NOTE ON THE CONVERGENCE OF THE SERIES PROVIDING DERIVATIVES OF A TABULATED FUNCTION

By

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Danti mihi sapientiam dabo gloriam; non recedet memoria eius, et nomen eius requiretur a generatione in generationem.
Eccli. LI,23 XXXIX, 13

Let us consider a function $y(x)$ whose values at $x = x_0 + nh$, n being any natural number, are given. In other words $y(x)$ is tabulated only for x increasing in a half-straight line. Suppose that from these given values of $y(x)$ we construct a difference table; we shall employ following common symbolism;

$$y(x+h) - y(x) = \Delta y(x);\ y(x+h) = y(x) + \Delta y(x) = [1+\Delta]y(x)$$

$$\Delta^{n-1} y(x+h) - \Delta^{n-1} y(x) = \Delta^n y(x);\ y(x+nh) = [1+\Delta]^n y(x)$$

It is easily shown that the derivative D is given by the symbolical formula

$$D = \frac{1}{h} l(1+\Delta) \tag{1}$$

We have also the higher derivatives from

$$D^r = \frac{1}{h^r}[l(1+\Delta)]^r \tag{2}$$

both second members being expanded and applied to $y(x)$.

In this paper we will suppose that the given tabulated function $y(x)$ makes the series (1′) convergent

$$y'(x_0) = Dy(x_0) = \frac{1}{h}\left[\Delta y(x_0) - \frac{\Delta^2 y(x_0)}{2} + \frac{\Delta^3 y(x_0)}{3} \dots\right] \tag{1'}$$

The well-know series

$$y(x+h) = y(x) + hy'(x) + \frac{1}{2!} h^2 y''(x) + \ldots$$

may be symbolically written

$$\Delta y(x) = hDy(x) + \frac{h^2}{2!} D^2 y(x) + \ldots$$

an equation that becomes

$$\Delta y(x) = (e^{hD} - 1)\, y(x)$$

Equalizing only the operators we have

$$e^{hD} = (1 + \Delta)$$

Hence we have equation (1) and in a like manner, raising to power r, equation (2), which will be more useful for us to write as follows

$$D^r y(x_0) = \frac{1}{h^r} [a_{rr} \Delta^r y(x_0) + a_{r\,r+1} \Delta^{r+1} y(x_0) + \ldots + a_{r\,r+k} \Delta^{r+k} y(x_0) + \ldots] \quad (2')$$

where the coefficients $a_{r\,r+k}$ are found by following formulae:

$$\begin{aligned} a_{r\,r+k} &= a_{r-1\,r+k-1} - \frac{1}{2} a_{r-1\,r+k-2} + \frac{1}{3} a_{r-1\,r+k-3} - \ldots \\ &+ (-1)^{k-1} \frac{1}{k} a_{r-1\,r} + (-1)^k \frac{1}{k+1} a_{r-1\,r-1} \end{aligned} \quad (3)$$

It is easy to show that

$$a_{11} = a_{22} = \ldots = a_{r-1\,r-1} = a_{rr} = 1$$

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Let us construct $F(\alpha)$ according to the so-called Newton-Gregory formula, which is also given by E. Stieffel in a slighty different form.

We will write it as follows

$$F(\alpha) = y(x_0) + \frac{(\alpha - x_0)^{(1)}}{h} \Delta y(x_0) + \frac{(\alpha - x_0)^{(2)}}{2!\, h^2} \Delta^2 y(x_0) + \ldots + \frac{(\alpha - x_0)^{(n)}}{n!\, h^n} \Delta^n y(x_0) + \ldots \quad (4)$$

where $(\alpha - x_0)^{(n)}$ is the so-called factorial function of order n, whose value is given by

$$(\alpha - x_0)^{(n)} = (\alpha - x_0)(\alpha - x_0 - h)(\alpha - x_0 - 2h) \ldots [\alpha - x_0 - (n-1)h]$$

We will suppose $\alpha > x_0$; of course this assumption will not impair the generality of the later obtained results.

Generally speaking all the equally spaced formulae for approximative integration are deduced from (4).

We will show that the series (4) converges for every case in which the numerical series (1′) is also convergent.

It will be enough to prove that for given x and ε there can be found an N such that for every $n > N$

$$\left| \sum_{i=n}^{i=n+p} \frac{(\alpha-x_0)^{(i)}}{i!} \Delta^i y(x_0) \right| < \varepsilon \qquad \text{will follow.}$$

(1′) being convergent such a N_1 can be found that for $n > N_1$

$$\left| \sum_{i=n}^{i=n+p} (-1)^{i+1} \frac{\Delta^i y(x_0)}{i} \right| < \varepsilon \frac{(\varrho-1)!}{(\alpha - x_0)^{(\varrho)}}$$

where ϱ is the highest natural number contained in $(\alpha - x_0)$

Let us take N as the highest number of $\varrho + 1$ and N_1

In the series (1′) let us sum the consecutive terms having the same sign writing the development (1′) as follows.

$$\pm Dy(x_0) = P_1 - P_2 + P_3 - P_4 + \dots \tag{1''}$$

(1″) being convergent, P_m must approach zero as a limit and as $n > N \geq N_1$ if P_m contains the nth term,

$$\left| \sum_{j=m+1}^{j=p} P_j \right| < \frac{\varepsilon(\varrho-1)!}{(\alpha - x)^{(\varrho)}} \qquad \text{will follow}$$

In a like manner let us sum in the development (4) the consecutive terms having the same sign obtaining

$$\pm F(\alpha) = Q_1 - Q_2 + Q_3 - Q_4 + \dots$$

but as $N > \varrho + 1$ one term of this development, Q_{l+1} for instance, will contain the terms of the series (4) corresponding to the terms of the series (1′) contained in P_{m+1}; let us suppose that both P_{m+1} and Q_{l+1} contain, in the respective series, the terms from the first *μth* to the last *νth*.

In the assummed hypotheses we may write

$$\frac{(\alpha-x_0)^{(\varrho)}}{(\varrho-1)!} > \frac{(\alpha-x_0)^{(\mu)}}{(\mu-1)!} > \left| \frac{Q_{l+1}}{P_{m+1}} \right| > \frac{(\alpha-x_0)^{(\nu)}}{(\nu-1)!}$$

From this follows

$$\left|\sum_{j=l+1}^{j=q+l-m} Q_j\right| < \frac{(\alpha-x_0)^{(\varrho)}}{(\varrho-1)!}\left|\sum_{j=m+1}^{j=q} P_j\right| < \varepsilon. \quad \text{q.e.d.}$$

* * *

Shown the convergence of the series (4) for every α, we may say that $F(\alpha)$ is an entire function.

From (4) we could consider only the n first terms and write them as a polynomial $F_n(\alpha) = A_0^n + A_1^n\,\alpha + A_2^n\,\alpha^2 + \ldots A_n^n\,\alpha^n$; $F_n(\alpha) - F(\alpha)$ will approach zero when n increases as previously shown. Every A_i^n shall converge to a certain value, which we will call A_i. $F(\alpha)$ also being convergent for any real when written as a power series, the well known Cauchy theorem assures that $\lim_{n\to\infty} \sqrt[n]{A_n} = 0$ and then $F(\alpha)$ will be convergent when α is not real but a complex variable.

The derivatives of $F(\alpha)$ are also entire functions convergent for every α.

Taking the rth derivative of the equation (4) the reader will find $\frac{d^r F(\alpha)}{d^r x}$ and equalizing α to x the resulting series will be easily identified with the development (2′); therefore this development is also convergent.

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Now we will directly show that the derivative of the sth difference

$$D\Delta^s y(x_0) = \frac{1}{h}\left(\Delta^{s+1}\,y(x_0) - \frac{\Delta^{s+2}\,y(x_0)}{2} + \frac{\Delta^{s+3}\,y(x_0)}{3}\ldots\right) \tag{5}$$

is also convergent for $s > 1$, assumming convergence for $s = 1$ (1′)

It will be enough to prove that given ε_s it is possible to find such a N_s (ε_s) that for every $\nu > N_s$

$$\left|\left[\frac{\Delta^{s+\nu}\,y(x_0)}{\nu} - \frac{\Delta^{s+\nu-1}\,y(x_0)}{\nu+1} + \ldots \pm \frac{\Delta^{s+\nu+p-1}\,y(x_0)}{\nu+p-1}\right]\right| < \varepsilon_s \tag{6}$$

irrespective of the chosen p.

(1′) being convergent we may take such a N_0 that for every $s + \nu > N_0$ should follow

$$\left|\left(\frac{\Delta^{s+\nu}\, y(x_0)}{s+\nu} - \frac{\Delta^{s+\nu-1} y(x_0)}{s+\nu-1} + \ldots \pm \frac{\Delta^{s+\nu+p-1}\, y(x_0)}{s+\nu+p-1}\right)\right| < \frac{\varepsilon_s}{s}$$

$$\text{or.} \left|\left(\frac{s}{s+\nu}\Delta^{s+\nu}y(x_0) - \frac{s}{s+\nu+1}\Delta^{s+\nu-1}y(x_0) + \ldots + \frac{s}{s+\nu+\varrho-1}\Delta^{s+\nu+\varrho-1}\, y(x_0)\right)\right| < \varepsilon_s \qquad (7)$$

Let us take as $N_s = N_0 - s$

If we compare term by term the sums (6) and (7) we will see that in both sums the corresponding terms have the same sign and that the ones from the sum (7) are bigger than the ones from the sum (6) and reasoning as above, eq. (6) inmediately follows.

Using the convergence of development (5) as we previsiously used the convergence of development (1′) we could prove the convergence of

$$D^r\Delta^s\, y(x_0) = \frac{1}{h^r}\left(a_{rr}\,\Delta^{s+r}\, y(x_0) + a_{rr+1}\,\Delta^{s+r+1}\, y(x_0) + \ldots\right) \qquad (8)$$

* * *

We have proved the convergence in the complex field of developments (4) and (8) assumming the convergence of (1′) when x and h are real. It is only too easy to show that we may also understand the series (1′) x and h being complex; then we can change the origin and the axis taking x_0 as the new origin and the straight line where h lies as the new real axis. The reasoning follows as above.

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Now, let us suppose that we admit the convergence of development (2′) for a precise r, it not being necessary that r be equal to 1.

Then, we can take $D^{r-1}\, y\,(x)$ as a new funcktion, let us say, $Y\,(x)$ in which differences at x_0 are

$$\Delta^s\, Y\,(x_0) = \frac{1}{h^{r-1}}\left(a_{rr-1}\,\Delta^{s+r-1}\, y\,(x_0) + a_{r-1\,r}\,\Delta^{s+r}\, y\,(x_0) + \ldots\right)$$

With these hypotheses we have proved the convergence of the constructed function according to development (4); This constructed function is an entire one, and therefore integrated r times gives also an entire function, let us say, $Y_r(\alpha)$; to this function an arbitrary polynomial of $r-1$ grade can be added and then it will equalize $F(\alpha)$ if the coefficients of this polynomial are chosen in order to have it vanish at x_0, $x_0+h, \ldots, x_0+(r-1)h$

Therefore we can say that $F(\alpha)$ is convergent only knowing the convergence of one development (2') ; and the convergence of one development (8) assures the convergence of the others.

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From the convergence of the series (1') we can infer the possibility of constructing an entire function $F(x)$ according to development (4).

Now, we can put ourselves the question that in a certain way is the reciprocical of the former one. Given any entire function $F(x)$ and the difference interval h, will it be found convergent the series (1') ?

Not always. For instance if we are dealing with the function $F(x) = e^{kx}$ by a very elementary operation we can write

$$\Delta F(x) = e^{k(x+h)} - e^{kx} = e^{kx}(e^{kh}-1) \qquad \text{and}$$
$$\Delta^n F(x) = e^{kx}(e^{kh}-1)^n$$

In this application the series (1') will converge if and only if

$$e^{kh} - 1 \leq 1$$
$$\text{or} \qquad k \leq \frac{l2}{h}$$

It is not hard to see that if $F(z)$ is of order less than 1 every difference interval h will be good enough and in such a case the series (1') will be always convergent.

But if $F(z)$ is of order larger than 1 no h will be small enough and then (1') will never be convergent.

In such a case we could divide the given tabulated function of order $m > 1$ by an entire function of order no lesser than m; we could then work on the resulting function as stated above and once the development (4) is obtained we can multiply it by the choosen functional dividend.

The author acknowledges his indebtedness to Professors Dr. Juan Augé and Dr. Enrique Linés (both at the University of Barcelona) for their valuable bibliographical suggestions.

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