REAL INVERSION THEOREMS FOR A GENERALIZED LAPLACE TRANSFORM

BY

J. M. C. JOSHI (Nainital - India)

1 — INTRODUCTION

Erdelyi [1] has used fractional integration to produce a kernel which has been made use of to define a transformation,

$$(1.1)\quad G(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^\infty (xy)^\beta {}_1F_1\left(\begin{matrix}\beta+\eta+1\\ \alpha+\beta+\eta+1\end{matrix}; -xy\right) f(y)\, dy$$

which may be called a generalization of Laplace integral,

$$(1.2)\qquad g(x) = \int_0^\infty e^{-xy} f(y)\, dy$$

For $\alpha = \beta = 0$, (1.1) reduces to (1.2).

2 — DEFINITION OF OPERATORS

Kober [2] defined the operators of fractional integration as follows

$$(2.1)\quad I^{+}_{\eta,\alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt.$$

$$(2.2)\quad K^{-}_{\zeta,\alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^{\zeta} \int_x^\infty (t-x)^{\alpha-1} t^{-\zeta-\alpha} f(t)\, dt.$$

$$(2.3)\quad I^{-}_{\eta,\alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_x^\infty (t-x)^{\alpha-1} t^{\eta} f(t)\, dt.$$

$$(2.4) \quad K_{\zeta,\alpha}^{+} f(x) = \frac{1}{\Gamma(\alpha)} x^{\zeta} \int_0^x (x-t)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt.$$

where $f(t) \in L_p(O, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 < p < \infty$ and $\frac{1}{p'}$ or $\frac{1}{p} = 0$ if p or $p' = 1$, $\alpha > 0$, $\eta > -\frac{1}{p'}$, $\zeta > -\frac{1}{p}$

If $\alpha < 0$, $g(x) = I_{\eta,\alpha}^{+} f(x)$ and $h(x) = K_{\zeta,\alpha}^{-} f(x)$ are defined as the solution, if any, of the integral equations,

$$(2.5) \quad f(x) = I_{\eta+\alpha,-\alpha}^{+} g(x), \; f(x) = K_{\zeta+\alpha,-\alpha}^{-} \mathrm{h}(x).$$

The Mellin Transform of a function $f(x)$ of $L_p(0, \infty)$ will be defined as:

$$\varnothing(t) = \overline{M} f = \int_0^{\infty} f(x)\, x^{it}\, dx \text{ when } p = 1$$

$$\varnothing(t) = \overline{M} f = \underset{X \to \infty}{\text{l.i.m.}} \int_{X}^{\frac{1}{X}} f(x)\, x^{it - \frac{1}{p'}}\, dx \text{ when } p > 1.$$

where l.i.m. indicates the limit in mean with index p' over the interval $(-\infty, \infty)$. The inverse Mellin Transform of a function $\varnothing(t)$ of $L_{p'}(-\infty, \infty)$ is correspondingly defined as:

$$(2.6) \quad f(x) = \overline{M}^{-1} \Phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(t)\, x^{-it}\, dt \text{ when } p' = 1$$

$$(2.7) \quad \mathrm{f}(x) = \overline{M}^{-1} \Phi = \frac{1}{2\pi} \underset{T \to \infty}{\text{l.i.m.}} \int_{-T}^{T} \Phi(t)\, x^{-it - \frac{1}{b}}\, dt \text{ when } p' > 1$$

where l.i.m. indicates the limit in mean with index p over the interval $(0, \infty)$.

Whe have; Kober [2, Theorem 5 (a)]

$$\overline{M} \left\{ I_{\eta,\alpha}^{+} f \right\} = \Gamma(\eta + p'^{-1} - it)\, \Gamma^{-1}(\eta + \alpha + p'^{-1} - it)\, \overline{M} f$$

$$\overline{M} \left\{ K_{\zeta,\alpha}^{+} f \right\} = \Gamma(\zeta + it + p^{-1})\, \Gamma^{-1}(\eta + \alpha + p^{-1} + it)\, \overline{M} f$$

But $\overline{M}\,(x^{\beta}\,e^{-x}) = \int_0^{\infty} e^{-x}\,x^{\beta+it-\frac{1}{p'}}\,d\,x = \Gamma\,(\beta + p^{-1} + it)$

provided $\beta + p^{-1} > 0$

$$(2.8)\ \begin{cases} \text{Therefore, } \overline{M}\left\{ I_{\eta,\alpha}\,(x^{\beta}\,e^{-x}) \right\} = \dfrac{\Gamma\,(\eta + p^{-1} - i\,t)\;\Gamma\,(\beta + p^{-1} + i\,t)}{\Gamma\,[\alpha + (\eta + p'^{-1} - i\,t)\,]} \\ \text{and} \\ \overline{M}\left\{ K_{\bar{\zeta},\alpha}\,(x^{\beta}\,e^{-x}) \right\} = \dfrac{\Gamma\,(\zeta + p^{-1} + i\,t)\;\Gamma\,(\beta + p^{-1} + i\,t)}{\Gamma\,\;[\alpha + (\,\zeta\, + p^{-1} - i\,t)\,]} \end{cases}$$

By (2.7) and (2.8) we have

$$I^{+}_{\eta,\alpha}\,(x^{\beta}\,e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma(\eta + p'^{\,-1} - it)\;\Gamma\,(\beta + p^{-1} + i\,t)}{\Gamma\,[\alpha + (\eta + p'^{\,-1}\; - i\,t)}\,x^{-it-p^{-1}}\,dt$$

$$\text{and } K^{+}_{\zeta,\alpha}\,(x^{\beta}\,e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma(\zeta + p^{-1} + i\,t)\;\Gamma(\beta + p^{-1} + i\,t)}{\Gamma\,[\alpha + (\zeta + p^{-1} - i\,t)\,]}\,x^{-it-p^{-1}}\,dt$$

provided $\beta + p^{-1} > 0$, $\eta + p'^{\,-1} > 0$ and $\zeta + p^{-1} > 0$.
Erdelyi has also shown that, by term by term integration,

$$(2.9)\qquad I_{\eta,\alpha}\,(x^{\beta}\,e^{-x}) = x^{\beta}\sum_{0}^{\infty}\frac{\Gamma\,(\beta + \eta + r + 1)}{[\alpha + (\beta + \eta + r + 1)\,]}\;\frac{(-\,x)^{r}}{r\,!}$$

so that

$$\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\frac{\Gamma\,(\eta + p'^{\,-1} - i\,t)\;\Gamma\,(\beta + p^{-1} + i\,t)}{\Gamma\,[\alpha + (\eta + p'^{\,-1} - i\,t\,)\,]}\,x^{-it-p^{-1}}\,dt$$
$$= x^{\beta}\sum_{0}^{\infty}\frac{\Gamma\,(\beta + \eta + r + 1\,)}{\Gamma\,[\alpha + (\beta + \eta + r + 1]}\;\frac{(-\,x)^{r}}{r\,!}$$

where $\beta + \eta + r\; + 1 \neq 0, -\,1, -\,2, \ldots$ for $r = 0, 1, 2, \ldots, \alpha > 0$, and $(\eta + \beta) > -\,1$

The object of the present paper is to give two inversion formulae and a representation theorem for the transform given by (1.1). In the first inversion formula I have used properties of Kober's operators while in the second we have changed our transform into generalized Stieltzes transform and have applied a differential operator which inverts the Stieltzes transform to get the real inversión formula. Incidentally we have been able to get a generalized form of Stieltzes Integral analogous to that studied by Arya [8].

3 — A Theorem

Theorem. Let $f(y) \epsilon L_p (0, \infty)$, $1 \leqslant p < \infty$, $x > 0$. If $\eta > -\frac{1}{p}$ $\beta > -\frac{1}{p'}$ when $\alpha > 0$ and $\eta + \alpha > -\frac{1}{p}$ $\beta > -\frac{1}{p'}$, when $\alpha < 0$ then $I^{+}_{\eta,\alpha} [h(x)]$ exists and is equal to $G(x)$ where

$$(3.1) \qquad h(x) = \int_0^\infty (xy)^{\beta} e^{-xy} f(y) \, \mathrm{d}y$$

and $G(x)$ is defined by (1.1)

Proof when $\alpha > 0, 1 < p < \infty$

If $f(y) \epsilon L_p (0, \infty)$ and $\beta > -\mathrm{p}'^{-1}$, we see that $h(x)$ exists and

$$I^{+}_{\eta,\alpha}[g'(x)] = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} h(t) \, \mathrm{dt}.$$

$$= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} \, dt \int_0^\infty (ty)^{\beta} e^{-yt} f(y) \, \mathrm{dy}$$

$$= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x y^{\beta} f(y) \, dy \int_0^x (x-t)^{\alpha-1} t^{\eta+\beta} e^{-yt} \, dt$$

by changing the order of integration which is justified under the conditions stated ; for by Holder's inequality

$$\int_0^\infty y^{\beta} e^{-yt} f(y) \, dy \leq \left\{ \int_0^\infty | f(y) |^{p} \right\}^{p-1} \left\{ \int_0^\infty e^{-p' yt} y^{\beta p'} \, dy \right\}^{p'-1}$$

$$= c\, t^{-\beta - p'-1} \qquad , \beta p' > -1$$

where c is a constant. Thus repeated integral is majorized by

$c \int_0^x (x-t)^{\alpha-1} t^{\eta-p'-1}$ which is convergent if $\eta - p'^{-1} + 1 > 0$ i.e.if $\eta < -p^{-1}$

$$\text{Now} \int_0^x (x-t)^{\alpha-1} t^{\eta-\beta} e^{-yt} \, dt =$$

$$= \int_0^1 x^{\alpha+\eta+\beta} (1-t)^{\alpha-1} t^{\eta+\beta} e^{-xyt} dt$$

$$= x^{\eta+\alpha+\beta} \sum_0^\infty \frac{\Gamma(\eta+\beta+r+1)\,\Gamma(\alpha)}{\Gamma(\alpha+\eta+\beta+r+1)} \frac{(-xy)^r}{r!}$$

$$= x^{\eta+\alpha+\beta} \frac{\Gamma(\alpha)\,\Gamma(\eta+\beta+1)}{\Gamma(\alpha+\eta+\beta+1)} {}_1F_1(\eta+\beta+1; \eta+\alpha+\beta+1, -xy)$$

Therefore $I^{+}_{\eta,\alpha} [h(x)] = \dfrac{\Gamma(\eta+\beta+1)}{\Gamma(\alpha+\eta+\beta+1)} \displaystyle\int_0^\infty (xy)^\beta \, {}_1F_1(\eta+\beta+1,$

$$\eta+\alpha+\beta+1; -xy) \times f(y)\, dy$$

If $p = 1$, it is similarly seen that the change in order of integration is justified if $\eta > -1$, $\alpha > 0$, $\beta > 0$

Proof when $\alpha < 0$, $1 < p < \infty$

If $\alpha < 0$ then by dy (2.5) $I^{+}_{\eta,\alpha}$ $[h(x)]$ is the solution, if any, of the

integral equation $g'(x) = I^{+}_{\eta+\alpha,-\alpha}$ $[G(x)]$

Now $I_{\eta+\alpha,-\alpha}$ $[G(x)]$

$$= \frac{x^{-\eta}}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} t^{\eta+\alpha} G(t)\, dt$$

$$= \frac{x^{-\eta}}{\Gamma(-\alpha)} \int_0^x \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} (x-t)^{-\alpha-1} t^{\eta+\alpha} dt \int_0^\infty (ty)$$

$$\beta_1 F_1 \left(\begin{matrix} \beta+\eta+1 \\ \alpha+\beta+\eta+1 \end{matrix} ; -ty) f(y)\, dy \right)$$

$$= \frac{x^{-\eta}}{\Gamma(-\alpha)} \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^\infty y^\beta f(y)\, dy \int_0^x t^{\eta+\alpha+\beta} (x-t)^{-\alpha-1}$$

$$ {}_1F_1 \left(\begin{matrix} \beta+\eta+1 \\ \alpha+\beta+\eta+1 \end{matrix} - yt \right) dt$$

Changing the order of integration which is justified if $\beta > -p'^{-1}$
$\eta + \alpha > -p^{-1}$

But $\displaystyle\int_0^x t^{\eta+\alpha+\beta} (x-t)^{-\alpha-1} {}_1F_1 \left(\begin{matrix} \beta+\eta+1 \\ \alpha+\beta+\eta+1 \end{matrix} - ty \right) dt$

$$= \int_0^1 t^{\eta+\beta+\alpha} (1-t)^{-\alpha-1} x^{\eta+\beta} \sum_0^\infty \frac{\Gamma(\eta+\beta+r+1)}{\Gamma(\eta+\beta+1)} \frac{\Gamma(\alpha+\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+r+1)} \frac{(-t\,y)^r}{r\,!}\, d\,t$$

$$= x^{\eta+\beta} \frac{\Gamma(\alpha+\beta+\eta+1)\ \Gamma(-\alpha)}{\Gamma(\beta+\eta+1)} e^{-xy}$$

Therefore $I_{\eta+\alpha,-\alpha}\, G(x) = h(x)$

If $p = 1$ change is justified if $\eta + \alpha > -1,\ \beta > 0$

4 — Inversion formula

We now define a differential operator as follows :

Definition (4.2) An operator $Q_{n,t}\,[G(x)]$ is defined for any positive number t and any positive integer n by

$$W_n\,[G(x)] = (-)^n x^n \left(\frac{d}{dx}\right)^n [x^{-\beta} G(x)] \tag{4.1}$$

$$Q_{n,t}\,[G(x)] = \frac{t^{-\beta-1}}{\Gamma(n)}\ [W_n \{ G(x) \}\ ]\, x = \frac{n}{t}$$

Theorem let $f(y) \in L_p(o,\ \infty)$

for every positive R. If integral given by (1.1) converges for $x > 0$ and $\eta > -p^{-1}$ when $\alpha > 0$ and $\eta + \alpha > -p^{-1}$ if $\alpha < 0$

then for almost all positive t $\underset{n\to\infty}{\text{l.i.m.}}\ Q_{n,t}\ [I_{\eta+\alpha,-\alpha} \{ G(x) \}\ ] = f(t)$

We have seen under conditions of the theorem,

$$I_{\eta+\alpha,-\alpha}\,[G(x)] = \int_0^\infty (x\,y)^\beta e^{-xy} f(y)\, d\,y$$

Therefore $W_n\,[I_{\eta+\alpha,-\alpha} G(x)] = (-)^n x^n \left(\frac{d}{dx}\right)^n \left[\int_0^\infty y^\beta e^{-xy} f(y)\, dy\right]$

$$= x^n \int_0^\infty y^{\beta+\eta} e^{-xy} f(y)\, d\,y$$

$$\text{Therefore } Q_{n,t}\,[G(x)] = \frac{t^{-\beta-1}}{\Gamma(\eta)} \left(\frac{n}{t}\right)^n \int_0^\infty y^{\beta+\eta} e^{-\frac{n}{t}y} f(y)\, dy$$

$$= \frac{t^{-\beta}}{n!} \left(\frac{n}{t}\right)^{n+1} \int_0^\infty y^n e^{-\frac{n}{t}y} y^\beta f(y)\, dy$$

proceeding as in Saksena [3, Theorem 3, page 603] our result is established.

5 — Another operator

We now give another integro-differential operator which will invert (1.1).

Definition (5.1). An operator is defined for any real positive number s by the equations,

$$(5.1)\quad L_{n,s}\,[\mathrm{H}\ (s)] = \frac{\Gamma(2n+\alpha+\beta+\eta)\,\Gamma(2n-\alpha)}{\Gamma(\beta+\eta+2n)\,\Gamma(2n)\,\Gamma(\mathrm{n}+1)\,\Gamma(n-\alpha-1)}(-)^{n-1}\times$$
$$\times D^n\, s^{2n-1}\, D^{n-1}\, s^{\beta+\eta+n-1}\, D^{n-1}\, s^{\alpha}\, D^{-n+1}\, s^{-(d+\beta+\eta+n-1)}\, H(s)\ (n=2,3,\cdots)$$

$$L_{1,S}\ [H\ (s)] = DsH\ (s).$$
$$L_{0,S}\ [H\ (s)] = H\ (s).$$

where $D = (d\,|\,ds)$, and $D^{-1}\,s^a = \int_0^s s^a\, ds$ if $Re\ (a+1) > 0$ and

$$= -\int_s^{\infty} s^a\, ds \ if\ Re\ (a+1) < 0$$

It is assumed that $H\ (s)$ has derivatives and integrals of all orders.

(5.2). *Theorem.* If $F\ (x)$ is defined by (1.1) then

$$(5.2)\qquad L_{n,s}\ [H\ (s)] \sim f\ (s) \qquad (n \to \infty)$$

where $$H\ (s) = \int_0^{\infty} e^{-ys}\, F\ (y)\, dy.$$

provided that $f\ (s) \in L$ in $0 \leq t \leq R$ for every positive R and is such that integral (1.1) converges.

Proof. We have,

$$H\ (s) = \int_0^{\infty} e^{-ys}\, F\ (y)\, d\,y.$$

$$= \int_0^{\infty} f\ (x)\, dx \int_0^{\infty} \frac{\Gamma\ (\beta+\eta+1)}{\Gamma\ (\alpha+\beta+\eta+1)}\, e^{-ys}\, {}_1F_1\left(\begin{matrix}\beta+\eta+1\\ \alpha+\beta+\eta+1\end{matrix};\ -x\,y\right) d\,y$$

$$= \frac{\Gamma\ (\beta+\eta+1)}{\Gamma\ (\alpha+\beta+\eta+1);}\cdot\frac{1}{s}\int_0^{\infty} F\left(\begin{matrix}\beta+\eta+1,1,\\ \alpha+\beta+\eta+1\end{matrix};\ -\frac{x}{s}\right) f\ (x)\, d\,x \quad (5.3)$$

since Erdelyi [6 Vol I page 219 (17)],

$$\int_0^\infty e^{-pt}\, t^{\sigma-1}\, {}_1F_1\,(\alpha_1,\, P_1,\, \lambda\, t) = \Gamma\,(\sigma)\, p^{-\sigma}{}_2\, F_1\left(\frac{\alpha_1}{p_1}\,;\, \sigma_1\,;\, \frac{\lambda}{p}\right)$$ provided that $Re\ \sigma > 0,\ Re\ p > Re\ \lambda.$

Therefore as in Arya [4, Theorem 7.2.1 page 100],

$$L_{n,s}\ \ H\,(s) \sim f\,(s) \qquad\qquad (n \to \infty)$$

Remark : — It may be noticed that (5.3) is a Generalized form of Stieltzes Integral analogous to that introduced by Varma [7].

6 — Representation theorem

We now give a representation theorem for the transform given by (1.1).

Theorem (6.1). The necessary and sufficient conditions for a function $G\,(x)$ to have representation (1.1) with $f\,(y) \in L_p\,(0,\ \infty)$ $p \geq 1$, $x > 1$, and with $\eta > -\, p^{-1}$ when $\alpha > 0$ and $\eta + \alpha > -\, p^{-1}$ if $\alpha < 0$ are (*i*) $I_{\eta+a,-a}\ [G\,(x)] = \varnothing\,(x)$ exists, has derivatives of all ordes in $0 < x < \infty$ and vanishes at infinity and
(*ii*) there exist constants M and p $(p \geq 1)$ such that

$$\int_0^\infty |\, Q_{n,\,t}\, G\,(x)\ |\ ^p dt < M \qquad\qquad (\eta = 1,\ 2,\, \ldots\ldots)$$

Proof. The proof follows the lines of Widder [5, Theorem 15 *a*, pp. 313 - 14].

I am indebted to Dr. K. M. Saksena for guidance and help in the preparation of this paper.

REFERENCES

1. ERDEYLI, A. *On some functional transformations*, Rend. Del. Semin. Mat. 10 (1950-51) 217-234.

2. KOBER, H. *On fractional integrals and derivatives*, Quart. Jour Math. 11, (1940) 193-211.

3. SAKSENA, K. M. *Inversions and Representation Theorem for a generalized Laplace Integral*, Pacific Journal of Mathematics Vol. 8 No. 3 (1958) 597-607.

4. ARYA. S. C. *on a Generalized Stieltzes Transform*, Ph. D. Thesis (Agra University India) 1959.

5. WIDDER, D. V. *The Laplace Transform*, 1941.

6. ERDEYLI, A. etc. *Tables of Integral Transform*, Vol. I.

7. VARMA, R. S. *on a Generalization of Laplace Integral*, Proc. Nat. Acad. Sc. Vol. 20, pp. 209-16. (1951).

8. ARYA, S. C. *Convergence Theorems etc. of a Generalized Stieltzes Trans form* Jour. Ind. Math. Soc. Vol. XXII (117-135). (1958).

Dr. S. M. Joshi
Ranikhet

D. S. B. Government College,
Naini Tal.
INDIA