# THE UNIQUENESS PROBLEM IN THE THEORY OF NUMERICAL DIVERGENT SERIES AND FORMAL LAWS OF CALCULUS I (*)

by

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## § 1. Characterization of powers series and LAPLACE'S transformation in u- and U-spaces.

### 1. u- *and* U-*spaces.*

Def. **1.** We call [1, 6, 11] (**) a **c**-space to the set of convergent sequences $\{s_t\}$ with the norm

$$\|s\| = \sup_{0 \leq t < +\infty} |s_t|$$

Def. 2. The set of sequences $\{u_t\}$ with convergent series:

$$\sum_{t=0}^{\infty} u_t = \lim_{T\to\infty} \sum_{t=0}^{T} u_t < 0$$

will be called **u**-space, adopting the norm

$$\|u\| = \sup_{0 \leq T < +\infty} \left| \sum_{t=0}^{T} u_t \right|$$

Def. 1′. We call **C**-space $[0, +\infty]$ or for short just **C**-space the set of functions $S(t)$ continuous at the compact positive real semi-axis $[0, +\infty]$ with the norm

$$\|s\| = \sup_{0 \leq t < +\infty} |S(t)|$$

Def. 2′. The set of functions $U(t)$ summable in every finite interval $0 \leq t \leq T$ and with convergent improper or generalized integral:

$$\int_0^{+\infty} U(t)\, dt = \lim_{T\to+\infty} \int_0^T U(t) dt < \infty$$

is called **U**-space [17, 119] [3, 226], [98, 57] [4, 444] when metrized with the norm:

$$\|U\| = \sup_{0 \leq T < +\infty} \left| \int_0^T U(t) dt \right|$$

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(**) First number indicates paper and last one the page. Numbers in between refers to the theorem, definition, etc., or other complementary reference. Parentheses refers to formulae or numbers in this very paper, as explained in [21, Previous specifications].

Here are some immediate or already known consequences that we have to use:

Prop. 1. *The sums*

$$s_t = \sum_{\nu=0}^{t-1} u_\nu \text{ for } t = 1, 2..., \text{ and } s_0 = 0$$

*corresponding to sequences* $\{u_t\} \in \mathbf{u}$ *compose the vectorial subspace* $\mathbf{c}_0$ *of* $\mathbf{c}$, *formed by the sequences* $\{s_t\}$ *of* $\mathbf{c}$ *with* $s_0 = 0$; *and conversely the differences* $\{\Delta s_t\}$ *of sequences* $\{s_t\} \in \mathbf{c}_0$ *fill all the* $\mathbf{u}$*-space.*

Prop. 1. *The primitives*

$$S(t) = \int_0^t U(\nu)d\nu \text{ for } 0 \leq t \leq +\infty$$

*of the functions* $U(t) \in \mathbf{U}$ *compose the vectorial subspace* $\mathbf{C}_0$ *of* $\mathbf{C}$ *formed by the functions absolutely continuous in every finite interval* $0 \leq t \leq T$, *with finite limit*

$$S(+\infty) = \lim_{t\to+\infty} S(t) < \infty$$

*and null at* 0; *and conversely, the derivatives* $S'(t)$ *in almost the whole finite interval* $0 \leq t \leq T$ *for* $0 < T < \infty$ *of* $\mathbf{C}_0$ *form exactly* $\mathbf{U}$.

Proof. Both propositions are immediate, since

$$\Delta s_t = u_t \text{ for } t = 0, 1, 2,...$$

$$S'(t) = U(t) \text{ for almost all } 0 \leq t \leq T$$

Remark. On the other hand if we write

$$s_t = \sum_{\nu=0}^{t} u_\nu \text{ for } t = 0, 1, 2,...$$

we have

$$\Delta s_t = u_{t+1} \text{ for } t = 0, 1, 2,...$$

So the sequence $\{\Delta s_t\}$ for $t = 0, 1, 2,...$ does not yield the $\{u_t\}$ for $t = 0, 1, 2,...$ but only for $t = 1, 2...$, and must be completed with the initial term $u_0 = s_0$. For the same reason the converse in Prop. 1 does not turn true for $\mathbf{c}$, but only for $\mathbf{c}_0$.

Def. 3. We will call

| | |
|---|---|
| $\mathbf{u}_a$ | $\mathbf{U}_a$ |

the vectorial subspace of

| | |
|---|---|
| $\mathbf{u}$ | $\mathbf{U}$ |

formed by the

| | |
|---|---|
| sequences $\{\alpha_t\}$ | functions $U(t)$ |

with

| | |
|---|---|
| series $\sum_{n=0}^{\infty} \alpha_t$ | integral $\int_0^{+\infty} U(t)dt$ |

absolutely convergent

$$\sum_{t=0}^{\infty} |\alpha_t| < \infty \qquad\qquad \int_0^{\infty} |U(t)|\, dt < \infty$$

Remark. — Only $\mathbf{u}_a$ -subspace (Def. 5 and Prop. 5, no. 2, § 1), (Prop. 2 and Theor. 1, no. 6, § 1) will be used in the sequel and not the $\mathbf{U}_a$ one.

Let us recall the expressions for the linear functionals [1, 23] in these spaces.

Theor, 1. *Any continuous linear functional $\mathcal{F}(s)$ defined in the space* $\mathbf{c}$ *of convergent sequences $\{s_t\}$ has the form:*

$$\mathcal{F}(s) = C \lim_{t\to\infty} s_t + \sum_{t=0}^{\infty} C_t\, s_t, \tag{1}$$

*$C$ being a constant and $\{C_t\}$ a sequence such that*

$$|C| + \sum_{t=0}^{\infty} |C_t| = |\mathcal{F}| < \infty \tag{2}$$

Proof. [1, 66-67].

Theor, 2. *Any continuous linear functional $\mathcal{F}(u)$ defined in* $\mathbf{u}$ *may be expressed as follows:*

$$\mathcal{F}(u) = \sum_{t=0}^{\infty} u_t\, K_t \tag{4}$$

*where $\{K_t\} \in \mathbf{c}$, i. e., with:*

$$\lim_{t\to\infty} K_t = C < \infty \tag{5}$$

Proof. Writing

$$\mathcal{F}_1(s) = \mathcal{F}(u) \tag{6}$$

for every sequence $\{s_t\}$ with

$$s_t = \sum_{\nu=0}^{t-1} u_\nu \text{ for } t = 1, 2\ldots, \text{ and } s_0 = 0, \tag{7}$$

we have a linear functional $\mathcal{F}_1(s)$ defined in the vectorial subspace $\mathbf{c}_0$ of $\mathbf{c}$ (Prop. 1), which extends [1, Theor. 2, 55] to the whole vectorial $\mathbf{c}$-space.

Theor. 1′. *Any continuous linear functional $\mathcal{F}(S)$ defined in the vectorial subspace $\mathbf{C}_0[0, +\infty]$ of $\mathbf{C}[0, +\infty]$ of the continuous functions in the compact interval $[0, +\infty]$ and null at 0 has the form:*

$$\mathcal{F}(s) = \int_0^{\infty} S(t)\, d\, C(t), \tag{1'}$$

*$C(t)$ being a function of limited variation in $[0, +\infty]$ such that*

$$\int_0^{\infty} |dC(t)| = |\mathcal{F}| < \infty, \tag{2'}$$

*which besides can be taken with*

$$C(+\infty) = \lim_{t\to+\infty} C(t) = 0 \tag{3'}$$

Proof. [19, III. 108].

Theor, 2′. *Any continuous linear functional $\mathcal{F}(U)$ in* $\mathbf{U}$ *may be expressed as follows:*

$$\mathcal{F}(U) = \int_0^{\infty} U(t)\, K(t)\, dt \tag{4'}$$

*with*

$$K(+\infty) = \lim_{t\to+\infty} K(t) = 0 \tag{5'}$$

Proof. [19, IV. 108].

It may, then, be expressed for any $\{s_t\} \in \mathbf{c}$ as follows:

$$\mathcal{F}_1(s) = C \lim_{t\to\infty} s_t + \sum_{t=0}^{\infty} C_t\, s_t \tag{8}$$

being, after (2)

$$|C| + \sum_{t=0}^{\infty} |C_t| < \infty \tag{9}$$

Writing now,

$$k_t = \sum_{v=t}^{\infty} C_v < \infty \text{ for } t = 0, 1, 2, \ldots$$

we have evidently

$$\Delta k_t = -C_t \text{ for } t = 0, 1, 2, \ldots \tag{10}$$

$$\lim_{t\to\infty} k_t = 0 \tag{11}$$

Therefore, after (6), (8), (10), (7) and (11) we have

$$\mathcal{F}(u) = C\sum_{t=0}^{\infty} u_t - \sum_{t=0}^{\infty} (\Delta k_t)\, s_t =$$

$$= C\sum_{t=0}^{\infty} u_t - \Big[k_t\, s_t\Big]_0^{\infty} + \sum_{t=0}^{\infty} k_{t+1}\, u_t =$$

$$= C\sum_{t=0}^{\infty} u_t + \sum_{t=0}^{\infty} k_{t+1}\, u_t$$

that is, (4) with

$$K_t = C + k_{t+1} \text{ for } t = 0, 1, 2, \ldots$$

which really verifies (5), after (11) and (9).

2. *Complete and closed sequences in* **u** *and* **U**.

Def. 1. We say that a sequence $\{\phi_n\} \in \mathbf{c}$ of sequences

$\{\phi_{n,t}\} \in \mathbf{c}$ for $n = 0,1,2,\ldots$ and $t = 0,1,2,\ldots$

is *complete* in *c* when the only constant $C$ and the only sequence $\{C_t\} \in \mathbf{u}$ satisfying all the conditions

$$C \lim_{t\to\infty} \phi_{n,t} + \sum_{t=0}^{\infty} \phi_{n,t}\, C_t = 0 \text{ for } n = 0, 1, 2, \ldots$$

are the

$$C = 0 \text{ and } C_t = 0 \text{ for } t = 0, 1, 2 \ldots$$

Def. 1′. We say that a sequence $\{\phi_n\} \in \mathbf{C}\,[0, +\infty]$ of functions

$\phi_n(t) \in \mathbf{C}$ for $n = 0,1,2\ldots$ and $0 \leq t \leq +\infty$

is *complete* in **C** when the only function of limited variation $C(t)$ in $[0, +\infty]$

$$\int_0^{\infty} |d\, C(t)| < \infty$$

and with a finite limit

$$C(+\infty) = \lim_{t\to+\infty} C(t) < \infty$$

satisfying all the conditions

$$\int_0^\infty \phi_n(t)\, d\, C(t) = 0 \text{ for } n = 0, 1, 2\ldots$$

is the constant

$$C(t) = C(+\infty) \text{ in } 0 \leq t \leq +\infty$$

except for a countable subset at the most.

Def. 2. We will say that a sequence $\{\varphi_n\} \subset \mathbf{u}$ of sequences

$$\{\varphi_{n,\,t}\} \in \mathbf{u} \text{ for } n = 0, 1, 2\ldots, t = 0, 1, 2, \ldots \quad (1)$$

is *complete* in

**u**

when the only

sequence $\{K_t\} \in \mathbf{c}$

satisfying all the conditions

$$\sum_{t=0}^{\infty} \varphi_{n,\,t}\, K_t = 0 \quad (2)$$

for $n = 0, 1, 2,\ldots$ is the identically null one

$$K_t = 0 \text{ for } t = 0, 1, 2,\ldots \quad (3)$$

Def. 2′. We will say that a sequence $\{\varphi_n\} \subset \mathbf{U}$ of functions

$$\varphi_n(t) \in \mathbf{U} \text{ for } n = 0, 1, 2,\ldots \text{ and } 0 \leq t \leq +\infty \quad (1')$$

**U**

function $C(t)$ of limited variation in $[0, +\infty]$

$$\int_0^\infty |d\, C(t)| < \infty$$

with null limit

$$C(+\infty) = \lim_{t \to +\infty} C(t) = 0$$

$$\int_0^\infty \varphi_n(t) C(t)\, dt = 0 \quad (2')$$

is the null constant

$$C(t) = 0 \text{ in } 0 \leq t \leq +\infty \quad (3')$$

except for a countable subset at the most,

Prop. 1. *In order that a sequence*

$$\{\varphi_n\} \subset \mathbf{u} \qquad | \qquad \{\varphi_n\} \subset \mathbf{U}$$

*be complete in*

$$\mathbf{u} \qquad | \qquad \mathbf{U}$$

*it is necessary and sufficient that the*

*sums*

$$\phi_{n,\,t} = \sum_{\nu=0}^{t} \varphi_{n,\,\nu} \quad (4)$$

*primitives*

$$\phi_n(t) = \int_0^t \varphi_n(\nu)\, d\nu \quad (4')$$

*for* $n = 0, 1, 2,\ldots$ *and*

$$t = 0, 1, 2,\ldots \qquad | \qquad 0 \leq t \leq +\infty$$

*form a sequence*

| | |
|---|---|
| $\{\phi_n\} \subset \mathbf{c}$ | $\{\phi_n\} \subset \mathbf{C}$ |

*complete in*

| | |
|---|---|
| $\mathbf{c}$ | $\mathbf{C}$ |

Proof, Necessity. Let

| | |
|---|---|
| $\{\varphi_n\} \in \mathbf{u}$ | $\{\varphi_n\} \in \mathbf{U}$ |

be complete in

| | |
|---|---|
| $\mathbf{u}$ | $\mathbf{U}$ |

If

| | |
|---|---|
| $C \lim\limits_{t\to\infty} \phi_{n,t} + \sum\limits_{t=0}^{\infty} \phi_{n,t}\, \phi_t = 0$ for $n = 0, 1, 2, \ldots$ | $\int_0^{\infty} \phi_n(t)\, d\, C(t) = 0$ |
| $C$ being a constant and $\{C_t\} \in \mathbf{u}$ | $\int_0^{\infty} \lvert d\, C_1(t)\rvert < \infty$ and $C_1(+\infty) = 0$ |

it is also

| | |
|---|---|
| $C \lim\limits_{t\to\infty} \phi_{n,t} - \sum\limits_{t=0}^{\infty} \phi_{n,t}\, \Delta k_t = 0$ for $n = 0, 1, 2, \ldots$ with | $\int_0^{\infty} \phi_n(t)\, d\, C_1(t) = 0$ |
| $k_t \in \mathbf{c}$ | $\int_0^{\infty} \lvert d\, C_1(t)\rvert < \infty$ and $C_1(+\infty) = 0$ |

setting

| | |
|---|---|
| $k_t = \sum\limits_{\nu=t}^{\infty} C_\nu + C$ for $t = 0, 1, 2, \ldots$ | $C_1(t) = C(t) - C(+\infty)$ for $0 \le t \le +\infty$ |

But, then

| | |
|---|---|
| $C \lim\limits_{t\to\infty} \phi_{n,t} - \sum\limits_{t=0}^{\infty} \phi_{n,t}\, \Delta k_t = C \lim\limits_{t\to\infty} \phi_{n,t} - \left[\phi_{n,t} k_t\right]_{t=0}^{\infty} + \sum\limits_{t=0}^{\infty} k_{t+1}\, \Delta_t\, \phi_{n,t} = \phi_{n,0}\, k_0 + \sum\limits_{t=0}^{\infty} k_{t+1}\, \varphi_{n,t+1} = \phi_{n,0}\, k_0 + \sum\limits_{t=1}^{\infty} k_t\, \varphi_{n,t} = \sum\limits_{t=0}^{\infty} k_t\, \varphi_{n,t} = 0$ | $\int_0^{\infty} \phi_n(t)\, d\, C_1(t) = \Big[\phi_n(t)\, C_1(t)\Big]_0^{+\infty} - \int_0^{\infty} C_1(t)\, \varphi_n(t)\, dt = - \int_0^{\infty} C_1(t)\, \varphi_n(t)\, dt = 0$ |

for $n = 0, 1, 2, \ldots$; and as $\{\varphi_n\}$ is complete in

| **u** | **U** |
|---|---|
| it results $k_t = 0$ for $t = 0, 1, 2, \ldots$ whence $C = \lim_{t \to \infty} k_t = 0$, $C_t = -\Delta k_t = 0$ for $t = 0, 1, 2, \ldots$ | $C_1(t) = 0$ i. e., $C(t) = C(+\infty)$ in $[0, +\infty]$ except for a countable subset at most |

Being, besides after

| (Prop. 1, no. 1), $\{\phi_n\} \subset$ **c**, | (Prop. 1', no, 1) $\{\phi_n\} \subset$ **C**, |
|---|---|

this sequence $\{\phi_n\}$ is complete in

| **c** | **C** |
|---|---|

*Sufficiency*. Let

| $\{\phi_n\} \subset$ **c** | $\{\phi_n\} \subset$ **C** |
|---|---|

be complete in

| **c** | **C** |
|---|---|

If

| $$\sum_{t=0}^{\infty} \varphi_{n,t} K_t = 0$$ | $$\int_0^{\infty} \varphi_n(t)\, C(t)\, dt = 0$$ |
|---|---|

for $n = 0, 1, 2, \ldots$, being

| $\{K_t\} \in$ **c** and we set $C = \lim_{t \to \infty} K_t < \infty$, $C_t = -\Delta K_t$ for $t = 0, 1, 2, \ldots$ | $\int_0^{\infty} \lvert d\, C(t) \rvert < \infty$ and $C(+\infty) = 0$ |
|---|---|

we have

$$\sum_{t=0}^{\infty} \varphi_{n,t} K_t = \sum_{t=1}^{\infty} (\Delta_t\, \phi_{n,t-1})\, K_t + \varphi_{n,0} K_0 =$$

$$= \Big[ \phi_{n,t-1} K_t \Big]_1^{\infty} - \sum_{t=1}^{\infty} \phi_{n,t}\, \Delta K_t + \phi_{n,0} K_0 =$$

$$= C \lim_{t \to \infty} \phi_{n,t} + \sum_{t=1}^{\infty} \phi_{n,t}\, C_t + \phi_{n,0}\, C_0 =$$

$$= C \lim_{t \to \infty} \phi_{n,t} + \sum_{t=0}^{\infty} \phi_{n,t}\, C_t = 0$$

$$\int_0^{\infty} \varphi_n(t)\, C(t)\, dt = \int_0^{\infty} C(t)\, d\, \phi_n(t) =$$

$$= \Big[ \phi_n(t)\, C(t) \Big]_0^{+\infty} - \int_0^{\infty} \phi_n(t)\, d\, C(t) =$$

$$= -\int_0^{\infty} \phi_n(t)\, d\, C(t) = 0$$

for $n = 0, 1, 2, \ldots$; and as $\{\phi_n\}$ is complete in

| | |
|---|---|
| **c** | **C** |

it results

| | |
|---|---|
| $C = 0$ and $C_t = 0$ for $t = 0, 1, 2, \ldots$ | $K(t) = K(+\infty) = 0$ within $0 \leq t \leq +\infty$ |
| wherefrom | except, at most, in a countable subset |
| $K_0 = 0$ and $K_t = 0$ for $t = 1, 2, \ldots$ | |
| that is, | |
| $K_t = 0$ for $t = 0, 1, 2, \ldots$ | |

And, therefore, being

| | |
|---|---|
| $\{\varphi_n\} \subset$ **u** | $\{\varphi_n\} \subset$ **U** |
| after (Prop. 1, no. 1): $\{\varphi_n\} \subset$ **u** | after (Prop. 1′, no. 1): $\{\varphi_n\} \subset$ **U** |

this sequence $\{\varphi_n\}$ is complete in

| | |
|---|---|
| **u** | **U** |

And the proposition is proved.

Prop. 2. *Any complete sequence in*

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|---|---|
| **c** or **u** | **C** or **U** |

*is a total set in the space, i. e.* [1, 58] *if a continuous linear functional vanishes in every element of the sequence it vanishes also in the whole space*

| | |
|---|---|
| **c** or **u** | **C** or **U** |

Proof. Same as in [1, 73]: if the functional vanishes at every element of the sequence, it follows from definitions

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|---|---|
| 1 and 2 | 1′ and 2′ |

precisely adapted to the expression of the functional, in the form of

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| series (Theor. 1 and 2, no. 1) | integral (Theor. 1′ and 2′, no. 1) |

that the kernel of this expression is

| | |
|---|---|
| nul | constant |

for

| | |
|---|---|
| **c** and **u**, | **C** and **U**, |

except for a countable subset, at the most; therefore the

| | |
|---|---|
| series | integral |

is identically zero and, consequently, the functional in the space is zero too.

| | |
|---|---|
| Def. 3. We say that a sequence $\{\phi_n\}$ of sequences of | Def. 3′. We say that a sequence of functions of |
| **c** | **C** |

is closed in

| | |
|---|---|
| **c** | **C** |

when, there is, for any

| | |
|---|---|
| sequence $\{s_t\} \epsilon$ **c** | function $S(t) \epsilon$ **C** |

a sequence of linear combinations

| | |
|---|---|
| $\left\{\sum_{\nu=0}^{k_n} \alpha_{n,\nu} \; \phi_{\nu,t}\right\}$ | $\left\{\sum_{\nu=0}^{k_n} \alpha_{n,\nu} \; \phi_\nu(t)\right\}$ |

of $\{\phi_n\}$ tending to the

| | |
|---|---|
| sequence $\{s_t\}$ | function $S(t)$ |

for $n \to \infty$, uniformely with respect to the variable

| | |
|---|---|
| $t = 0, 1, 2, \ldots$ | $t \epsilon [0, +\infty]$ |
| Def. 4. We will say that a sequence of sequences of | Def. 4'. We will say that a sequence of functions of |
| **u** | **U** |

is *closed* in

| | |
|---|---|
| **u** | **U** |

when there is, for every

| | |
|---|---|
| sequence $\{u_t\} \epsilon$ **u** | function $U(t) \epsilon$ **U** |

a sequence of linear combinations

| | |
|---|---|
| $\left\{\sum_{\nu=0}^{k_n} a_{n,\nu} \; \varphi_{\nu,t}\right\}$ | $\left\{\sum_{\nu=0}^{k_n} a_{n,\nu} \, \varphi_\nu(t)\right\}$ |

of the $\{\varphi_n\}$, whose

| | |
|---|---|
| sums $\left\{\sum_{\mu=0}^{t} \; \sum_{\nu=0}^{k_n} a_{n,\nu} \, \varphi_{n,\mu}\right\}$ | primitives $\left\{\int_0^t \sum_{\nu=0}^{k_n} a_{n,\nu} \, \varphi_\nu(\mu) d\mu\right\}$ |

(which are evidently linear combinations

| | |
|---|---|
| $\left\{\sum_{\nu=0}^{k_n} \alpha_{n,\nu} \sum_{\mu=0}^{t} \varphi_{n,\mu}\right\}$ | $\left\{\sum_{\nu=0}^{k_n} \alpha_{n,\nu} \int_0^t \varphi_n(\mu) d\mu\right\}$ |

of the

| | |
|---|---|
| sums $\left\{\sum_{\mu=0}^{t} \varphi_{n,\mu}\right\}$ | primitives $\left\{\int_0^t \varphi_n(\mu) d\mu\right\}$ |

of the $\{\varphi_n\}$) tend to the

| | |
|---|---|
| sum $\sum_{\mu=0}^{\infty} u_\mu$ | primitive $\int_0^t U(\mu) \, d\mu$ |

for $n \to \infty$, uniformly with respect to

| | |
|---|---|
| $t = 0, 1, 2, \ldots$ | $t \in [0, +\infty]$ |

Prop. 3. *Closed sequences in*

| | |
|---|---|
| **c** and **u** | **C** and **U** |

*are fundamental subsets in the space, i. e.,* [1, 58], *the set of linear combinations of their elements in dense in the space.*

Proof. Same of [1, 73], as in Prop. 2. Actually, the difinitions

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|---|---|
| 3 and 4 | 3′ and 4′ |

of closed sequences, mean that for every element $P$ of the space, there is a linear combination of sequence elements whose distance from $P$ is less than $\varepsilon$ with the metric of the space.

Prop. 4. *In the spaces*

| | |
|---|---|
| **c** and **u** | **C** and **U** |

*any closed sequence is complete, and conversely.*

Proof. It follows immediately, after Props. 2 and 3, by virtue of theorem [1, Theor. 7, 58].

Cor. *For* $\{\varphi_n\}$ *to be closed in*

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|---|---|
| **u** | **U** |

*it occurs and suffices that the sequence* $\phi_n$ *of*

| | |
|---|---|
| *sums* (4) | *primitives* (4′) |

*be closed in*

| | |
|---|---|
| **c** | **C** |

roof. It is an obvious consequence from Prop. 1, by virtue of Prop. 4.

In order to simplify further propositions (Prop. 2, Theor. 1, and Cor. no. 6) it is convenient to generalize the concept of complete sequences in **u** for sequences not contained in **u**. Although Prop. 2 does not apply any more and be lost therefore the equivalence with the concept of closed sequences in **u**. Such equivalence is not necessary for our main purpose.

Def. 5. We will say that a sequence $\{\varphi_n\}$ of arbitrary sequences $\{\varphi_{n,t}\}$, although not belonging to **u**, now even to **c**, is,

| | | |
|---|---|---|
| *complete* | *weakly complete* | *very weakly complete* |

with respect to **u**, when the only sequence

| | | |
|---|---|---|
| $\{K_t\} \in$ **c** | $\{K_t\} \in$ **u** | $\{K_t\} \in \mathbf{u}_a$ |

satisfying all conditions (2) is the indentically nul one (3).

Prop. 5. *Any sequence*

| | |
|---|---|
| *weakly complete* | *very weakly complete* |

*whith respect to* **u** *is surely*

| | |
|---|---|
| *complete* | *weakly complete* |

*with respect to* **u**.

Proof. Being

| | |
|---|---|
| $\mathbf{u} \subset \mathbf{c}$ (Defs. 1 and 2, no, 1), | $\mathbf{u}_a \subset \mathbf{u}$ (Defs. 2 and 3, no. 1), |

if there is in

| | |
|---|---|
| **c** | **u** |

no sequence different from (3) verifying all conditions (1), there cannot either be it in

| | |
|---|---|
| **u** | $\mathbf{u}_a$ |

Examples: We give here those which will be used afterwards

1. *The sequence*

$$\{2^{-t}\, t^{(n+p)}\} \subset \mathbf{u} \quad (*)$$

*is complete in u (Def. 2)*

1′. *If* $\{r_t\}$ *is an arbitrary increasing sequence, with limit* $+\infty$, *the sequence*

$$\left\{e^{-r_t}\, r_t^{n+p}\right\} \subset \mathbf{c}$$

*is complete with respect to u (Def. 5).*

*whatever the integre n may be independent of t and u.*

1″. *The sequence*

$$\{e^{-t}\, t^{n+p}\} \subset \mathbf{U}$$

*is complete in U (Def. 2′)*

Proof. $C(t)$ is of limited variation within $[0, +\infty]$:

$$\int_0^\infty |d\, C(t)| < \infty$$

with a zero limit:

$$C(+\infty) = \lim C(t) = 0,$$

Proof. *If* $\{K_t\} \in \mathbf{c}$,

| | | |
|---|---|---|
| the powers series $f(z) = \sum_{t=p}^{\infty} K_t\, t^{(p}\, (z+1)^{t-p}$ | the Dirichlet series $f(z) = \sum_{t=0}^{\infty} K_t\, r_t^p\, e^{-r_t z}$ | the Laplace-Stieltjes transform: $f(z) = \int_0^\infty e^{-tz}\, t^p\, d\, C(t)$ |

has the

convergency circle $|z+1| < 1$ | convergency half-plan $\Re_e\, z > 0$:

where

| | | |
|---|---|---|
| $f^{(n)}(z) = \sum_{t=n+p}^{\infty} K_t\, t^{(n+p}\, (z+1)^{t-p-n}$ | $f^{(n)}(z) = (-1)^n \sum_{t=0}^{\infty} K_t\, r_t^{n+p}\, e^{-r_t z}$ | $f^{(n)}(z) = (-1)^n \int_0^\infty e^{-tz}\, t^{n+p}\, d\, C(t)$ |

for $n = 1, 2, \ldots$; and, specially, we have

| | | |
|---|---|---|
| $f^{(n)}(-\tfrac{1}{2}) = 2^{n+p} \sum_{t=n+p}^{\infty} K_t\, 2^{-t}\, t^{(n+p}$ | $f^{(n)}(1) = \sum_{t=0}^{\infty} K_t\, e^{-r_t}\, r_t^{n+p}$ | $f^{(n)}(1) = \int_0^\infty e^{-t}\, t^{n+p}\, dC(t)$ |

for $n = 1, 2, \ldots$ Therefore, if it is

| | | |
|---|---|---|
| $\sum_{t=0}^{\infty} K_t\, 2^{-t}\, t^{(n+p} = \sum_{t=n+p}^{\infty} K_t\, 2^{-t}\, t^{(n+p} = 0$ | $\sum_{t=0}^{\infty} K_t\, e^{-r_t}\, r_t^{n+p} = 0$ | $\int_0^\infty e^{-t}\, t^{n+p}\, d\, C(t) = 0$ |

(*) Let us recall the definitions: $t^{(n} = t(t-1)\ldots(t-n+1)$ for $t = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$; $t^{(0} = 1$ for $t = 1, 2, \ldots$

| | |
|---|---|
| for $n = 0, 1, 2, \ldots$ it will surely be | |
| $f(-\frac{1}{2}) = 0$ and $f^{(n)}(-\frac{1}{2}) = 0$ | $f(1) = 0$ and $f^{(n)}(1) = 0$ |
| for $n = 1, 2, \ldots$, i. e., | |
| $f(z) = 0$ and, therefore, $K_t = 0$ for $t = 0, 1, 2, \ldots$ | $C(t) = C(+\infty)$ for $0 \leq t \leq +\infty$ up to a countable subset, at the most. |

2. *If* $\{\varphi_n\}$ *is complete* ***with*** *respect to*

| | |
|---|---|
| $\mathbf{u}$ and $\{k_t\} \in \mathbf{c}$ | $\mathbf{U}$ *and* $\int_0^\infty \lvert d\ k(t) \rvert < \infty,\ k(+\infty) = 0$ |

*being*

| | |
|---|---|
| $k_t \neq 0$ *for every* $t = 0, 1, 2, \ldots$ *the sequence* | $k(t) \neq 0$ *for* $t \in [0, +\infty]$ *up to a countable subset, at the most,* |

$\{k\varphi_n\}$ *is also complete in*

| | |
|---|---|
| $\mathbf{u}$ | $\mathbf{U}$ |

Proof. Let us set

| | |
|---|---|
| $\sum_{t=0}^{\infty} k_t\, \varphi_{n,t}\ K_t = 0$ | $\int_0^\infty k(t)\ \varphi_n(t)\ C(t)\ dt = 0$ |

for $n = 0, 1, 2, \ldots$ being

| | |
|---|---|
| $\{K_t\} \in \mathbf{c}$ | $\int_0^\infty \lvert d\ C(t) \rvert < \infty,\ C(+\infty) = 0$ |

Since

| | |
|---|---|
| $\{K_t\ K_t\} \in \mathbf{c}$ | $\int_0^\infty \lvert d[k(t)C(t)] \rvert \leq \max_{0 \leq t \leq +\infty} \lvert k(t) \rvert \cdot \int_0^\infty \lvert d\ C(t) \rvert + {}$ $+ \max_{0 \leq t \leq +\infty} \lvert C(t) \rvert \int_0^\infty \lvert dk(t) \rvert \leq$ $\leq 2 \int_0^\infty \lvert dk(t) \rvert \cdot \int_0^\infty \lvert dC(t) \rvert < \infty,$ $\lim_{t \to +\infty} [k(t)\ C(t)] = k(+\infty)\ C(+\infty) = 0$ |

it follows

| | |
|---|---|
| $k_t K_t = 0$ and, therefore $K_t = 0$ for $t = 0, 1, 2, \ldots$ | $k(t)C(t) = 0$ and, therefore, $C(t) = 0$ for $t \in [0, +\infty]$, up to a countable subset at the most. |

3. *Characterization of the powers series and Laplace's transformation*

The characterization of Laplace's transformation as a linear functional through the convolution and derivation laws, stablished for various spaces in some other publications

[14], [3], [16], [17], [18], [19], [4], will be exposed here in the form which will be used for the space **U**, supplemented with the correlative characterization of powers series in the **u**-space.

Theor. 1. *In order that a (continuous) linear functional*

$\mathcal{F}_z(u)$ | $\mathcal{F}_z(U)$

*defined in*

**u** | **U**

*and depending on a complex parameter z, with*

$|z| < 1$, | $R_e z > 0$,

*be a*

*power series*

$$P_z(u) = \sum_{t=0}^{\infty} u_t\, z^t, \tag{1}$$

| *Laplace's transformation*

$$\mathcal{L}(U) = \int_0^{\infty} U(t)\, e^{-tz}\, dt, \tag{1'}$$

*it occurs and suffices that the linear function satisfy the*

*incrementation law*

$$z\mathcal{F}_z(\Delta u_t) = (1 - z)\mathcal{F}_z(u) - u_0 \tag{2}$$

| *derivation law*

$$\mathcal{F}_z(U') = z\, \mathcal{F}_z\,(U) - U(0) \tag{2'}$$

*for every*

*sequence* $\{u_t\} \epsilon$ **u**

*or at least for every term of a complete sequence* $\{\varphi_n\}$ *in* **u** *of sequences* $\{\varphi_{n,\,t}\} \epsilon$ **u** *with zero limits*

$$\varphi_{n,\,\infty} = \lim_{t\to\infty} \varphi_{n,\,t} = 0 \text{ for } n = 0, 1, 2, \ldots \tag{3}$$

| *function* $U(t)\ \epsilon$ **U**

*or at least for every term of a sequence* $\{e^{-at}\phi_n(t)\}$, *where* $a > 0$ *or constant*

$$\phi_n(t) = \int_0^t \varphi_n(v)dv \text{ for } n = 0, 1, 2, \ldots \tag{3'}$$

*and* $\{\varphi_n\}$ *is complete sequence in* **U**.

Proof. The condition is necessary,

since

$$P_z(u) = \frac{1}{z-1}\sum_{t=0}^{\infty} u_t\,\Delta\, z^t = \frac{1}{z-1}\left\{\left[u_t z^t\right]_0^{\infty} - \sum_{t=0}^{\infty}(\Delta u_t) z^{t+1}\right\} = \frac{1}{1-z}\left[u_0 + z\sum_{t=0}^{\infty}(\Delta u_t) z^t\right] =$$

$$= \frac{1}{1-z}\left[u_0 + z\, P_z(\Delta\, u_t)\right]$$

| as we know [2,99].

Conversely, after

(Theor. 2, no. 1) | (Theor. 2′, no. 1)

we have

$$\mathcal{F}_z(u) = \sum_{t=0}^{\infty} u_t\, K_t(z) \tag{5}$$

| $$\mathcal{F}_z(U) = \int_0^{\infty} U(t)\, K(t,z)dt \tag{5'}$$

for every

sequence $\{u_t\} \in \mathbf{u}$ | function $U(t) \in \mathbf{U}$

being

$K(t, z)$, for every $z \in \{R_e z > 0\}$, a function of limited variation in $[0, +\infty]$:

$$\int_0^{+\infty} |d\, K(t, z)| < \infty \tag{6'}$$

(7)

with

$$K_\infty(z) = \lim_{t \to \infty} K_t(z) < \infty \text{ for every } z \in \{|z| < 1\}$$

$$K(+\infty, z) = \lim_{t \to +\infty} K(t, z) = 0 \tag{7'}$$

Specially, since

$\{\varphi_n\} \in \mathbf{u}$ | $\{\varphi_n\} \subset \mathbf{U}$

and also therefore

$\{\Delta_t\ \varphi_{n,\,t}\} \subset \mathbf{u}$,

for $n = 0, 1, 2, \ldots$

$e^{-at}\,\phi_n(t) \in \mathbf{U}$ and $\dfrac{d}{dt}[e^{-at}\,\phi_n(t)] = [e^{-at}\,\varphi_n(t) - - ae^{-at}\,\phi_n(t)] \in \mathbf{U}$

for $n = 0, 1, 2, \ldots$ and $a > 0$,

we have

$$\mathcal{F}_z(\varphi_n) = \sum_{t=0}^{\infty} \varphi_{n,\,t}\, K_t(z) \tag{8}$$

$$\mathcal{F}_z(\Delta_t\, \varphi_n) = \sum_{t=0}^{\infty} (\Delta_t\, \varphi_{n,\,t}) K_t(z) \tag{9}$$

$$\mathcal{F}_z[e^{-at}\,\phi_n(t)] = \int_0^\infty e^{-at}\,\phi_n(t)\ K(t, z)\, dt \tag{8'}$$

$$\mathcal{F}_z\left(\frac{d}{dt}[e^{-at}\,\phi_n(t)]\right) = \tag{9'}$$

$$= \int_0^\infty e^{-at}\,\varphi_n(t) K(t, z)\, dt - a\int_0^\infty e^{-at}\,\phi_n(t) K(t, z)\, dt$$

But, according to

(3) and (7) | (3′) and (7′)

it is:

$$\sum_{t=0}^{\infty} (\Delta_t\, \varphi_{n,\,t}) K_t(z) = \Big[\varphi_{n,\,t}\, K_t(z)\Big]_0^\infty - $$

$$- \sum_{t=0}^{\infty} \varphi_{n,\,t+1}\, \Delta K_t(z) = -\varphi_{n,\,0}\, K_0(z) -$$

$$- \sum_{t=1}^{\infty} \varphi_{n,\,t} \Delta_t\, K_{t-1}(z) \tag{10}$$

$$\int_0^\infty e^{-at}\,\phi_n(t)\, K(t, z)\, dt =$$

$$= -\int_0^\infty \phi_n(t)\, d\int_t^\infty e^{-av}\, K(v, z)\, dv =$$

$$= \left[-\phi_n(t)\int_t^\infty e^{-av}\, K(v, z)\, dv\right]_0^\infty +$$

$$+ \int_0^\infty \varphi_n(t)\left[\int_t^\infty e^{-av}\, K(v, z)\, dv\right] dt =$$

$$= \int_0^\infty \varphi_n(t)\left[\int_t^\infty e^{-av}\, K(v, z)\, dv\right] dt \tag{10'}$$

By applying now:

| (2) | (2′) |
|---|---|
| to every term of the sequence | |
| $\{\varphi_n\} \subset \mathbf{u}$ | $\{e^{-at}\phi_n, (t)\} \subset \mathbf{U}$ |
| we deduce, then, by means of | |
| (8), (9) and (10), | (8′), (9′) and (10′), |

the equation

$$- z\,\varphi_{n,0}\,K_0(z) - z\sum_{t=1}^{\infty}\varphi_{n,t}\,\Delta_t\,K_{t-1}(z) = (1-z)\sum_{t=0}^{\infty}\varphi_{n,t}\,K_t(z) - \varphi_{n,0} \qquad (11)$$

$$\int_0^{\infty}\varphi_n(t)\left[e^{-at}K(t,z) - a\int_t^{\infty}e^{-av}K(v,z)dv\right]dt = z\int_0^{\infty}\varphi_n(t)\left[\int_t^{\infty}e^{-av}K(v,z)\,dv\right]dt \qquad (11')$$

for $n = 0, 1, 2, \ldots$

But being $\{\varphi_n\}$ complete in

| $\mathbf{u}$ | $\mathbf{U}$ |
|---|---|

and also

after (7) and (Prop. 1, no. 1):

$$(1-z)K_t(z) + z\Delta_t K_{t-1}(z)\} \in \mathbf{c}$$

for every fixed $z$ with $|z| < 1$

for every fixed $z$ with $R_e z > 0$ the function of $t$

$$e^{-at}K(t,z) - (a+z)\int_t^{\infty}e^{-av}K(v,z)\,dt$$

of limited variation in $[0, +\infty]$:

$$\int_0^{\infty}|d[e^{-at}K(t,z) - (a+z)\int_t^{\infty}e^{-av}K(v,z)\,dv]| \leq$$

$$\leq \int_0^{\infty}e^{-at}\,|dK(t,z)| + a\int_0^{\infty}e^{-at}\,|K(t,z)|\,dt +$$

$$+ |a+z|\int_0^{\infty}e^{-at}\,|K(t,z)|\,dt \leq \int_0^{\infty}|dK(t,z)| +$$

$$+ \max_{0\leq t\leq+\infty}|K(t,z)| + \frac{|a+z|}{a}\max_{0\leq t\leq+\infty}|K(t,z)| \leq$$

$$\leq \int_0^{\infty}|dK(t,z)| + \int_0^{\infty}|dK(t,z)| +$$

$$+ (1 + |\frac{z}{a}|)\int_0^{\infty}|dK(t,z)| =$$

$$= (3 + |\frac{z}{a}|)\int_0^{\infty}|dK(t,z)| < \infty$$

it follows:

$$(z-1)K_0(z) = z\ K_0(z) - 1$$

$(z-1)\ K_t(z) = z\Delta_t\ K_{t-1}(z)$ for $t = 1, 2, \ldots$
i. e.:

$$K_0(z) = 1$$

$$K_t(z) = zK_{t-1}(z) \text{ for } t = 1, 2, \ldots$$

or:

$$K_t(z) = z^t \text{ for } t = 0, 1, 2, \ldots$$

$$e^{-at}K(t,z) - (a+z)\int_t^{\infty} e^{-av}K(v,z)\,dv = 0$$

in $0 \leq t \leq +\infty$
up to a countable subset; therefrom

$$K'_t(t,z) + zK(t,z) = 0$$

or:

$$K(t,z) = C(z)e^{-at}$$

in almost everywhere in $0 \leq t \leq T < +\infty$ for every $R_e\ z > 0$, being $C(z)$ a function not depending on $t$.

By replacing now in

(5) | (5′)

we obtain

$$\mathcal{F}_z(u) = \sum_{t=0}^{\infty} u_t\, z^t \text{ for } |z| < 1 \text{ and } \{u_t\} \in \mathbf{u} \qquad (12)$$

q. e. d.

$$\mathcal{F}_z(U) = C(z)\int_0^{\infty} U(t)e^{-tz}\,dt \text{ for } R_e z > 0 \qquad (12')$$

and $U(t) \in \mathbf{U}$

Specially, for

$$U(t) \equiv e^{-t} \in \mathbf{U}$$

it follows from (2′) and the linearity of $\mathcal{F}_z(U)$:

$$-\mathcal{F}_z(e^{-t}) = z\,\mathcal{F}(e^{-t}) - 1 \text{ for } R_e z > 0$$

i. e.

$$\mathcal{F}_z(e^{-t}) = \frac{1}{z+1} \text{ for } R_e z > 0$$

while from (12′) it follows obviously

$$\mathcal{F}_z(e^{-t}) = \frac{C(z)}{z+1} \text{ for } R_e\ z > 0$$

then, it is

$$C(z) \equiv 1 \text{ for } R_e z > 0$$

or, after (12′)

$$\mathcal{F}_z(U) = \int_0^{\infty} U(t)\,e^{-tz}\,dt \text{ for } R_e z > 0 \text{ and}$$

$$U(t) \in \mathbf{U}$$

q. e. d.

Theor. 2. *All conditions of Theor.* 1 *hold if the functional is defined in the*

| Sequences | Functions |
|---|---|
| *constant sequence* | *constant function* |
| $\xi_t = 1 \; for \; t = 0, 1, 2, \ldots \qquad (13)$ | $\xi(t) = 1 \; for \; 0 \leq t \leq +\infty \qquad (13')$ |
| *(although* $\{\xi_t\} \notin \mathbf{u}$*)* | $\xi(t) \notin \mathbf{U}$*)* |
| *with the value* $\mathcal{F}_z(\xi_t) = \dfrac{1}{1-z} \; for \; \lvert z \rvert < 1; \qquad (14)$ | $\mathcal{F}_z(\xi) = \dfrac{1}{z} \; for \; R_e z > 0; \qquad (14')$ |
| *in the unitary sequence* $\xi_0^{(0)} = 1 \; and \; \xi_t^{(0)} = 0 \; for \; t = 0, 1, 2, \ldots \qquad (15)$ *with the value* $\mathcal{F}_z(\xi_t^{(0)}) = 1 \qquad (16)$ | |
| *verifying, besides, the convolution law* $\mathcal{F}_z(u_t * v_t) = \mathcal{F}_z(u_t) . \mathcal{F}_z(v_t) \qquad (17)$ | $\mathcal{F}_z(U * V) = \mathcal{F}_z(U) . \mathcal{F}_z(V) \qquad (17')$ |
| *for every pair* $\{u_t\} \in \mathbf{U}(\mathbf{u}, \{\xi_t\}, \{\xi_t - \xi_t^{(0)}\})$ and $\{v_t\} \in \mathbf{U}(\mathbf{u}, \{\xi_t\}, \{\xi_t - \xi_t^{(0)}\})$ | $U(t) \in [\mathbf{U} \cup \xi(t)]$ and $V(t) \in [\mathbf{U} \cup \xi(t)]$ |
| *such that also* $(u_t * v_t) \in \mathbf{U}(\mathbf{u}, \{\xi_t\}, \{\xi_t - \xi_t^{(0)}\})$ | $(U * V) \in [\mathbf{U} \cup \xi(t)]$ |
| *or, at least, for the products* $\xi_t * \xi'_t = \xi_t - \xi_t^{(0)} \; for \; t = 0, 1, 2, \ldots \qquad (18)$ $\xi_t * \Delta_t \varphi_{n,t} = \varphi_{n,t+1} \; for \; t = 0, 1, 2, \ldots \qquad (19)$ $\xi'_t * \varphi_{n,t+1} = \varphi_{n,t} \; for \; t = 0, 1, 2, \ldots \qquad (20)$ | $e^{-at} * [e^{-at} \varphi_n(t)] = e^{-at} \phi_n(t) \qquad (19')$ $for \; 0 \leq t \leq +\infty$ |
| *and* $n = 0, 1, 2, \ldots$*, being* $\{\varphi_n\}$ *a complete sequence in* $\mathbf{u}$ *of sequences* $\{\varphi_{n,t}\}$ *with* $\varphi_{n,0} = 0 \; for \; n = 0, 1, 2, \ldots \qquad (21)$ *and, on the other hand* $\{\xi'_t\} \in \mathbf{u}$ *the sequence with* $\xi'_0 = 0, \; \xi'_1 = 1 \; and \; \xi'_t = 0 \; for \; t = 2, 3, \ldots \qquad (22)$ | $\mathbf{U}$*,* $a > 0$ *a constant and* $\phi_n(t) = \int_0^t \varphi_n(v) dv \; for \; n = 0, 1, 2, \ldots \qquad (21')$ $and \; 0 \leq t \leq +\infty$ |

Proof. The conditions are necessary as we know [2, 104].

Conversely, after

(21), | (21′)

we obtain actually

(19) | (19′)

since

| | |
|---|---|
| $\xi_t * \Delta_t \varphi_{n,t} = \sum_{v=0}^{t} \Delta_t \varphi_{n,t} = \varphi_{n,t+1}$ for $t = 0, 1, 2, \ldots$ | $e^{-at} * [e^{-at} \varphi_n(t)] = \int_0^t e^{-a(t-v)} e^{-av} \varphi_n(v) \, dv = e^{-at} \phi_n(t)$ for $0 \leq t \leq +\infty$ |

and $n = 0, 1, 2, \ldots$; hence, after the convolution law

(17) for (19) | (17′) for (19′)

and the initial condition

(14), | (14′)

it results:

$$\frac{1}{1-z} \mathcal{F}_z(\Delta_t \varphi_{n,t}) = \mathcal{F}_z(\varphi_{n,t+1}) \qquad (23)$$

$$\frac{1}{z} \mathcal{F}_z[e^{-at} \varphi_n(t)] = \mathcal{F}_z[e^{-at} \phi_n(t)] \qquad (23')$$

for $n = 0, 1, 2, \ldots$

which is not yet the incrementation law for the sequence

$$\{\varphi_n\} \in \mathbf{u}$$

But, after (13) and (21) it is

$$\xi_1 * \xi'_t = \xi_0 \xi'_0 = 0 \text{ for } t = 0$$

$$\xi_t * \xi'_t = \sum_{\nu=0}^{t} \xi'_t \xi_{t-\nu} = \xi'_1 \xi_{t-1} = 1 \text{ for}$$

$$t = 1, 2, \ldots$$

or, after (13) and (15):

$$\xi_t * \xi'_t = \xi_t - \xi_t^{(0)} \text{ for } t = 0, 1, 2, \ldots$$

as expressed in (18); and apply in (17), we have, because of the functional linearity:

$$\mathcal{F}_z(\xi_t) \cdot \mathcal{F}_z(\xi'_t) = \mathcal{F}_z(\xi_t) - \mathcal{F}_z(\xi_t^{(0)})$$

and solving with respect to $F_z(\xi'_t)$, and, by virtue of the initial conditions (14) and (16), it results

$$\mathcal{F}_z(\xi'_t) = z \qquad (24)$$

On the other and, after (22) we prove (20), since

$$\xi'_t * \varphi_{n,t+1} = \sum_{\nu=0}^{t} \xi'_\nu \varphi_{n,t+1-\nu} = \xi'_1 \varphi_{n,t} = \varphi_{n,t}$$

for $t$ and $n = 0, 1, 2, \ldots$ whereby, applying (17), through (24), we deduce:

$$z\mathcal{F}_z(\varphi_{n,t+1}) = \mathcal{F}_z(\varphi_{n,t}) \text{ for } n = 0, 1, 2, \ldots \qquad (25)$$

which substituted in (23) yields

$$z\mathcal{F}_z(\Delta_t \varphi_{n,t}) = (1-z)\mathcal{F}_z(\varphi_{n,t})$$

which is the incrementation law (2) for the sequence $\{\varphi_n\}$, by virtue of (21).

which is the derivations law for the sequence

$$\{e^{-at} \phi_n(t)\} \in \mathbf{U}$$

and it suffices to apply Theor. 1.

Theor. 3. *Conclusion of Theors.* 1 *and* 2 *is verified if we take*

$\varphi_{n,t} = 2^{-t}\, t^n$ *for* $t = 0, 1, 2, \ldots$ | $\varphi_n(t) = e^{-t}\, t^n$ *for* $0 \leq t \leq +\infty$

*and* $n = 0, 1, 2 \ldots,$

Proof. Because this sequence $\{\varphi_n\}$ is complete in

**u** (ex. 1, no. 2) | **U** (ex. 1′, no. 2)

and evidently fulfils condition (21).

NOTES. 1st. Lack of correlation in hypotheses of theorems in both colums is due to the fact that these hypotheses have been chosen in order to facilitate application of Theor. 3 to demonstration of (Theor. 1, no. 5, § 2).

2nd. In theorem 1 we could have adopted, for the left column, an assumption correlative to the right one, but it turns more difficult the proof of Theor. 1, no. 5, § 2). On the other hand, the proof for right starting from the derivation law, would not be useful for the sequence $\{\varphi_n\}$, complete in **U**, since after partial integration it would appear the derivative $K'_t\,(t, z)$, which certainly exists for almost the whole finite interval, but may not be of limited variation.

3rd. Equation (24) could be adopted directly as initial condition in stead of (16) and the convolution law for the products (18), (19) and (20); furthermore it could be deduced out of (14), which correlative is (14′) through the products convolution law:

$$\xi_t * \varphi_{n,\,t+1} = \sum_{\nu=0}^{t+1} \varphi_{n,\,\nu} \qquad \text{and} \qquad \xi_t * \varphi_{n,\,t} = \sum_{\nu=0}^{t} \varphi_{n,\,\nu},$$

whose difference is evidently $\varphi_{n,\,t+1}$; and (25) would be easily obtained. But we should have to assume that these products belong to **u** $\cup$ $\{\xi_t\}$ and Theor. 3, would not hold any more.

4th. If we admit the infinite additivity of the functional for sequences $\{\xi_t^{(n)}\}$ with

$$\xi_n{}^{(n)} = 1 \quad \text{and} \quad \xi_t{}^{(n)} = 0 \quad \text{for} \quad t \neq n,$$

the sum of which is evidently

$$\sum_{n=0}^{\infty} \xi_t^{(n)} = \xi_t \quad \text{for} \quad t = 0, 1, 2, \ldots \tag{26}$$

i. e., the functional is continuous in $\{\xi_t\} \notin$ **u** initial condition (14) results as a consequence of (16), (24) and (26), for being, evidently

$$\xi_t^{(n)} * \xi'_t = \xi_t^{(n+1)} \quad \text{for} \quad t = 0, 1, 2, \ldots \quad \text{and} \quad n = 0, 1, 2, \ldots, \tag{27}$$

we have:

$$\mathcal{F}_z(\xi_t^{(n)}) = z^n$$

and it follows

$$\mathcal{F}_z(\xi_t) = \sum_{n=0}^{\infty} \mathcal{F}_z\,(\xi_t^{(n)}) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for} \quad |z| < 1$$

5th. Finally, from infinite additivity for (26) and from the convolution laws for (27) we deduce in general:

$$\mathcal{F}_z(\xi_t) = \mathcal{F}_z(\xi_t^{(0)}) + \frac{\mathcal{F}_z(\xi'_t)}{1 - \mathcal{F}_z(\xi'_t)}$$

which gives also condition (14) for other values of $\mathcal{F}_z(\xi_t^{(0)})$ and $\mathcal{F}_z(\xi'_t)$; for instance

$$\mathcal{F}_z(\xi_t^{(0)}) = z^2 \quad \text{and} \quad \mathcal{F}_z(\xi'_t) = \frac{1 - z^2 + z^3}{2 - z - z^2 + z^3}$$

Therefore, conditions (16) and (24) are independent from (14).

But, we repeat, none of these conclusions are necessary in the present work and the proof of these various assumptions in demonstrations of (Theor. 1, no. 5, $\xi$ 2) results uneasier than those of foregoing Theor. 1,2 and 3.

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