THE TYCHONOFF PRODUCT THEOREM FOR COMPACT HAUSDORFF SPACES DOES NOT IMPLY THE AXIOM OF CHOICE: A NEW PROOF. EQUIVALENT PROPOSITIONS.

by

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In most usual axiomatic set theories (1) the axiom of choice (briefly, AC) is used to prove the classical Tychonoff's theorem on products of compact spaces:

AT. *The product* $\prod_{i \in I} E_i$ *of any family* $(E_i)_{i \in I}$ *of compact spaces is a compact space.*

Kelley has shown in [14] that AT implies AC and therefore, AT and AC are logically equivalent. In this paper we raise the question of studying the role of the following axioms, if the axiom of choice is removed from the above mentioned axiomatics:

AT_1. *The product* $\prod_{i \in I} E_i$ *of any family* $(E_i)_{i \in I}$ *of compact* T_1*-spaces is a compact space.*

AT_2. *The product* $\prod_{i \in I} E_i$ *of any family* $(E_i)_{i \in I}$ *of compact* T_2*-spaces is a compact space.*

AT_m. *The product* $\prod_{i \in I} E_i$ *of any family* $(E_i)_{i \in I}$ *of compact metric spaces is a compact space.*

AT_r. *The product* $\prod_{i \in I} E_i$ *of any family* $(E_i)_{i \in I}$ *of compact real intervals is a compact space.*

(1) Explicitly, we are referring to Zermelo-Fraenkel and Neumann-Bernays-Godel axiomatics, this latter with or without A. P. Morse's modifications. (See Cohen [4], p. 50-53 and 73-75, and Rubin [20], p. 28-83).

AT_f. *The product* $\prod_{i \in I} E_i$ *of any family* $(E_i)_{i \in I}$ *of finite topological spaces is a compact space.*

Clearly, AT implies AT_1, and as Kelley's proof also shows, AT_1 implies the axiom of choice. Therefore, AT_1 and AC are equivalent.

It is shown in theorem 1 that AT_2 and AT_r are equivalent, and, consequently, to AT_m. Also, it is shown that AT_2 is logically equivalent to some basic theorems in Topology and Functional Analysis, commonly proved using the axiom of choice.

In the literature on this subject is proved the equivalence of the following propositions:

P_1. *Boolean Prime Ideal Theorem* (BPI): *Every Boolean algebra has a prime ideal.*

P_2. *In every Boolean algebra, there exists a two-valued measure.*

P_3. *In every Boolean algebra every proper ideal is included in some prime ideal.*

P_4. *In every commutative ring with unit, every proper ideal is included in some prime ideal.*

P_5. *The restricted Stone representation theorem: Every Boolean algebra is isomorphic to a Boolean algebra of sets.*

P_6. *The Tychonoff theorem for* T_2*-spaces: The product of compact Hausdorff-spaces is compact in the product topology* ($= AT_2$).

P_7. *Alaoglu's theorem: The closed unit ball of the topological dual* E' *of a Banach space* E *is a compact Hausdorff space in the weak topology* $\sigma(E', E)$.

P_8. *The Stone-Čech compactification theorem.*

P_9. *Every complete and totally bounded uniform space is compact.*

P_{10}. *Alexander's lemma: A space is compact if there is a subbase for the open sets enjoying the Heine-Borel property.*

P_{11}. *The principle of consistent choices: If* $\mathcal{E} = (E_i)_{i \in I}$ *is a family of compact spaces,* ϱ *a relation of consistency for* $\mathcal{E}$, *and if, moreover, for every finite set* $J \subset I$ *there exists a* ϱ*-consistent choice from* $(E_i)_{i \in J}$, *then there exists a* ϱ*-consistent choice from* $\mathcal{E}$.

P_{12}. *The completeness theorem for* 1 *st.-order languages: Let* Σ *be a set of* 1 *st-order sentences with arbitrary many non-logical constants. If* Σ *is consistent, then it has a model.*

P_{13}. *The compactness theorem for* 1 *st-order languages: Let* Σ *be as in* P_{12}. *If every finite subset of* Σ *has a model, then* Σ *has a model.*

Proofs are found in the following papers: Łos-Ryll Nardzewski [16], [17], Ł. A. Henkin [11], H. Rubin-D. Scott [21], D. Scott [23] A. Tarski [26], [27], and R. Sikorski [24].

A consequence of the BPI is the following:

OE. *Order Extension Principle: If* A *is a set and* R_0 *a partial ordering on* A, *then there exists a total ordering* R *on* A *such that* $R_0 \subset R$.

The OE has been discovered by Banach, Kuratowski and Tarski (see W. Sierpinski [25] p. 158). The first proof which appeared in print is due to E. Marzewski [19]. Marczewski used the lemma of Zorn-Kuratowski in order to deduce OE. Łos, Ryll-Nardzewski and Ł. Henkin observed, that OE is already a consequence of the BPI.

An evident consequence of the OE is:

OP. *Ordering principle: Every set can be totally ordered.*

Thus we have:

$$ZF \vdash AC \to BPI \to OE \to OP.$$

In the Zermelo-Fraenkel axiomatic (ZF), BPI is strictly weaker than AC: $ZF \nvdash (BPI \to AC)$ (See D. Halpern - A. Łevy [10], U. Felgner [7], p. 128-146 and V. Dalen and Monna [6], p. 61). Therefore, $ZF \nvdash (AT_2 \to AC)$. According to Adrian R. Mathias, $OP \to OE$ is not a theorem of (ZF). It is not known whether $OE \to BPI$ is provable in ZF or not.

In this paper we show BPI and AT_2 are equivalent to the following propositions:

P_1'. *The Stone representation theorem for the Boolean algebra* $\mathcal{A} = \mathcal{P}(S)$ *of all subsets of a set* S.

P_2'. *Every proper ideal in the Boolean algebra* $\mathcal{P}(S)$ *of all subsets of a set* S *is included in a prime* (*or maximal*) *ideal.* (2).

P_3'. *Every filterbase on a set is included in an ultrafilter on that set.*

P_4'. *A topological space* E *is compact if and only if every ultrafilter on* E *is convergent.*

(2) The equivalence of P_3 and P_2' has been pointed by A. Tarski [28].

P_5'. *Alaoglu-Bourbaki theorem.*

P_6'. *Kakutani's theorem on representation of abstract M-spaces.*

P_7'. *Kakutani's theorem on representation of abstract M-spaces with an element.*

P_8'. *The unrestricted Stone representation theorem for a Boolean algebra A (with or without unit element).*

P_9'. *Every proper ideal in a Boolean algebra A (with or without unit element) is included in a prime ideal.*

The implication $BPI \to P_3'$ allows the use of the classical proof of Bourbaki [3] to show that $BPI \to AT_2$. To this end, Łos and Ryll-Nardzewski make use, in a slightly more complicated way, [17], the following proposition:

A topological space E is compact if and only if for every two-valued measure μ defined for all subsets of E, there exists precisely one point $x \in E$ such that $\mu(V) = 1 = \mu(E)$ for every neighbourhood V of x.

Łos and Ryll-Nardzewski use the principle of consistent choices to show that BPI implies the Hahn-Banach extension theorem. We achieve this result showing directly that AT_r implies the Hahn-Banach theorem. Proofs using AT_2 were already known for the existence of Haar measures on locally compact topological groups, the Alaoglu-Bourbaki theorem, the Mackey-Arens theorem, etc.

It is worth noting that, BPI and P_7' being equivalent, the following theorem of Nachbin is a consequence of BPI or AT_2:

Every normed space whose collection of spheres has the binary intersection property and whose unity sphere contains an extreme point is isomorphic in the vector and norm sense to the normed space of all real continuous functions over an extremally disconnected compact Hausdorff space, which is unique up to homeomorphisms.

His proof hinges crucially on the assumption of existence of an extremal point in the unit sphere. Kelley's proof ([15]), where the above assumption is removed, might become unfeasible using only the BPI.

The non-equivalence of AT_1 and AT_2 suggests that a wide family of axioms could ve obtained assuming the validity of Tychonoff's product theorem for certain classes of compact spaces. These axioms are, in a certain sense, similar to the axiom of choice (equivalent to AT_1). In particular, given a class α of sets, the following axioms can be considered:

$AT_{1,\alpha}$. *The product* $\prod_{i\in I} E_i$ *of a amily of compact* T_1*-spaces is a compact space if* $I \in \alpha$.

$AT_{2,\alpha}$. *The product* $\prod_{i\in I} E_i$ *of a family of compact* T_2*-spaces is a compact space if* $I \in \alpha$.

It is shown in theorem 2 that, if for every set Ω the set $B(\Omega)$ of all bounded real functions on Ω belongs to α, then $AT_{2,\alpha}$ is equivalent to AT_2. Particularly, this is true if every complete lattice belongs to α (3).

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THEOREM 1. *The following propositions are equivalent to* AT_2:

1.1. *Tychonoff's theorem for the product* $\prod_{i\in I} E_i$ *of compact real intervals.*

1.2. *The closed unit ball of the topological dual* E' *of a Banach space* E, *is compact for the weak topology* $\sigma(E', E)$.

1.3. *Stone-Čech compactification theorem for a completely regular Hausdorff space.*

1.4. *Stone representation theorem for the Boolean algebra* $\mathcal{A} = \mathcal{P}(\Omega)$ *of all subsets of a set* Ω.

1.5. *Ever proper ideal in the Boolean algebra* $\mathcal{A} = \mathcal{P}(\Omega)$ *of all subsets of a set* Ω, *is contained in a prime* (*or maximal*) *ideal.*

1.6. *Every filterbase on a set is included in an ultrafilter.*

1.7. *A topological space* E *is compact if and only if every ultrafilter on* E *is convergent.*

PROOF. $AT_2 \Rightarrow 1.1$. It is obvious.

$1.1 \Rightarrow 1.2$. See, for instance, Bachman-Narici [1], p. 339.

$1.2 \Rightarrow 1.3$. Let Ω be a completely regular Hausdorff space, $E = C_b(\Omega)$ the Banach space of continuous bounded real functions on Ω: $||f|| = \mathrm{Sup}\,\{|f(x)| : x \in \Omega\}$ $(f \in C_b(\Omega))$, and B' the closed unit

(3) It is easily proved that $AT_{1,\alpha}$ is equivalent to AT_1 if α contains every complete lattice.

ball in E' (dual space of E). Defining $j : \Omega \to B'$ by $j(x) = \delta_x$ (the evaluation at x) and endowing B' with the topology induced by $\sigma(E', E)$, and assuming 1.2, it is easyly shown that Ω and $J(\Omega)$ are homeomorphic, and that the $\sigma(E', E)$-closure $\overline{j(\Omega)}$ of $j(\Omega)$ is the Stone-Čech compactification of Ω. (See, for instance, Day [5], Definition 1 and Theorem 1, p. 85, and Badrikian [2], p. 63-64).

1.3 $\Rightarrow$ 1.4. It is shown as in Kakutani [13], § 14, p. 1013-1014.

1.4 $\Rightarrow$ 1.5. Let $\mathcal{J}$ be a proper ideal in $\mathcal{A} = \mathcal{P}(\Omega)$. If 1.4 is assumed, there exists an isomorphism Θ between $\mathcal{A}$ and the open-compact sets of a totally disconnected compact Hausdorff space E. As $\mathcal{J}$ is a proper ideal, there exists $A_0 \in \mathcal{A}$, $A_0 \notin \mathcal{J}$ (we can take $A_0 = E$), and so

$$\Phi = \{\Theta(A_0) \setminus \Theta(A) : A \in \mathcal{J}\}$$

is a filter base on E. As every $\Theta(A)$ is open in E, and $\Theta(A_0)$ is compact, there exists a point

$$x \in \bigcap \{\Theta(A_0) \setminus \Theta(A) : A \in \mathcal{J}\}.$$

Consider $\mathcal{P} = \{A \in \mathcal{A} : x \notin \Theta(A)\}$. Clearly, $\mathcal{P}$ is a proper ideal in $\mathcal{A}$, and $\mathcal{P} \supset \mathcal{J}$. If A and B belong to $\mathcal{A}$ and $A \cap B \in \mathcal{P}$, then

$$x \notin \Theta(A \cap B) = \Theta(A) \cap \Theta(B).$$

Therefore, $x \notin \Theta(A)$ or $x \notin \Theta(B)$, i. e., $A \in \mathcal{P}$ or $B \in \mathcal{P}$. This shows that $\mathcal{P}$ is a prime ideal containing $\mathcal{J}$.

1.5 $\Rightarrow$ 1.6. It is a consequence of the duality between filters on a set Ω and proper ideals in the Boolean algebra $\mathcal{P}(\Omega)$ of all subsets of Ω:

Φ is a filter on $\Omega \iff \{A : \Omega \setminus A \in \Phi\}$ is a proper ideal in $\mathcal{P}(\Omega)$.

1.6 $\Rightarrow$ 1.7. Suppose $(F_i)_{i \in I}$ is a family of closed sets in the topological space E with the finite intersection property. Then $\Phi = \{F_i : i \in I\}$ is a filterbase on E. If 1.6 is assumed, there exists an ultrafilter $\mathcal{U} \supset \Phi$, and, if every ultrafilter converges, $\mathcal{U}$ converges to a point x. Therefore, $x \in \bigcap_{i \in I} F_i$ and $\bigcap_{i \in I} F_i \neq \phi$. This shows that E is a compact space. Conversely, if E is compact, it is clear that every ultrafilter on E is convergent.

$1.7 \Rightarrow AT_2$. Let $(E_i)_{i\in I}$ be a family of compact Hausdorff spaces, and $E = \prod_{i\in I} E_i$. If $\mathcal{U}$ is an ultrafilter on E, every projection $p_i(\mathcal{U}) = \{p_i(U) : U \in \mathcal{U}\}$ $(i \in I)$ is an ultrafilter basis on E_i, and so, $p_i(\mathcal{U})$ converges to a single point $x_i \in E_i$. Then $\mathcal{U}$ converges to $x = (x_i)_{i\in I}$ and, if 1.7 is assumed, E is a compact space. (4)

THEOREM 2. *Let $B(\Omega)$ be the set of all bounded real functions on a set Ω. If $B(\Omega)$ belongs to the class α for every set Ω, then $AT_{2,\alpha}$ is equivalent to AT_2.*

PROOF. Proceeding as in theorem 1, we successively show that $AT_{2,\alpha}$ implies 1.1 for $I = B(\Omega)$, 1.2 for $E = B(\Omega)$, and 1.3 for E a discrete space. This clearly implies 1.4, and according to theorem 1, AT_2 too. Finally, it is obvious that AT_2 implies $AT_{2,\alpha}$.

THEOREM 3. *The Hahn-Banach theorem on extension of linear forms is a consequence of AT_2.*

PROOF. Let E be a real vector space, E_0 a subspace of E, f_0 a linear form on E_0, and p a sublinear functional on E such that $f_0(x) \leq p(x)$ for every $x \in E_0$. Let $S_x = [-p(-x), p(x)] \subset \mathbf{R}$ and $S = \prod_{x\in E} S_x$ be endowed with the product topology. If AT_2 is assumed, S is a compact space. Every point in S is an application $f : E \to \mathbf{R}$ such that $f(x) \in S_x$ for every $x \in E$. Let us denote $(E_i)_{i\in I}$ the family of all finite extensions of E_0 in E, i. e., each E_i is a vector subspace of E containing E_0 and such that E_i/E_0 has finite dimension. Let A_i be the set of members of S which restrictions to E_i are linear forms extending f_0. It is a well known fact that for every $i \in I$ there exists a linear form g_i, extending f_0 to E_i, that satisfies $g_i(x) \leq p(x)$ for every $x \in E_i$, so

$$-p(-x) \leq g_i(x) \leq p(x)$$

for every $x \in E_i$ (see, for instance, Bachman-Narici, [1], p. 176). If f_i is defined by

$$\begin{aligned} f_i(x) &= g_i(x) \quad \text{for} \quad x \in E_i \\ f_i(x) &= p(x) \quad \text{for} \quad x \notin E_i, \end{aligned}$$

then $f_i \in A_i$, and therefore $A_i \neq \emptyset$ for every $i \in I$.

(4) If the spaces E_i are not Hausdorff, the proof is not valid unless the axiom of choice is assumed, for $p_i(\mathcal{U})$ can have more than one limit point x_i in E_i. See Bourbaki [3], p. 88. (See the note added in proof).

For every pair α, β of real numbers, and every pair x, y of elements of E, the set

$$M(\alpha, \beta; x, y) = \{f \in S : f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)\}$$

is clearly a closed set in S. In a similar way, for every $x \in E_0$, the set

$$N(x) = \{f \in S : f(x) = f_0(x)\}$$

is closed in S. Then, every

$$A_i = [\cap \{M(\alpha, \beta; x, y) : \alpha, \beta \in \mathbf{R},\ x, y \in E_i\}] \cap [\cap \{N(x) : x \in E_0\}]$$

is closed in S. As the family of closed sets $(A_i)_{i \in I}$ in the compact space S has the finite intersection property since

$$A_i \cap A_j \supset A_k \neq \emptyset$$

for $E_k = E_i + E_j$, there exists an $f \in \bigcap A_i$. Let us show f is a linear form. If $x, y \in E$, there exists an $i \in I$ such that $x, y \in E_i$. As $f \in A_i$,

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Finally, as $f \in N(x)$ for every $x \in E_0$, f is an extension of f_0 to E and verifies $f(x) \leq p(x)$ for every $x \in E$, since $f \in S$.

THEOREM 4. *The following propositions are equivalent to AT_2:*

4.1. *Kakutani's representation theorem for abstract M-spaces.*

4.2. *Kakutani's representation theorem for abstract M-spaces with unit element.*

4.3. *Stone's representation theorem for a Boolean algebra A (with or without unit element).*

4.4. *Every proper ideal in a Boolean algebra A is included in a prime ideal.*

PROOF. $AT_2 \Rightarrow 4.1$. In view of theorems 1 and 3, AT_2 implies 1.2 and the Hahn-Banach theorem. Therefore, Kakutani's proof is valid here (Kakutani, [13], p. 1000-1005). (5)

(5) Proofs for 4.1 and 4.3 based on the Krein-Milman theorem are not valid for $AT_2 \Rightarrow 4.1$ (or 4.2), as it is not yet known whether or not AT_2 implies the Krein-Milman theorem. A proof of this kind is given in Schaefer [22], p. 247.

4.1 $\Rightarrow$ 4.2. See Kakutani, [13], § 11, p. 1005-1006.

4.2 $\Rightarrow$ 4.3. If A has unit element the proof is given in Kakutani, [13], § 14, p. 1014-1015. Suppose A has no unit element, and let A' be a Boolean algebra with unit element e that contains A as a subalgebra and has the property:

$$x \in A \quad \text{or} \quad e - x \in A \quad \text{for every} \quad x \in A'.$$

Then, there exist a totally disconnected compact Hausdorff space Ω' and an isomorphism Θ between A' and the Boolean algebra of all open-compact sets in Ω'. Writing $\Omega = \bigcup \{\Theta(a) : a \in A\}$ and endowing Ω with the topology whose open sets are the subsets G of Ω such that $G \cap \Theta(a)$ is open in Ω' for every $a \in A$, it is true that Ω is a totally disconnected locally compact Hausdorff space such that A is isomorphic to the Boolean algebra of open-compact sets in Ω.

4.3 $\Rightarrow$ 4.4. It is shown as 1.4 $\Rightarrow$ 1.5.

4.4 $\Rightarrow AT_2$. As 4.4 $\Rightarrow$ 1.5, it suffices to remember that, according to theorem 1, 1.5 implies AT_2.

Corollary 1. *AT_2 is not equivalent to axiom AT_1.*

Proof. As Kelley [14] shows, the axiom of choice is equivalent to AT_1, and, as proved in theorem 4, the BPI axiom is equivalent to AT_2. Now, if the Zermelo-Fraenkel axiomatic set theory is consistent, it is also consistent if the axiom of choice is denied and the BPI is included: $ZF \not\vdash (BPI \Rightarrow AC)$. (See *v.* Dalen and Monna [6], p. 61).

Corollary 2. *The Hahn-Banach extension theorem does not imply the axiom of choice.*

Proof. It is a consequence of theorem 3 and corollary 1. (See v. Dalen and Monna, [6], p. 34 and 61).

Theorem 5. *The following assertions are consequences* of AT_2:

5.1. *If $(E_i)_{i \in I}$ is a family of non-empty compact Hausdorff spaces, then $\prod_{i \in I} E_i$ is not empty. That is: there exists a choice function f such that $f(i) \in E_i$ for every $i \in I$.*

5.2. *If $(E_i)_{i \in I}$ is a family of non-empty finite sets, then there exists a choice function f such that $f(i) \in E_i$ for every $i \in I$.*

5.3. *If* $(E_i)_{i \in I}$ *is a family of finite topological spaces, then* $\prod_{i \in I} E_i$ *is a compact space.*

PROOF. $AT_2 \Rightarrow 5.1$. Let e be an element not belonging to $\bigcup_{i \in I} E_i$, $e_i = e$ and $E_i' = E_i \cup \{e_i\}$ endowed with the topology that has the open sets G_i of E_i and $\{e_i\}$ as a basis. Clearly, $(E_i')_{i \in I}$ is a family of compact Hausdorff spaces. Therefore, if AT_2 is assumed, $E' = \prod_{i \in I} E_i'$ is a compact space.

For every finite subset J of I we denote $A_i(J)$ the set

$$A_i(J) = E_i \quad \text{if} \quad i \in J$$
$$A_i(J) = E_i' \quad \text{if} \quad i \notin J,$$

and let $A(J) = \prod_{i \in I} A_i(J)$ $(\subset E')$. Then, every $A(J)$ is a non-empty closed set, since

$$A(J) = \bigcap_{i \in J} p_i^{-1}(A_i(J)),$$

where p_i is the projection of E' on E_i', E_i is non-empty, $e_i \in E_i'$ and J is finite. As

$$A(J') \cap A(J'') \supset A(J' \cup J''),$$

the $A(J)$'s form a family of closed sets with the finite intersection property. If AT_2 is assumed, the existence of a point x belonging to every $A(J)$ follows, and, in particular, the existence of a point belonging to every $A(\{i\})$. This proves that

$$f : f(i) = p_i(x) \in E_i, \quad \forall\, i \in I$$

is a choice function which fulfills the required conditions [6].

5.1 $\Rightarrow$ 5.2. Evidently, all we need is to endow every E_i with the discrete topology.

$AT_2 \Rightarrow 5.3$. Recalling theorem 1 and $AT_2 \Rightarrow 5.2$, it suffices to show that 1.6 and 5.2 implie 5.3. Let $(E_i)_{i \in I}$ be a family of finite topological spaces and $E = \prod_{i \in I} E_i$. If $\mathcal{U}$ is an ultrafilter on E, every projection $p_i(\mathcal{U})$ $(i \in I)$ is an ultrafilter basis on E_i. Therefore, for every i, the set A_i of limit points of $p_i(\mathcal{U})$ is finite and non-empty. It follows from 5.2 that there exists a choice function f such that $f(i) \in A_i$ for every $i \in I$. Consequently, $x = (f(i))_{i \in I}$ is a limit point of $\mathcal{U}$, and then from 1.6, it is clear that E is a compact space.

(6) This proof is similar to that given by Kelley [14], for proving $AT_1 \Rightarrow AC$.

Added in proof.

J. L. Bell and D. H. Fremlin have showed that each of the following three propositions is equivalent to the axiom of choice: (1) *The unit ball of the dual E' of a normed linear space E has an extreme point.* (2) *The conjunction of the Krein-Milman theorem and the Boolean prime ideal theorem.* (3) *The conjunction of the Hahn-Banach theorem and a slightly strengthened version of the Krein-Milman theorem.* (See: *A geometric form of the axiom of choice.* Fund. Math. *77* (1972), 167-170). The strengthened version of the Krein-Milman theorem in (3) is essential, for D. Pincus has recently shown that the conjunction of the Hahn-Banach theorem and the usual Krein-Milman theorem does not imply the axiom of choice.

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