AN EXPANSION THEOREM FOR NONANALYTIC FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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SUMMARY. For complex functions in several complex variables which are not necessarily analytic, but with components u and v of class C^{m+1}, a TAYLOR formula is derived. If u and v are of class C^∞ and those functions satisfy certain additional conditions, a TAYLOR series expansion is valid. The analytic case, as well as the special case of one independent variable, are briefly discussed.

1. NOTATION.

Our notation is standard. See, for instance, [3]. Let $z_j = x_j + iy_j$, $dz_j = dx_j + idy_j$, $j = 1,2, \dots, n$, and let

$$f(z_1, \dots, z_n) = u(x_1, y_1, \dots, y_n) + iv(x_1, y_1, \dots, y_n)$$

be a complex function of the n complex variables $z_1, \dots, z_n$ with differentiable components u, v in a polydisc $D_r(z) = D_{r_1}(z_1) \times \dots \times D_{r_n}(z_n)$, where $z = (z_1, \dots, z_n)$, $r = (r_1, \dots, r_n)$.

Let

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

be the complx differential operators, and define

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j .$$

2. *A Taylor formula.*

THEOREM 1. If u and v are of class C^{m+1} in $D_r(z)$ then we have

(1) $$f(z_1+dz_1, \ldots, z_n+dz_n) = f(z_1, \ldots, z_n) + \sum_{k=1}^{m} \frac{1}{k!} (\partial f + \bar{\partial} f)^{(k)} + R_m$$

where the symbolic power has the usual interpretation, R_m stands for the remainder term, and $z + dz \in D_r(z)$.

Proof. From Taylor's formula for real functions of several real variables we have

(2) $$u(x_1+dx_1, \ldots, y_n+dy_n) = u(x_1, \ldots, y_n) + \sum_{k=1}^{m} \frac{1}{k!} d^k u + R_{1,m}$$

(3) $$v(x_1+dx_1, \ldots, y_n+dy_n) = v(x_1, \ldots, y_n) + \sum_{k=1}^{m} \frac{1}{k!} d^k v + R_{2,m}$$

where

$$R_{1,m} = \frac{1}{(m+1)!} d^{m+1} u \Big]_{\zeta} \qquad R_{2,m} = \frac{1}{(m+1)!} d^{m+1} v \Big]_{\zeta'}$$

and $\zeta = z + \theta dz, \quad \zeta' = z + \theta' dz, \quad 0 < \theta < 1, \quad 0 < \theta' < 1.$

From (2) and (3) it follows that

(4) $$f(z_1+dz_1, \ldots, z_n+dz_n) = f(z_1, \ldots, z_n) + \sum_{k=1}^{m} \frac{1}{k!} (d^k u + i d^k v) + R_m$$

where $R_m = R_{1,m} + i R_{2,m}$.

But

$$du + i dv = \sum_{j=1}^{n} (u_{x_j} dx_j + u_{y_j} dy_j) + i \sum_{j=1}^{n} (v_{x_j} dx_j + v_{y_j} dy_j)$$
$$= \partial f + \bar{\partial} f$$

Similarly,

$$d^2 u + i d^2 v = (\partial f + \bar{\partial} f)^{(2)}$$

and by mathematical induction it can be easily shown that

$$d^k u + i d^k v = (\partial f + \bar{\partial} f)^{(k)}$$

Hence, (4) may be written in the form (1). This is Taylor formula for complex functions which are of class C^{m+1} in a neigborhood of z.

3. *A Taylor series expansion.*

THEOREM 2. If u and v are of class C^∞ and $R_m \to 0$ as $m \to \infty$ then

$$f(z_1 + dz_1, \ldots, z_n + dz_n) = f(z_1, \ldots, z_n) + \sum_{k=1}^{\infty} \frac{1}{k!} (\partial f + \bar{\partial} f)^{(k)} \tag{5}$$

Proof. It follows at once from (1).

We shall show that the condition $R_m \to 0$ holds in the following two cases:

a) The partial derivatives of all orders of u as well as those of v are in absolute value uniformly bounded by $M > 0$ in $D_r(z)$. For in this case we have

$$\begin{aligned} |R_{1,m}| &\leq \frac{M}{(m+1)!} \left(\sum_{j=1}^{n} |dx_j| + \sum_{j=1}^{n} |dy_j| \right)^{m+1} \\ &\leq \frac{(2n)^{m+1} M}{(m+1)!} \, ||dz||^{m+1} \end{aligned}$$

where $||dz|| = \max_j |dz_j|$.

Similarly,

$$|R_{2,m}| \leq \frac{(2n)^{m+1} M}{(m+1)!} \, ||dz||^{m+1}$$

so that

$$|R_m| \leq \frac{2\,(2n)^{m+1} M}{(m+1)!} \, ||dz||^{m+1} \tag{6}$$

and the right-hand side of (6) tends to zero as $m \to \infty$.

b) The functions u and v together with their partial derivatives of all orders are nonnegative in $D_r(z)$.

Then, by a theorem of J. T. DAY [2], we have $R_{1,m} \to 0$ and $R_{2,m} \to 0$ as $m \to \infty$ in some $D_{r'}(z) \subset D_r(z)$. Hence $R_m \to 0$ also in some neighborhood of z.

4. *Two special cases.*

a) If f is holomorphic in $D_r(z)$ then u and v are of class C^∞, and $\bar{\partial} f = 0$ in $D_r(z)$. Also, we have

$$f_{z_1} = u_{x_1} + i v_{x_1} = v_{y_1} - i u_{y_1}$$
$$f_{z_1 z_2} = u_{x_1 x_2} + i v_{x_1 x_2} = v_{y_1 x_2} - i u_{y_1 x_2} = - u_{y_1 y_2} - i v_{y_1 y_2}$$

etc. It follows that

$$\left| \frac{\partial^{m+1} u}{\partial x_1^{h_1} \dots \partial x_n^{h_n} \partial y_1^{k_1} \dots \partial y_n^{k_n}} \right| \leq \left| \frac{\partial^{m+1} f}{\partial z_1^{h_1+k_1} \dots \partial z_n^{h_n+k_n}} \right|$$

where the h_j and k_j are nonnegative integers satisfying $\sum_{j=1}^{n} (h_j + k_j) = = m + 1$, and similarly for the partial derivatives of v.

If $D_\varrho(\zeta) \subset D_r(z)$ is a polydisc with center at ζ and radius $\varrho = (\varrho, \dots, \varrho)$, and if $|f(z^*)| \leq M$ for all $z^* \in \bar{D}_\varrho(z)$, it is known [1] that

$$\left| \left(\frac{\partial^{m+1} f}{\partial z_1^{h_1+k_1} \dots \partial z_n^{h_n+k_n}} \right)_\zeta \right| \leq \frac{(h_1 + k_1)! \dots (h_n + k_n)! \, M}{\varrho^{m+1}}$$
$$\leq \frac{(m+1)! \, M}{\varrho^{m+1}}$$

Thus,

$$\left| \left(\frac{\partial^{m+1} u}{\partial x_1^{h_1} \dots \partial x_n^{h_n} \partial y_1^{k_1} \dots \partial y_n^{k_n}} \right)_\zeta \right| \leq \frac{(m+1)! \, M}{\varrho^{m+1}} \tag{7}$$

Then we have

$$R_{1,m} = \frac{1}{(m+1)!} d^{m+1} u \Big]_\zeta = \frac{1}{(m+1)!} \left(\sum_{j=1}^{n} u_{x_j} \, dx_j + \sum_{j=1}^{n} u_{y_j} \, dy_j \right)^{m+1} \Bigg]_\zeta$$
$$= \frac{1}{(m+1)!} \sum \frac{(m+1)!}{h_1! \dots h_n! \, k_1! \dots k_n!} \left(\frac{\partial^{m+1} u}{\partial x_1^{h_1} \dots \partial x_n^{h_n} \partial y_1^{k_1} \dots \partial y_n^{k_n}} \right)_\zeta \cdot$$
$$\cdot \, dx_1^{h_1} \dots dx_n^{h_n} \, dy_1^{k_1} \dots dy_n^{k_n}$$

and letting $||dz|| = \max_j |dz_j|$ we obtain, using (7),

$$|R_{1,m}| \leq \sum \frac{(m+1)!}{h_1! \dots k_n!} \frac{M}{\varrho^{m+1}} ||dz||^{m+1}$$
$$= (2n)^{m+1} \frac{M}{\varrho^{m+1}} ||dz||^{m+1}$$

The last expression tends to zero as $m \to \infty$ provided $||dz||$ is small enough, namely, $(2n)\, ||dz||/\varrho < 1$. If $r' = \min\, (r_1, \dots, r_n)$ we may choose $\varrho = r' - ||dz||$, and the preceding inequality gives

$$||dz|| < \frac{r'}{2n+1}.$$

Similarly, $R_{2,m} \to 0$ as $m \to \infty$ for $||dz|| < r'/(2n+1)$.

Hence, if f is holomorphic in $D_r(z)$ we have

$$f(z_1 + dz_1, \dots, z_n + dz_n) = f(z_1, \dots, z_n) + \sum_{k=1}^{\infty} \frac{1}{k!} (\partial f)^{(k)}$$

valid for $||dz||$ sufficiently small.

b) For $n = 1$ (i. e. for complex functions of one independent variable) we have, provided $R_m \to 0$,

$$f(z + dz) = f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} (f_z\, dz + f_{\bar z}\, d\bar z)^{(k)}$$

$$= f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(f_z + f_{\bar z}\, \frac{d\bar z}{dz}\right)^{(k)} dz^k$$

Letting $dz = |dz|\, e^{i\theta}$ the above expansion can be written as

$$f(z + dz) = f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} (f_z + f_{\bar z}\; e^{-2i\theta})^{(k)}\, dz^k \tag{8}$$

$$= f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} f_\theta^{(k)}(z)\, dz^k$$

denoting by $f_\theta^{(k)}(z)$ the *k-th* complex rectilinear directional derivative of f at z in the direction θ.

In the holomorphic case $f_\theta^{(k)}(z)$ reduces to $f^{(k)}(z)$ (the ordinary *k-th* derivative), and (8) becomes the well-known CAUCHY-TAYLOR expansion of f about the point z. In this case our estimate of the region of validity can be easily improved. If f is holomorphic in $D_r(z)$, r' is any real number satisfying $0 < r' < r$, and $M = \max\, |f(\xi)|$, $\xi \in \overline{D}_{r'}(z)$, it is known that [4]

$$|R_m| \leq \frac{r' M}{r' - |dz|} \left(\frac{|dz|}{r'}\right)^{m+1}$$

so that $R_m \to 0$ as $m \to \infty$, provided $|dz| < r'$.

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