p'_k so that $I'_k = [(1 - \sigma) p'_k, (1 + \sigma) p'_k]$ with $\sigma = (1 + \theta)^{1/2} - 1$. It follows that $I'_k \subset [(1 - \theta) n_k, (1 + \theta) n_k]$; and hence the maximal density of those indices $n + p$ at which sign changes of $f'_{n+p} = R(f_{n+p})$ occur in the intervals I'_k $(k = 1, 2, \ldots)$ is Δ' where $0 \leq \Delta' \leq \Delta < 1$. This follows from the fact that the maximal density of a subsequence is never greater than the maximal density of the sequence. [19]. Also, by condition 1, $|R(f_{p'_k})| \geq |f_{p'_k}| \cos\alpha$ so that

$$|R(f_{p'k})|^{1/p'k} \geq |f_{p'k}|^{1/p'k} (\cos\alpha)^{1/p'k}. \text{ As } k \to \infty$$

we see that

$$\overline{\lim_{n\to\infty}} |f'_n|^{1/n} = 1, \; n \varepsilon \{p'_k\}.$$

The conditions of Theorem I are then satisfied with sequence $\{n_k\}$ replaced by $\{p'_k\}$, θ by σ, β_n by 0, and Δ by Δ'. The conclusion follows immediately on application of Theorem I.

Theorem III. If $f(z) = \Sigma f_n z^n$ has radius of convergence 1, is regular on the arc $|z| = 1$, $|\arg z| \leq \Delta_0\pi$ with $0 \leq \Delta_0 < 1$, and if conditions 1 and 3 of the preceeding theorem are satisfied for some θ, $0 < \theta < 1$, some sequence $\{n_k\}$ and $\Delta = \Delta_0$, then there is a subsequence of partial sums of the series $\Sigma f_n z^n$ which converges uniformly to $f(z)$ in a neighborhood of each regular point of $f(z)$ on $|z| = 1$.

Proof: By the previous theorem there must be some δ, $0 < \delta < 1$ such that

$$\lim_{k\to\infty} \left\{ \sup_{(1-\theta)^{1/2} n_k \leq i \leq (1+\theta)^{1/2} n_k} |f_i|^{\frac{1}{i}} \right\} \leq \delta$$

so that for a suitable $\varrho < 1$ and for all $k \geq k_0 = k_0(\varrho)$ we have $|f_i|^{\frac{1}{i}} \leq \varrho < 1$ whenever i is in the intervals $[(1-\theta)^{1/2} n_k, (1+\theta)^{1/2} n_k]$ $(k = 1, 2, \ldots)$. This means that $f(z)$ has a representation as the sum of two power series $\Sigma g_n z^n$ and $\Sigma h_n z^n$, the first of which has HADAMARD gaps and radius of convergence 1 and the second has radius of convergence greater than 1. Bracketing the terms of these two series in the natural way and adding the resulting series term by term, we see that by OSTROWSKI's fundamental theorem that $\Sigma f_n z^n$ has the property stated in the conclusion of Theorem III.

If one refers back to Theorem II and there makes the assumption (in place of 3) that the number of changes of sign of $R(f_{n+p})$ for $n + p \varepsilon I_k$) is $q(n_k) = 0(n_k)$, then he may conclude that 1 is a singular point of $f(z)$. Effectively this is the case $\Delta = 0$. To see this it needs only to be noted that a subsequence $\{N_k\}$ of $\{n_k\}$ may be chosen such that $N_{k+1} > 2N_k$ $(k = 1, 2, \ldots)$. The indices $n + p$ at which changes of sign of $R(f_{n+p})$ occur in $-\theta n \leq p \leq \theta n$, $n \varepsilon \{N_k\}$ are arranged according to size and denoted by $\{r_\nu\}$. If r_m lies in $[(1-\theta) N_k, (1+\theta) N_k]$, then

$$m \leq q(N_1) + q(N_2) + \ldots + q(N_k) = 0(N_1 + N_2 + \ldots N_k),$$

so that

$$m = 0\left(N_k \sum_{i=0}^{\infty} 1/2^i\right) = 0(r_m).$$

Hence $\{r_m\}$ is of density zero and therefore of maximal density zero. Applying Theorem II to this situation with $\{N_k\}$ ins place of $\{n_k\}$ and with $\Delta = 0$ yields

THEOREM IV. Let all of the conditions of Theorem II hold except 3 which is replaced by

3'. The number of changes of sign of $R(f_{n+p})$ occuring for $n + p$ in $[(1-\theta) n_k, (1+\theta) n_k]$ is $q(n_k) = 0(n_k)$.

Then 1 is a singular point of $f(z)$.

The corresponding overconvergence statement follows from the last theorem just as Theorem III was obtained from Theorem II. We have namely.

THEOREM V. Let $f(z) = \Sigma f_n z^n$ have radius of convergence 1, be regular at $z = 1$ and satisfy for some increasing sequence of positive integers $\{n_k\}$ and some θ, $0 < \theta < 1$, conditions 1 of Theorem II and 3' of Theorem IV. Then there is a subsequence of partial sums of $\Sigma f_n z^n$ which converges uniformly to $f(z)$ in a neighborhood of each regular point of $|z| = 1$.

The distribution of the non-zero coefficients has an effect on the singularities on the circle of convergence similar to that produced by the distribution of the sign changes discussed above. An immediate consequence of Theorem I is for example:

THEOREM VI. Let $f(z) = \Sigma f_n z^n$ have radius of convergence 1, and let $\{n_k\}$ be an increasing sequence of positive integers satisfying for some θ, $0 < \theta < 1$, the following conditions:

1. $$\lim_{k \to \infty} (\sup_{i \varepsilon I_k} |f_i|^{1/i}) = 1$$

where I_k is the interval $[(1 - \theta)^{i/2} n_k, (1 + \theta)^{1/2} n_k]$.

2. The maximal density of the indices of the non-zero coefficients f_n with n in the intervals

$$[(1 - \theta) n_k, (1 + \theta) n_k], (k = 1, 2, ...) \text{ is } \Delta \text{ where } 0 \leq \Delta < 1.$$

Then on every arc of $|z| = 1$ of length $2\pi\Delta$, $f(z)$ has at least one singular point.

Proof: Consider $\Sigma f_n e^{i\omega n} z^n$ with ω a real number chosen arbitrarily. Now, as in the proof of Theorem II, select that i in I_k for which

$$\sup_{i \varepsilon I_k} (|f_i|^{1/i})$$

is reached and call it p'_k. Determine β_k in each interval I_k so that $R(f_{p'_k} e^{ip'_k\omega} e^{i\beta_k}) = f_{p'_k} e^{ip'_k\omega} e^{i\beta_k}$.

Therefore $|R(f_{p'_k} e^{ip'_k\omega} e^{i\beta_k})| = |f_{p'_k}|$ and the number of changes of sign of $R(f_{n+p} e^{i(n+p)\omega} e^{i\beta_k})$ for $-\sigma n \leq p \leq \sigma n$, $n \varepsilon \{p'_k\}$ is no greater than the number of non-zero coefficients in the corresponding intervals.

(σ is chosen as in Theorem II.) Applying Theorem I, one obtains the result that $f(ze^{i\omega})$ has at least one singular point on the arc $|z| = 1$, $|\arg z| \leq \pi\Delta$. Therefore $f(z)$ has at least one singular point on the arc $|z| = 1$, $|\arg z - \omega| \leq \pi\Delta$. Since ω is arbitrary, Theorem VI is established.

The overconvergence counterpart of the foregoing theorem may now be stated in the following way:

THEOREM VII. Let $f(z) = \Sigma f_n z^n$, $\{n_k\}$ and θ satisfy all of the conditions of Theorem VI except 1 which is replaced by.

1'. There is some arc of length $2\pi\Delta$ on $|z| = 1$ along which $f(z)$ is regular.

Then there is a subsequence of partial sums of $\Sigma f_n z^n$ which converges to $f(z)$ uniformly in a neighborhood of each regular point of $f(z)$ on $|z| = 1$.

It is clear from Theorem IV how one goes about proving.

THEOREM VIII. Let all conditions of Theorem VI hold except 2 which is replaced by.

2'. The number of non-zero coefficients with indices in the intervals $[(1 - \theta)n_k, (1 + \theta)n_k]$ is $q(n_k) = 0(n_k)$.

Then $|z| = 1$ is a natural boundary for the function $f(z)$. This leads of course to

THEOREM IX. Let $f(z) = \Sigma f_n z^n$ have radius of convergence 1, be regular at some point on $|z| = 1$ and satisfy for some increasing sequence $\{n_k\}$ of positive integers and some θ, $0 < \theta < 1$, conditions 1 of Theorem VI and 2' of Theorem VIII.

Then there is a subsequence of partial sums $\Sigma f_n z^n$ which converges uniformly to $f(z)$ in a neighborhood of each regular point of $f(z)$ on $|z| = 1$.

The next theorem is set forth by LANDAU and is shown to be a consequence of his version of Theorem I (essentially the case $\Delta = 0$) which he states and proves in [9]. It is included here without proof.

THEOREM X. Let $f(z) = \Sigma f_n z^n$ have radius of convergence 1 and set $f_n = |f_n| e^{i\varphi_n}$, $\varphi_n \gtreqless 0$. Let there be a θ, $0 < \theta < 1$ and an increasing sequence $\{n_k\}$ of positive integers such that.

1. $(\varphi_{n+1} - \varphi_n) \to 0$ if n runs through the ordered positive integers for which

$$(1 - \theta) p_k \leq n, \; n + 1 \leq (1 + \theta) p_k \qquad (k = 1, 2, \ldots)$$

2.

$$\overline{\lim_{n \to \infty}} \, | f_{p_n} |^{1/p_n} = 1.$$

Then 1 is a singular point of $f(z)$.

A similar theorem, suited to our purpose, may be stated in the following way.

THEOREM XI. Let $f(z) = \Sigma f_n z^n$, $\{n_k\}$ and θ be given satisfying all of the conditions of Theorem X except 2 which is replaced by

2'. $$\overline{\lim_{k \to \infty}} \, (\sup_{i \in I_k} | f_i |^{1/i}) = 1$$

where I_k is the interval $[(1 - \theta)^{1/2} p_k, (1 + \theta)^{1/2} p_k]$.

Then 1 is a singular point of $f(z)$.

Proof: As before, we determine p'_k in each interval I_k such that $| f_{p'_k} |^{1/p'_k} = \sup_{i \in I_k} | f_i |^{1/i}$ and select σ as in Theorem II. Certainly condition 1 is satisfied if θ is replaced by σ, and n_k by p'_k. Applying Theorem X established the conclusion.

The overconvergence result which then follows immediately is

THEOREM XII. Let $f(z)$, $\{n_k\}$, and θ be given satisfying all of the conditions of Theorem X except 2 which is replaced by.

2'. $z = 1$ is a regular point for $f(z)$.

Then there is a subsequence of partial sums of $\Sigma f_n z^n$ which converges uniformly to $f(z)$ in a beighborhood of each regular point of $f(z)$ on $|z| = 1$.

It would be inappropriate to conclude this section without making mention of the work done by J. R. BRAITZEV in this connection. [3] The result which he arrives at is very similar to Theorem I, with the exception that condition 2 there is replaced by the much simpler

$$\underline{\lim} \, (q_n / 2 \, \theta \, n) = \Delta,$$

where q_n is the number of changes of sign in $R(f_{n+p} \, e^{i\beta n})$ occuring in the interval $-\theta n \leq p \leq \theta n$.

The article referred to above in which BRAITZEV proves his assertion is quite long and even more tedious than FABRY'S proof of a related theorem that no less a mathematician than A. PRINGSHEIM called « complicated and indeed not without objection ». The overconvergence theorem which one would expect to get out of BRAITZEV'S is not immediately evident, at least it does not seem to be. In order to develop such a result, a close examination of the inner workings of his proof is probably required, with the aid perhaps of an article by Subbotin [22] in which it is claimed that several theorems of BRAITZEV'S are simplified.

4. A GENERALIZATION OF HADARMARD'S GAP THEOREM

It is easy to prove HADAMARD'S gap theorem (see the Introduction) once we have OSTROWSKI'S first overconvergence theorem (Theorem I, Section 1). For this latter result implies, the gap condition being satisfied, that the sequence of partial sums

$$s_k(z) = \sum_{n=0}^{k} a_n \, z^{p_n}$$

of the series

$$\sum_{n=0}^{\infty} a_n \, z^{p_n} \tag{1'}$$

converges in a full neighborhood of each regular point on the circle of convergence of (1'). But the sequence $\{s_k(z)\}$ is the full sequence of partial sums of (1'), which we know diverges everywhere exterior to the circle of convergence. Therefore every point on this circle is a singular point of the function defined by (1'). This is a much older result than that of overconvergence, having been published by HADAMARD about 30 years before OSTROWSKI'S first theorem was proved.

What if there are two non-zero coefficients separating the ranges of zero coefficients instead of one as in the preceeding example? Or any finite number? How infrequently must the gaps occur when some point on the circle of convergence of a gap series is regular? A partial answer to these questions is given in the main theorem of this section. And as the title indicates, HADAMARD'S theorem is shown to be an immediate consequence of the ideas here.

The following result from the theory of determiants will be found useful and is stated as.

LEMMA I. The determinant

$$(1)\qquad \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-q-1} & x_1^{n-q+1} & \dots & x_1^{n-1} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-q-1} & x_2^{n-q+1} & \dots & x_2^{n-1} & x_2^n \\ & \dots & & & \dots & & & \dots & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-q-1} & x_n^{n-q+1} & \dots & x_n^{n-1} & x_n^n \end{vmatrix}$$

is equal to

$$(2)\qquad S_q(x_1, x_2, \dots, x_n). \prod_{i\leq j, k\leq n}^{j>k} (x_j - x_k)\,;\ q = 1, 2, \dots, n,$$

where $S_q(x_1, x_2, \dots, x_n)$ is the q^{th} elementary symmetric function of the n variables $x_1, x_2, \dots, x_n$.

This lemma is given as an exercise in the second volume of [20] and, as shown there, may be verified by solving the system of equations (n a fixed positive integer):

$$(3)\qquad \begin{aligned} (-1)^{n-1} S_n + x_1(-1)^{n-2} S_{n-1} + x_1^2(-1)^{n-3} S_{n-2} + \dots + x_1^{n-1} S_1 &= x_1^n \\ (-1)^{n-1} S_n + x_2(-1)^{n-2} S_{n-1} + x_2^2(-1)^{n-3} S_{n-2} + \dots + x_2^{n-1} S_1 &= x_2^n \\ \dots \qquad \dots \qquad \dots \qquad & \dots \\ (-1)^{n-1} S_n + x_n(-1)^{n-2} S_{n-1} + x_n^2(-1)^{n-3} S_{n-2} + \dots + x_n^{n-1} S_1 &= x_n^n \end{aligned}$$

for $(-1)^{q-1} S_q$. The system (3) is consistent and each equation in it follows by induction and the definition of the S_q. It may be noted that (1) is valid for $q = 0$ by taking $S_0 \equiv 1$; the determinant in (1) then being the VANDERMONDE determinant for the n variables $x_1, x_2, \dots, x_n$.

THEOREM I. Let the series of polynomials

$$(4)\qquad a_0 + \sum_{\nu=0}^{\infty} \sum_{k=1}^{q_\nu} a_{p_\nu+k}\, z^{p_\nu+k},\ (p_{\nu+1} + 1 > p_\nu + q_\nu)$$

converge uniformly on a circular arc λ, of radius R and center $z = 0$.

Assume that $q_n/p_n \to 0$ as n increases through the positive integers. Then the power series

$$(5) \qquad \sum_{i=0}^{\infty} a_i z^i$$

obtained from (4) by dropping the brackets has radius of convergence $r \geq R$.

Proof: A selection of q_n points (n fixed) is made in the following way. If q_n is odd, divide λ into $q_n - 1$ equal arcs so that the points of division $z_1, z_2, \ldots, z_{q_n}$ are arranged on λ so as to satisfy:

1. z_1 is taken as the midpoint of λ.

2. The remaining z_i having even indices are assigned to the points of subdivision on one side of z_1 so that i increases as the distance along the arc from z_i increases.

3. The z_i with odd indices are arranged in a similar manner on the other half of the arc λ.

The end points of λ are in this case z_{q_n} and z_{q_n-1}.

If q_n is even divide λ into q_n equal arcs by means of $z_1, z_2, \ldots, z_{q_n}$ as described above, with the exception that the end point on the 'odd' side of λ is left unassigned.

The arc λ may be taken smaller than a semicircle without disturbing the results. We suppose that this is done and denote the length of λ by L. Now consider the expression

$$(6) \quad (z_{q_n} - z_\sigma)\,(z_{q_n-1} - z_\sigma)\,\ldots\,(z_{\sigma+1} - z_\sigma)\,(z_\sigma - z_{\sigma-1})\,\ldots\,(z_\sigma - z_1)$$

with $1 \leq \sigma \leq q_n$. Note that there are $q_n - 1$ factors in such a product. Because of the manner in which the points z_i were assigned on λ, (6) has its smallest absolute value for a given q_n when $\sigma = 1$. Noting the geometric fact that the chord of an arc less than a semicircle is greater than half the arc's length, we have when q_n is odd:

$$\prod_{1<i\leq q_n} |z_1 - z_i| \geq \left(\frac{L}{\frac{q_n-1}{2}}\right)^2 \left(\frac{2L}{\frac{q_n-1}{2}}\right)^2 \cdots \left(\frac{\left(\frac{q_n-1}{2}\right).\,L}{\frac{q_n-1}{2}}\right)^2$$

or

$$(7) \qquad \prod_{1<i\leq q_n} |z_1 - z_i| \geq \frac{\left(\frac{L}{2}\right)^{q_n-1}\left[\left(\frac{q_n-1}{2}\right)!\right]^2}{(q-1)^{(q_n-1)}}\,.$$

If q_n is even, the estimate becomes

$$(8) \qquad \prod_{1<i\leq q_n} |z_1 - z_i| \geq \frac{\left(\frac{L}{2}\right)^{q_n-1}\left[\left(\frac{q_n-2}{2}\right)!\right]^2}{2q_n^{(q_n-1)}} .$$

For convenience, the results of the past few sentences are combined in

LEMMA II. The expressions in (7) and (8) bound the absolute value of (6) $(1 \leq \sigma \leq q_n)$ from below, depending on whether q_n is odd or even respectively.

If we now set $f_n(z) = a_0 + \sum_{\nu=0}^{n} \sum_{k=1}^{q_\nu} a_{p_\nu+k} z^{p_\nu+k}$, then $f_n(z) - f_{n-1}(z) = = \sum_{k=1}^{q_n} a_{p_n+k} z^{p_n+k} \equiv \varepsilon_n(z)$. So that by hypothesis

$$(9) \qquad \sum_{k=1}^{q_n} a_{p_n+k} z_i^{p_n+k} = \varepsilon_n(z_i) \equiv {}_i\varepsilon_n ; \; i = 1, 2, \dots, q_n,$$

where ${}_i\varepsilon_n$ tends to zero uniformly with increasing n. The system (9) may be thought of as determining the a_{p_n+k} $(1 \leq k \leq q_n)$, so we have by CRAMER'S rule:

$$(10) \quad a_{p_n+k} = \frac{| z_i^{p_n+1} \; z_i^{p_n+2} \dots z_i^{p_n+k-1} \; {}_i\varepsilon_n \; z_i^{p_n+k+1} \dots z_i^{p_n+q_n} |}{| z_i^{p_n+1} \; z_i^{p_n+2} \dots z_i^{p_n+q_n} |},$$

for $k = 2, 3, \dots, q_n - 1$ and where the notation employed replaces all of the entries of the q_n —rowed determinants with a typical row. (The cases $k = 1$ and $k = q_n$ are handled in a completely similar way throughout). From (10), we have

$$a_{p_n+k} \prod_{i=1}^{q_n} z_i^{p_n+1} \prod_{1\leq i, j\leq n}^{i>j} (z_i - z_j) = | z_i^{p_n+1} \dots z_i^{p_n+k-1} \, {}_i\varepsilon_n \, z_i^{p_n+k+1} \dots z_i^{p_n+q_n} | .$$

Expanding the determinant on the right side of the last expression according to the k^{th} column yields

$$\sum_{\nu=1}^{q_n} (-1)^{k+\nu} \, {}_\nu\varepsilon_n \, | z_i^{p_n+1} \dots z_i^{p_n+k-1} \, z_i^{p_n+k+1} \dots z_i^{p_n+q_n} |_{(i\neq\nu)}$$

which may be written

$$\sum_{\nu=1}^{q_n} (-1)^{k+\nu} \, {}_\nu\varepsilon_n \left(\prod_{1\leq i\leq q_n}^{i\neq\nu} z_i^{p_n+1} \right) | 1 \; z_i, \dots, z_i^{k-2} \; z_i^{k}, \dots, z_i^{q_n-1} |_{i\neq\nu}$$

Dividing both sides by $\prod_{i=1}^{q_n} z_i^{p_n+1}$ gives

$$a_{p_n+k} \prod_{1\leq i,\, j\leq q_n}^{i>j} (z_i - z_j) = \sum_{\nu=1}^{q_n} (-1)^{k+\nu}\ {}_\nu\varepsilon_n\, z^{-p_n-1}.$$

$$\cdot \mid 1\ z_i,\ \ldots,\ z_i^{k-2}\ z_i^k,\ \ldots,\ z_i^{q_n-1} \mid i \neq \nu$$

$$= \sum_{\nu=1}^{q_n} (-1)^{k+\nu}\ {}_\nu\varepsilon_n\, z^{-p_n-1} \left(\sum_{1\leq i,\, j\leq q_n}^{j>k;\, j,\, k\neq\nu} (z_j - z_k) \right) .$$

$$\cdot\ S_{q_n-k} (z_1,\ \ldots,\ \underset{\wedge}{z_\nu},\ \ldots,\ z_{q_n})$$

upon application of Lemma I, suitable identifications being made. Finally,

$$(11)\quad a_{p_n+k} = \sum_{\nu=1}^{q_n} (-1)^{k+\nu}\ {}_\nu\varepsilon_n\ z^{-p_n-1}.\ S_{q_n-k}(z_1, z_2, \ldots, \underset{\wedge}{z_\nu}, \ldots z_{q_n}) .$$

$$\cdot \left(\prod_{i>\nu} (z_i - z) \right)^{-1} \left(\prod_{j<\nu} (z - z_j) \right)^{-1}.$$

By inspection we see that (11) holds for the cases $k = 0$ and $k = q_n$ as well, if as agreed on above $S_0 = 1$. By the definition of $S_{q_n-k}(z_1, z_2, \ldots, \underset{\wedge}{z_\nu}, \ldots, z_{q_n})$ we have

$$(12)\quad | S_{q_n-k}(z_1, z_2, \ldots, \underset{\wedge}{z_\nu}, \ldots, z_{q_n}) | \leq \binom{q_n - 1}{q_n - k} R^{q_n-k};\ 1 \leq \nu \leq q_n.$$

Certainly then

$$| S_{q_n-k}(z_1, z_2, \ldots, \underset{\wedge}{z_\nu}, \ldots, z_{q_n}) | \leq 2^{q_n-1} R^{q_n-k}.$$

Observing that there are only q_n terms on the right side of (11), making use of Lemma II, (13), and the hypothesis on the ${}_i\varepsilon_n$, we get

$$| a_{p_n+k} | \leq \frac{q_n K \left(\frac{2}{L}\right)^{(q_n-1)}}{R^{(p_n+1)}} \frac{(q_n - 1)^{(q_n-1)}\ 2^{(q_n-1)}\ R^{(q_n-k)}}{\left[\left(\frac{q_n-1}{2}\right)!\right]^2}$$

when q_n is odd and n is sufficiently large. K is some constant.

A similar expression (using the estimate in (8)) holds when q_n is even. Since $k\,! > (k/e)^k e$, it follows that

$$| a_{p_n+k} | \leq \frac{q_n \; C^{(q_n-1)}}{R^{(p_n+1-q_n+k)}}$$

in either case, C being a suitable constant. So that

$$| a_{p_n+k} |^{1/p_n+k} \leq \frac{q_n{}^{1/p_n+k}\; C^{(q_n-1)/p_n+k}\; R^{(q_n-1)/p_n+k}}{R}$$

for q_n even or odd. In general

$$1 \leq q_n{}^{1/p_n+k} \leq q_n{}^{1/p_n} = (q_n{}^{1/q_n})^{q_n/p_n} \leq e^{q_n/p_n}$$

and

$$1 \leq (CR)^{(q_n-1)/p_n+k} \leq (CR)^{q_n/p_n} \text{ or } (CR)^{q_n/p_n} \leq (CR)^{(q_n-1)/p_n+k} \leq 1$$

as CR is greater than or less than one respectively.

Applying the hypothesis $q_n/p_n \to 0$ as $n \to \infty$ we obtain

$$\overline{\lim_{n\to\infty}} \; | a_{p_n+k} |^{1/p_n+k} \leq \frac{1}{R}$$

$k = 1, 2, \ldots, q_n$. This completes the proof of Theorem I, and we now can go on to establish the result which is of principal interest to us here.

THEOREM II. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{14}$$

have radius of convergence 1. Let $\{n_k\}$ and $\{N_k\}$ be two sequences of positive integers such that

$$(1 + \theta)\, n_k \leq N_k, \; n_k < N_k < n_{k+1}$$

and

$$c_n = 0 \text{ for } n_k < n \leq N_k\,; \; k = 1, 2, \ldots, p, \ldots$$

where θ is a fixed positive number. Suppose that $n_{k+1}/N_k \to 1$ as $k \to \infty$. Then no point of $|z| = 1$ is a regular point of $f(z)$.

Proof : Suppose that $|z_0| = 1$ and that z_0 is a regular point of $f(z)$. Then by Theorem I (sect. 1) the sequence of partial sums

$$f_{n_k}(z) = \sum_{n=0}^{n_k} c_n z^n$$

converges uniformly to $f(z)$ in a neighborhood of z_0.

Accordingly, there is an arc λ of radius $R > 1$ having as center the origin, along which the sequence of partial sums converges uniformly to $f(z)$. This is equivalent to saying that the series of polynomials

$$\sum_{k=0}^{\infty} (a_{N_k+1} z^{N_k+1} + \dots + a_{n_{k+1}} z^{n_{k+1}}), \ (N_0 \equiv -1)$$

converges uniformly to $f(z)$ along λ. Applying the theorem just proved, we take $q_k = n_{k+1} - N_k$ and $p_k = N_k$. Then

$$q_k/p_k = (n_{k+1} - N_k)/N_k = -1 + n_{k+1}/N_k$$

which tends to zero by the assumption made above. Therefore (14) converges in $|z| < R$. This is a contradiction of our hypothesis on the series in (14), so that z_0 can not be a regular point of the function $f(z)$.

HADAMARD's gap theorem states that the series $\Sigma a_n z^{n_k}$ having radius of convergence 1, has the unit circle as a natural boundary when $n_{k+1}/n_k \geq 1 + \theta$ for $(k = 1, 2, \dots)$, θ a positive number. We can get this out of the previous result by noting that $n_{k+1} = N_k + 1$ here. That is

$$n_{k+1}/N_k = (N_k + 1)/N_k \to 1 \text{ as } k \to \infty.$$

The two assertions which follow are immediate consequences of the previous theorems.

1. Let $f(z) = \Sigma a_i z^i$ have radius of convergence 1.

If the series (4) obtained by bracketing the terms of the given series converges uniformly to $f(z)$ on some arc of $|z| = R > 1$, then

$$\overline{\lim} \ (q_\nu/p_\nu) > 0.$$

2. Let $f(z) = \Sigma c_n z^n$ have radius of convergence 1. Let the sequences $\{n_k\}$ and $\{N_k\}$ and the coefficients c_n satisfy the conditions

of Theorem II. If some point of the unit circle is a regular point of $f(z)$, then

$$\overline{\lim}\, (n_{k+1}/N_k) > 1. \tag{15}$$

5. Remarks on Dienes' proof of Ostrowski's second theorem

As pointed out in the introduction, any series $\Sigma\, c_n\, z^n$ having a subsequence of partial sums $\{s_{n_k}\,(z)\}$ converging uniformly in the neighborhood of a regular point on the circle of convergence has a gap structure. More precisely, there is a $\theta > 0$ and a ϱ, $0 < \varrho < 1$ such that $|c_n| < \varrho^n$ for $n_k \leq n \leq n_k(1 + \theta)$ and all k sufficiently large. This is Ostrowski's second theorem essentially. It is also possible to conclude that $|c_n| < \varrho^n$ for $(1 - \theta)n_k \leq n \leq n_k$; but the first statement is all that is required. See [4] and [18].

In the preceding section (remark 2. on page 27) we have seen that for a gap series to be overconvergent at all, the gaps cannot occur 'too frequently'. That is to say, the length of the separation between gaps may not be arbitrarily small. Indeed there is a certain dependence on the region in which the overconvergence is desired. The method used in proving Ostrowski's second theorem can be applied to this problem to yield a numerical estimate of a lower bound for

$$\overline{\lim}\; n_{k+1}/N_k,$$

dependent on the region of overconvergence. However in [4], the proof rests — somewhat insecurely it would seem — on a mapping theorem stated earlier in the text. Whatever misgivings there may be can be avoided by making some adjustments in the first part of the proof. Also, by using a slightly modified majorant, we obtain for the lower bound mentioned above a value which is a little better than that which Dienes' majorant would have allowed.

Let D be a simply connected domain containing the origin whose boundary is the contour C. If

$$u = g\,(z) = z + a_2\, z^2 + a_3\, z^3 + \ldots \tag{1}$$

is schlicht for $z \,\varepsilon\, D$ and maps D conformally onto $|\,\mu\,| < r$, then r is said to be the inner mapping radius of D with respect to $z = 0$. Let ϱ be the radius of the largest circle about $z = 0$ whose interior is contained in D. Consider the inverse of (1)

$$z = u + b_2\, u^2 + b_3\, u^3 + \ldots$$

which maps $|u| \leq r$ onto D. Setting $u = rw$, $z' = z/r$ in the last expression and applying KOEBE's theorem shows that $\varrho \geq r/4$. Returning to (1), we put $z = t\varrho$ so that

$$(2) \qquad G(t) = g(t\varrho) = t\varrho + a_2 t^2 \varrho^2 + \dots \text{ and}$$

$$(2') \qquad G(t)/\varrho = t + a_2 t^2 \varrho + \dots$$

are schlicht and regular for $|t| < 1$. From (2'), and the last remark it follows that $|a_2\varrho| \leq 2$; and from (2) and CAUCHY's inequality we have $|a_n \varrho^n| \leq r/(1-\varepsilon)^n$ for $n \geq 3$ and ε such that $0 < \varepsilon < 1$. Since $\varrho \geq r/4$ and $|a_2| \leq 8/r$, it follows that $|a_n| \leq 4^n/r^{n-1}(1-\varepsilon)^n$, $n \geq 3$. Now $4^n \leq 8^{n-1}$ when $n \geq 3$, so that $|a_n| \leq (8/r)^{n-1}/(1-\varepsilon)^n$ for all $n \geq 2$. Let r' be chosen as close to r as we please such that $0 < r' < r$. Then $r' = (1-\varepsilon')r$ where $0 < \varepsilon' < 1$. Choose the ε above so that $(1-\varepsilon') = (1-\varepsilon)^2$. This makes $r' = (1-\varepsilon)^2 r$ and hence $r'/r = (1-\varepsilon)^2 \leq (1-\varepsilon)^{n/n-1}$ for $n \geq 2$. In other words $r'^{n-1} \leq r^{n-1}(1-\varepsilon)^n$, which yields $|a_n| \leq 8^{n-1}/r'^{n-1}$ for $n \geq 2$. Therefore

$$g(z) << \frac{z}{1 - \dfrac{8z}{r'}}$$

where the symbol $<<$ is defined as in [5]. That is,

$$f(z) = \Sigma\, a_n z^n << F(z) = \Sigma\, A_n z^n$$

means $A_n > 0$ and $|a_n| \leq A_n$ for alls n.

Consider now $f(z) = \Sigma\, c_n z^n$ with radius of convergence 1.

Let $\{n_k\}$ and $\{N_k\}$ be two sequences of positive integers satisfying

$$(1+\theta)n_k \leq N_k \quad \text{and} \quad n_k < N_k \leq n_{k+1}$$

such that $c_n = 0$ for $n_k < n \leq N_k$ $(k = 0, 1, \dots)$ where θ is a fixed positive number. Suppose that the sequence of partial sums $\{S_{n_k}(z)\}$ converges in some neighborhood of $z = z_0$ on the circle of convergence. But $s_{n_k}(z) = \sum_0^{n_k} c_n z^n = \sum_0^{N_k} c_n z^n$ here. This means that there is a contour C bounding a simply connected domain D having the following properties: 1. D contains $z = 0$.

2. D has inner mapping radius $r > 1$. 3. The sequence of functions $\{s_{n_k}(z)\}$ converges uniformly to $f(z)$ in D.

We may apply to this D and r the remarks made above. Now $v_k(z) = \sum_{N_k}^{\infty} c_n z^n$ is regular and such that $|v_k(z)| \leq 1$ in D for sufficiently large k. Under the mapping (1), $v_k(z)$ is transformed into $\bar{v}_k(u) = d_{N_k} u^{N_k} + d_{N_k+1} z^{N_k+1} + \dots$ Since $|\bar{v}_k(u)| \leq 1$ for $|u| < r$ we have $|d_m| \leq 1/r'^m$, where r' may be chosen as close to r as we please. Or

$$\bar{v}_k(u) << \frac{u^{N_k}}{r'^{N_k}(1 - u/r')}. \tag{3}$$

Replacing u in the expression (3) by means of (1) yields

$$v_k(z) << \frac{z^{N_k}}{r'^{N_k}\left(1 - \dfrac{8z}{r'}\right)^{N_k}} + \frac{z^{N_k+1}}{r'^{N_k+1}\left(1 - \dfrac{8z}{r'}\right)^{N_k+1}} + \dots$$

This means that

$$|c_{N_k+p}| \leq \frac{1}{r'^{N_k+p}}\left[\binom{N_k+p-1}{p} 8^p + \binom{N_k+p-1}{p-1} 8^{p-1} + \right.$$
$$\left. + \dots + \binom{N_k+p-1}{0}\right]$$

for $p \geq 0$. For $p \leq 8N_k$, we may write

$$|c_{N_k+p}| \leq \frac{8^p p}{r'^{N_k+p}}\binom{N_k+p}{p} < \frac{8^p p\,(1 + p/N_k)^{(N_k+p)}}{r'^{N_k+p}\,(p/N_k)^p}.$$

Consider p fixed for the moment, take the $(N_k + p)th$ root, and set $p/N_k = \lambda \leq 8$. One now has

$$|c_{N_k+p}|^{1/N_k+p} \leq \frac{8^{\lambda/1+\lambda}\,(p)^{1/N_k(1+\lambda)}\,(1+\lambda)}{r'\,(\lambda)^{\lambda/1+\lambda}}.$$

The function $h(\lambda) \equiv (1 + \lambda)\, 8^{\lambda/1+\lambda}/\lambda^{\lambda/1+\lambda}$ is continuous and differentiable for $\lambda > 0$. Also $h(\lambda)$ increases monotonically with λ for $0 < \lambda \leq 8$, while $\lim_{\lambda \to 0+} h(\lambda) = 1$.

Therefore if $(N_k + p) \to \infty$ in such a way that p/N_k remains less than $\Phi \leq 8$, one may write

$$\lim_{(N_k+p)\to\infty} |c_{N_k+p}|^{1/N_k+p} \leq h(\Phi)/r'. \tag{4}$$

Restrict the r of the previous discussion to $1 < r \leq 9$ and define $\Phi(r)$ as yhat value Φ such that $h(\Phi) = r$, and for which $0 < \Phi \leq 8$. Now suppose that

$$(\overline{\lim_{k\to\infty}}\ n_{k+1}/N_k) - 1 = A < \Phi(r),\ (A \geq 0).$$

Choose $\varepsilon > 0$ so that $1 + A + \varepsilon < \Phi(r) + 1$. Consequently for k sufficiently large, $n_{k+1} \leq (1 + A + \varepsilon) N_k$. Because of the monotonicity of the function $h(\lambda)$, we way select an $r'(1 < r' < r)$ so that $r' < h(A + \varepsilon)$. From (4)

$$\overline{\lim_{(N_k+p)\to\infty}}\ | c_{N_k+p} |^{1/N_k+p} \leq h(A + \varepsilon)/r' < 1$$

when $0 \leq p \leq (A + \varepsilon) N_k$, which is to say that for k sufficiently large

$$| a_n |^{1/n} \leq B \leq 1$$

when $N_k \leq n \leq n_{k+1}$. But this would mean that the radius of convergence of $\Sigma\, c_n z^n$ is greater than or equal to $1/B > 1$. Therefore $\overline{\lim}\ n_{k+1}/N_k \geq \Phi(r) + 1$.

Some values of r between 0 and 4 and the corresponding lower bound for $\overline{\lim}\ n_{k+1}/N_k$ are given in the table below.

r	$\overline{\lim}\ n_{k+1}/N_k$
1.2	1.028
1.4	1.060
1.6	1.093
1.8	1.127
2.0	1.161
2.2	1.196
2.4	1.232
2.6	1.268
2.8	1.304
3.0	1.342
3.2	1.381
3.4	1.421
3.6	1.462
3.8	1.504
4.0	1.548

If one knows the function (1) for a particular domain D, a better (larger) value for the lower bound of $\overline{\lim}\,(n_{k+1}/N_k)$ can reasonably be expected. As an example, let D be the crescent-shaped domain between the circles $|\,z\,| = 1$ and $|\,z - a\,| = 1 + a$ ($a \quad 0$). The inner radius of this domain is $(2a + 1)/(a + 1)$ and the function (1) is

$$u = \frac{z}{1 + \dfrac{za}{2a + 1}} \,.$$

When $a = 3/2$ we have $r = 1.6$ and $\overline{\lim}\,(n_{k+1}/N_k) \geq 1.143$.

6. Related theorems and remarks

The purpose of this section is to place the preceding considerations in their proper setting against the background of some fundamental results on overconvergence; and, as mentioned in the introduction, to indicate areas in which further development might be possible.

The two theorems of Ostrowski quoted so far (6 and 7 in the Introduction) at first appear to offer a complete solution to the problem of determining the relation between lacunary structure and overconvergence. That is to say (7), is the converse of (6). As far as the bare fact of overconvergence in some neighborhood of a regular point on the circle of convergence is concerned, this is true. Another point of view is expressed by saying that the property of overconvergence is reflected in the sequence of coefficients of the series in much the same way as the radius of convergence is.

Let

$$f(z) = \sum a_n z^n \tag{1}$$

have radius of convergence 1 and be analytically continuable in a simply connected domain D containing $|\,z\,| < 1$. If we wish to become precise and ask for the exact quantitative relations between the lacunary structure of (1) and the domain in which some subsequence of partial sums of (1) converges, the answer is far from clear. The domain of overconvergence referred to could be the whole of D or some prescribed subdomain. The lacunary structure of the coefficients necessary and sufficient for overconvergence in such a domain is not known. In speaking of lacunary structure we should include ranges of 'small coefficients' (relative to the radius of convergence) as well as gap length.

One might hope that a careful analysis of PORTER's examples would shed some light on this problem. Thus let a be a fixed real number between 0 and $1 + \sqrt{2}$.

Form the series

$$\sum \frac{x^{n_i}}{(2a+1)^{n_i}}, \quad n_{i+1} = 2n_i + 1$$

and replace x by $z(2a + z)$. This situation is discussed in the same way as that corresponding to the series in (1) of the Introduction. For each a in the given range, the CASSINIAN (fig. 1) defined by $|z(2a+z)| = 2a + 1$ contains $|z| = 1$ and bounds a simply connected domain.

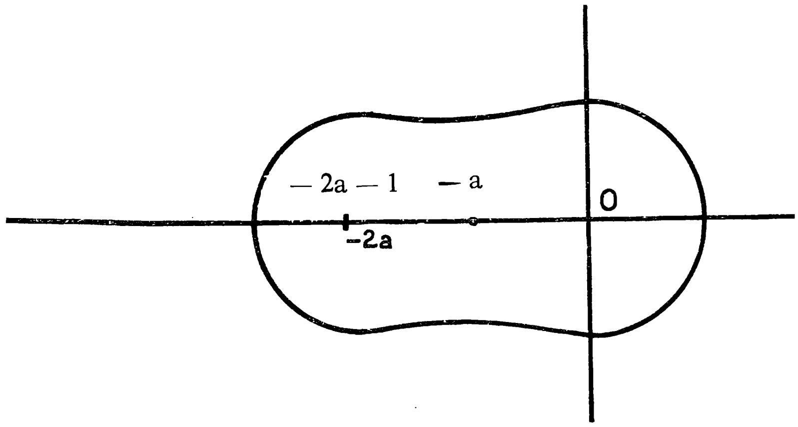

fig. 1

Examing the coefficient b_i of the power z^i in the series

(2) $$\sum \frac{z^{n_i}\,(2a+z)^{n_i}}{(2a+1)^{n_i}}$$

reveals that $\overline{\lim}\, |b_i|^{1/i} = 1$ and $\underline{\lim}\, |b_i|^{1/i} = 2a/(2a+1)$ or $1/(2a+1)^{1/2}$ according as $a \leq (1 + 5^{1/2})/4$, or $a > (1 + 5^{1/2})/4$. The exact relationship between the sequence of coefficients and the domain in which the series of polynomials (2) converges is by no means obvious. If this is true for specific examples, the prospects for information of this kind in the general situation are no better. The third overconvergence theorem of OSTROWSKI provides the answer in a special case. We have namely

(3) If in the series (1), $a_n = 0$ for $n_k < n \leq N_k$ and $N_k/n_k \to \infty$, then the subsequence $\{s_{n_k}(z)\}$ converges uniformly to $f(z)$ in some neighborhood of every regular point of $f(z)$.

Other investigations in this area have been made by Professors MACINTYRE [12] and SUNYER BALAGUER [23]. SUNYER BALAGUER'S theorem follows.

(4) Let D be any domain containing the origin and let $f(z)$ be a function regular in D with the expansion $\Sigma\, a_n z^n$. Let D_1 be a domain and its boundary contained in D. Then there exists a positive number $t_0 = t_0\,(D, D_1)$ such that if $a_n = 0$ for a sequence of intervals $n_k \leq n \leq t n_k$ with $t > t_0$ then the subsequence of partial sums $s_{n_k}(z)$ converges uniformly to $f(z)$ in D_1.

BOURION'S theorem [2], however, even for very short gap lengths concludes that the domain of overconvergence of a subsequence of the series has at least a sort of local agreement with the domain of regularity of the function. See also MACINTYRE [13] for this theorem. There are even certain domains in which overconvergence is assured throughout provided that the gap lengths are of a certain size. Thus if in (1), $a_n = 0$ for $n_k < n \leq N_k$ and $N_k \geq (1 + \theta)\, n_k$; and if $f(z)$ is assumed regular at $z = 1$ and in the domain E corresponding to $|w| < R$, $(R > 1)$ under the transformation $z = w^p\,(1 + w)/2$, then $\{s_{n_k}(z)\}$ converges to $f(z)$ in the whole of E if $\theta > 1/p$. [2, p. 17]. The regions E for various p and R are difficult to describe geometrically, nevertheless they along with the lemniscates of PORTER'S examples, illustrate the fact that the condition $N_k/n_k \to \infty$ is not necessary for the conclusion of (3). Finally, lest the remarks above convey the impression that overconvergence is restricted to domains of a very special character, the following theorem is included.

(5) If D is a simply connected domain containing the origin but not the point at infinity and having at least two boundary points, then there is a function $f(z)$ such that (a) D is the domain of existence of $f(z)$ and (b) the series $f(z) = \Sigma\, c_n z^n$ has a subsequence of partial sums which (over) converges to $f(z)$ in D, uniformly on compact sets. [7]

Several other possibilities which invite further work are these:

(a) Can be theorem of BRAITZEV (see the end of section 2) be recast in an overconvergence form? Indeed there is in this connection a need for a simple proof of BRAITZEV'S theorem itself and an explanation with specific examples of the way in which it generalizes POLYA'S theorem.

(*b*) It should be possible to interpret the developments in section 2 as properties of the entire functions which interpolate the coefficients of the series.

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