FINITELY GENERATED IDEALS IN THE BANACH ALGEBRA H^∞

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In the Banach algebra H^∞ of all bounded analytic functions in the open unit disc D we consider the structure of algebraically finitely generated ideals. We give sufficient «analytic» conditions, that a function f belongs to the finitely generated ideal $(f_1, ..., f_n)$. We introduce ideals $W(f_1, ..., f_n)$ and $H(f_1, ..., f_2)$, that are related in a natural way to the ideal $(f_1, ..., f_n)$, and show that there are many connections between them. We prove that no free maximal ideal is finitely generated and use this fact in the last section for an aplication to ring automorphisms of H^∞.

1. INTRODUCTION

Let H^∞ be the ring of all bounded analytic functions in the open unit disc D. We denote by $(f_1, ..., f_n)$ the ideal generated by the functions $f_1, ..., f_n \in H^\infty$. It turns out to be convenient to define the following set:

$$W(f_1, ..., f_n) := \{f \in H^\infty : |f(z)| \leq C \sum_{i=1}^{n} |f_i(z)|$$

in D for some positive constant $C\}$.

It is obvious that $W(f_1, ..., f_n)$ is an ideal in H^∞ containing $(f_1, ..., f_n)$. Then the corona theorem of L. Carleson [2] (see also Duren [3], p. 201 ff. and Hörmander [5], p. 948) can be stated in the following terms.

THEOREM 1.1. Let $f_1, ..., f_n$ be functions in H^∞, then we have: $1 \in (f_1, ..., f_n) \Longleftrightarrow 1 \in W(f_1, ..., f_n)$.

The main unsolved problem in this direction is the so-called Rubel problem (see Birtel [1], p. 347, problem 12).

Rubel problem: Given finitely many functions $f_1, ..., f_n \in H^\infty$. Find neccessary and sufficient conditions that a function f belongs to the ideal $(f_1, ..., f_n)$.

Of course, we are looking for such «analytic» conditions. Some time ago, it was conjectured, that we have:

$$f \in (f_1, ..., f_n) \Leftrightarrow f \in W(f_1, ..., f_n). \tag{1.1}$$

The conjecture is false, as shown by the following slightly modified counterexample of Rajeswara Rao [7]. In general we have $(f_1, ..., f_n) \subset W(f_1, ..., f_n)$ with proper inclusion.

COUNTEREXAMPLE. Let F_1 be the outer function $F_1(z) = (z - 1)$ and F_2 be the inner function $F_2(z) = \exp[(z + 1)/(z - 1)]$. We put $f_1 = F_1^2$, $f_2 = F_2^2$, $f = F_1 F_2$.

From $0 \leq (|F_1| - |F_2|)^2$ we have $F_1 F_2 \in W(F_1^2, F_2^2)$. We show that $F_1 F_2 \notin (F_1^2, F_2^2)$. Otherwise there exists $g_1, g_2 \in H^\infty$ with

$$F_1 F_2 = F_1^2 g_1 + F_2^2 g_2.$$

Since F_1 and F_2 are relatively prime, we see immediately, that g_1 has a factor F_2 and g_2 has a factor F_1. These factors we pull out and have

$$F_1 F_2 = F_1^2 F_2 h_1 + F_2^2 F_1 h_2 \quad \text{with} \quad h_1, h_2 \in H^\infty.$$

This implies $1 = F_1 h_1 + F_2 h_2$. For real $x \in D$ and $x \to 1$ the right hand side of the equation tends to zero, because $\lim_{x \to 1} F_1(x) = \lim_{x \to 1} F_2(x) = 0$. This is a contradiction.

Even if f is such a nice polynomial as $f(z) = 1 - z$ the conjecture (1.1) is also false, as shown by J.P. Rosay [8]. In section 2 a condition will be given on f, such that (1.1) is true.

2. SUFFICIENT CONDITIONS FOR (1.1)

THEOREM 2.1. Let $f, f_1, ..., f_n \in H^\infty$. Suppose that there exists a neighbourhood of the boundary ∂D in which we have the inequality $|f(z)| \geq \delta \sum_{i=1}^{n} |f_i(z)|$ for some $\delta > 0$. If $f \in W(f_1, ..., f_n)$, then it follows that $f \in (f_1, ..., f_n)$.

PROOF. By assumption there exists an $\varepsilon > 0$, such that the following is valid:

$$\delta \sum_{i=1}^{n} |f_i(z)| \leq |f(z)| \quad \text{in} \quad 1 - \varepsilon < |z| < 1 \tag{2.1}$$

and f has no zeros on $|z| = 1 - \varepsilon$. Let $\{a_k\}$ be the sequence of such zeros of f inside $|z| \leq 1 - \varepsilon$, in which f vanishes with a higher power than $\sum_{i=1}^{n} |f_i(z)|$. Let B be the finite Blaschke product with only zeros in a_k, with the right multiplicities, such that f/B and $\sum_{i=1}^{n} |f_i|$ vanish with the same power at every a_k. Let $\{z_k\}$ be all the zeros of f inside $|z| \leq 1 - \varepsilon$. By construction we have locally on these points the inequalities $c_k \sum_{i=1}^{n} |f_i| \leq |f/B| \leq d_k \sum_{i=1}^{n} |f_i|$ with positive constants c_k, d_k. Because $\{z_k\}$ is a finite sequence, there exist positive constants $\varepsilon_1, \delta_1, C_1$ with $1 - |z_k| > \varepsilon + \varepsilon_1$ and

$$\delta_1 \sum_{i=1}^{n} |f_i(z)| \leq |(f/B)(z)| \leq C_1 \sum_{i=1}^{n} |f_i(z)| \quad \text{in} \quad |z - z_k| \leq \varepsilon_1 \tag{2.2}$$

For z with $|z - z_k| > \varepsilon_1$ and $|z| \leq 1 - \varepsilon$ the functions f and B are bounded away from zero; therefore, in consideration of (2.2), we have for some $\delta_2 > 0$:

$$\delta_2 \sum_{i=1}^{n} |f_i(z)| \leq |(f/B)(z)| \quad \text{in} \quad |z| \leq 1 - \varepsilon. \tag{2.3}$$

From (2.1) it follows for some $\delta_3 > 0$

$$\delta_3 \sum_{i=1}^{n} |f_i(z)| \leq |(f/B)(z)| \quad \text{in} \quad 1 - \varepsilon < |z| < 1, \tag{2.4}$$

because B is bounded away from zero there. (2.3) and (2.4) provide for $\delta_4 = \min(\delta_2, \delta_3)$

$$\delta_4 \sum_{i=1}^{n} |f_i(z)| \leq |(f/B)(z)| \quad \text{in} \quad D. \tag{2.5}$$

Then we have $\delta_4 |f_i(z)| \leq |(f/B)(z)|$ in D for every $i \in \{1, \ldots, n\}$, and therefore there exist functions $F_1, \ldots, F_n \in H^\infty$ with

$$f_i = (f/B) F_i. \tag{2.6}$$

The other assumption $f \in W(f_1, \ldots, f_n)$ says that

$$|f(z)| \leq C \sum_{i=1}^{n} |f_i(z)| \quad \text{in} \quad D \tag{2.7}$$

for some $C > 0$. We claim

$$|f(z)| \leq C_2 |B(z)| \sum_{i=1}^{n} |f_i(z)| \quad \text{in} \quad |z| < 1 \tag{2.8}$$

for another constant C_2.

B is in $|z - z_k| > \varepsilon_1$ bounded away from zero, therefore from (2.7) we can deduce

$$|f(z)| \leq C_3 |B(z)| \sum_{i=1}^{n} |f_i(z)| \quad \text{in} \quad |z - z_k| > \varepsilon_1 \tag{2.9}$$

for some $C_3 > 0$. For $|z - z_k| \leq \varepsilon_1$ we have in consideration of (2.2) the same inequality with the constant C_1. Hence we have established (2.8).

Using (2.6) in (2.8) yields $1 \leq C_2 \sum_{i=1}^{n} |F_i(z)|$. By theorem 1.1 there exist $g_1, \ldots, g_n \in H^\infty$ with $1 = \sum_{i=1}^{n} F_i g_i$. This gives $f = \sum_{i=1}^{n} (F_i f) g_i = \sum_{i=1}^{n} f_i (B g_i)$ and therefore $f \in (f_1, \ldots, f_n)$.

Corollary 2.2. Let f be a function continuous in the closed disc $\bar{D}$, holomorphic in D and without zeros on the boundary ∂D. Then we have

$$f \in (f_1, \ldots, f_n) \iff f \in W(f_1, \ldots, f_n).$$

Now we will give a condition on the ideal $(f_1, \ldots, f_n)$, such that (1.1) holds. Recall that $\{z_k\}$, with $z_k \in D$, is an interpolation sequence in H^∞, if and only if for every sequence $\{W_k\} \in l^\infty$ there exists a function $f \in H^\infty$ with $f(z_k) = W_k$. About interpolation sequences in H^∞ a great deal is known (see e.g. Hoffman [4], p. 194 ff.).

THEOREM 2.3. Assume that in the ideal $(f_1, \ldots, f_n)$ there is a Blaschke product, whose zeros are an interpolation sequence in H^∞. Then we have $(f_1, \ldots, f_n) = W(f_1, \ldots, f_n)$.

PROOF. Let $B \in (f_1, \ldots, f_n)$ be such a Blaschke product with zeros $\{z_k\}$. Let $f \in W(f_1, \ldots, f_n)$. We define $W_{ik} := [f(z_k)\overline{f_i(z_k)}] / \sum_{i=1}^{n} |f_i(z_k)|^2$. The W_{ik} are well defined. If the denominator is zero, then the singularity on the right hand side of the equation is removable, and W_{ik} is defined in the obvious way. Our assumption implies $\{W_{ik}\} \in l^\infty$. Because $\{z_k\}$ is an interpolation sequence in H^∞, for every $i \in \{1, \ldots, n\}$ there exists a function $g_i \in H^\infty$ with $g_i(z_k) = W_{ik}$. We conclude that

$$(f - \sum_{i=1}^{n} f_i g_i)(z_k) = f(z_k) - \sum_{i=1}^{n} f_i(z_k) W_{ik}$$

$$= f(z_k) - \sum_{i=1}^{n} \{[f(z_k)\, |f_i(z_k)|^2] / \sum_{i=1}^{n} |f_i(z_k)|^2\} = 0.$$

According to the factorization theorem of F. Riesz (see Duren [3], p. 20) B divides $f - \sum_{i=1}^{n} f_i g_i$, i.e. there exists a function $g \in H^\infty$ with $f - \sum_{i=1}^{n} f_i g_i = gB$. It follows that $f \in (B, f_1 \ldots, f_n)$; since $B \in (f_1, \ldots, f_n)$ we actually have $f \in (f_1 \ldots, f_n)$.

REMARK. Theorem 2.3 is a generalization of the corollary in Hoffman [4], p. 206. We note that our condition does not depend on the system of generators; for example, it may happen that no function f_i is a Blaschke product.

3. A PRINCIPAL IDEAL RELATED TO $(f_1, \ldots, f_n)$

DEFINITION 3.1. By $H(f_1, \ldots, f_n)$ we define the intersection of all principal ideals containing the ideal $(f_1, \ldots, f_n)$.

THEOREM 3.2. For $f_1, \ldots, f_n \in H^\infty$ we have:

(1) $W(f_1, \ldots, f_n) \subset H(f_1, \ldots, f_n)$.

(2) $H(f_1, \ldots, f_n)$ is a principal ideal.

PROOF. Let h be a function in H^∞ with $(f_1, \ldots, f_n) \subset (h)$. This implies that there exist functions $h_i \in H^\infty$ with $f_i = h h_i$ $(i = 1, \ldots, n)$. Assume that f is in $W(f_1, \ldots, f_n)$, in other words there exists a constant C, such that we have $|f(z)| \leq C \sum_{i=1}^{n} |f(z)|$ in D. Therefore we get $|f(z)| \leq C\,|h(z)| \sum_{1=n}^{n} |h_i(z)|$, and this implies $f \in (h)$. Thus we have established (1).

Now we show that $H(f_1, \ldots, f_n)$ is a principal ideal. We can assume that every $f_k \neq 0$ $(k = 1, \ldots, n)$. The canonical factorization theorem in H^p spaces (see Duren [3], p. 24) gives $f_k = G_k F_k$ with inner functions G_k and outer functions

$$F_k(z) = \lambda_k \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log |f_k(e^{it})|\, dt\right)$$

with $\lambda_k \in \mathbf{C}$, $|\lambda_k| = 1$. It is known that any non-empty collection of inner functions has a greatest common divisor (see Hoffman [4] p. 85). Therefore let G be the greatest common divisor of the functions $G_1, \ldots, G_n$. Let us define $h(e^{it}) := \max\{|f_k(e^{it})| : 1 \leq k \leq n\}$. Then we have $h(e^{it}) \geq 0$, $\log h(e^{it}) \in L^1(\partial D)$, $h(e^{it}) \in L^\infty(\partial D)$.

Consequently

$$F(z) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log h(e^{it})\, dt\right)$$

is an outer function in H^∞. F divides every F_k. Clearly by construction F is a greatest common divisor of the outer functions $F_1, \ldots, F_n$.

Therefore (FG) is the smallest principal ideal over $(f_1, \ldots, f_n)$; it follows that $(FG) = H(f_1, \ldots, f_n)$.

COROLLARY 3.3. If $(f_1, \ldots, f_n)$ is a principal ideal in H^∞, then so is $W(f_1, \ldots, f_n)$. In fact the two ideals coincide.

Our question is now: Can $W(f_1, \ldots, f_n)$ be a principal ideal, when $(f_1, \ldots, f_n)$ is not? A negative answer is provided in the following theorem.

THEOREM 3.4. $(f_1, \ldots, f_n)$ is a principal ideal in H^∞, if and only if $W(f_1, \ldots, f_n)$ is a principal ideal.

PROOF. For one direction see the corollary above. Now let $W(f_1, \ldots, f_n)$ be a principal ideal in H^∞, i.e. $(d) = W(f_1, \ldots, f_n)$ for a function $d \in H^\infty$. Then by theorem 3.2. we have $(d) = H(f_1, \ldots, f_n)$ and it follows that d is a divisor of every function f_i, i.e. we have $f_i = d\, F_i$ with $F_i \in H^\infty$ $(i = 1, 2, \ldots, n)$. Because $d \in W(f_1, \ldots, f_n)$ there exists some constant $C > 0$ with $|d(z)| \leq C \sum_{i=1}^{n} |f_i(z)|$ in D. Consequently we have $C^{-1} \leq \sum_{i=1}^{n} |F_i(z)|$ in D. In observation of theorem 1.1, this implies $1 \in F_1, \ldots, F_n)$ and therefore $d \in (f_1, \ldots, f_n)$. Thus we have established that $(d) = (f_1, \ldots, f_n)$.

If the function 1 is in the ideal $W(f_1, \ldots, f_n)$, then it is also in the ideal $(f_1, \ldots, f_n)$. What occurs, when we only have $1 \in H(f_1, \ldots, f_n)$? Does it follow that $1 \in W(f_1, \ldots, f_n)$ and therefore $1 \in (f_1, \ldots, f_n)$? That is not the case, as the following counterexample shows.

COUNTEREXAMPLE. Let $n = 2$, let f_1 the zero free inner function $f_1(z) = \exp[(z+1)/(z-1)]$ and f_2 a Blaschke product, whose real zeros have a cluster point at $z = 1$. For every function $f \in W(f_1, f_2)$ there exists a sequence $\{z_k\}$, $z_k \in D$, $\lim_{k\to\infty} z_k = 1$, such that $\lim_{k\to\infty} f(z_k) = 0$. Therefore we have $W(f_1, f_2) \neq H(f_1, f_2) = H^\infty$.

4. PRIME AND MAXIMAL IDEALS RELATED TO THE IDEALS $(f_1, \ldots, f_n)$ and $W(f_1, \ldots, f_n)$

As we have seen, $W(f_1, \ldots, f_n)$ is an ideal which in general properly contains $(f_1, \ldots, f_n)$. In consideration of the relationship between these two ideals the following question arises naturally: How different can these ideals be? One may guess, that a function f in $W(f_1, \ldots, f_n)$ cannot be too far away, in some sense, from $(f_1, \ldots, f_n)$. We will see, that this is true. Before we proceed, let us give a definition.

DEFINITION 4.1. Let I be an ideal in a commutative ring R with identity element. Then prim rad (I) [max rad (I)] denote the intersection of all prime [maximal] ideals containing I.

Note that prim rad (I) and max rad (I) are again ideals. prim rad (I) is usually called the radical of the ideal I. Kelleher and Taylor [6] show for various rings of holomorphic functions restricted by growth conditions, that we have $W(f_1, \ldots, f_n) \subset$ prim rad $(f_1, \ldots, f_n)$. In particular if $(f_1, \ldots, f_n)$ is a prime ideal, then it follows that $(f_1, \ldots, f_n) = W(f_1, \ldots, f_n)$. Note the analogy to the Hilbert's Nullstellensatz in algebraic geometry. Unfortunately their techniques exclude the ring H^∞ (see also Hörmander [5]), and they stated the question, whether the same result holds for the ring H^∞. We will give some results in this direction.

THEOREM 4.2. Let $f_1, \ldots, f_n \in H^\infty$. Then we have $W(f_1, \ldots, f_n) \subset$ max rad $(f_1, \ldots, f_n)$.

PROOF. Let $(f_1, \ldots, f_n)$ be a proper ideal in H^∞, the other case is trivial. Assume that there exists a maximal Ideal M with $(f_1, \ldots, f_n) \subset M$ but $W(f_1, \ldots, f_n) \not\subset M$. That is, there exists a function $g \in W(f_1, \ldots, f_n)$ with $g \notin M$. This implies $(g, M) = H^\infty$. Therefore there exists $f \in M$ with $\delta \leq |f(z)| + |g(z)|$ in D for some positive δ. Now $g \in W(f_1, \ldots, f_n)$ tells us that $|g(z)| \leq C \sum_{i=1}^{n} |f_i(z)|$ in D for some $C > 1$. We deduce that $(\delta/C) \leq |f(z)| + \sum_{i=1}^{n} |f_i(z)|$ in D. By theorem 1.1 we have $(f, f_1, \ldots, f_n) = H^\infty$, consequently $M = H^\infty$, a contradiction.

THEOREM 4.3. If $(f_1, \ldots, f_n)$ is a prime ideal in H^∞, then either $(f_1, \ldots, f_n) = W(f_1, \ldots, f_n)$ or a chain of infinitely many ideals lies between $(f_1, \ldots, f_n)$ and $W(f_1, \ldots, f_n)$.

PROOF. Throughout the proof we will use the notation $I = (f_1, \ldots, f_n)$ and $W = W(f_1, \ldots, f_n)$. Let $I \neq W$. First we show the existence of one ideal W_1 properly between I and W. Assume that there does not exist such an ideal. We consider the quotient ideal $Q := I : W$. Since the inclusion $I \subset W$ is proper, we have $1 \notin Q$, i. e. $H^\infty \setminus Q \neq \emptyset$. Choose any function $g \in H^\infty \setminus Q$. Then there exists a function $f \in W$ with $fg \notin I$. Therefore we have $I \subset (I, fg) \subset W$ and the first inclusion is proper. By assumption no ideal lies between I and W, hence

$$(I, fg) = W \tag{4.1}$$

From $(I, fg) \subset (I, f) \subset W$ and 4.1 it follows that

$$(I, f) = W \tag{4.2}$$

For every $h \in H^\infty$ there exist, in view of (4.1), functions $g_1, \ldots, g_n$, $H \in H^\infty$ with

$$fh = \sum_{i=1}^{n} f_i g_i + fgH,$$

hence we get

$$f(h - gH) \in I \tag{4.3}$$

Now (4.2) and (4.3) give:

$$(1 - gH) W = (1 - gH) I + (1 - gH) f \subset I$$

This implies $1 - gH \in Q$, i.e. $1 \in Q + (g) = (Q, g)$ Therefore Q must be a maximal ideal in H^∞. By theorem 4.2 we have $W \subset Q$ and consequently

$$W^2 \subset I \tag{4.4}$$

Let $F, G \in W \setminus I$, then by (4.4) it follows that $F \cdot G \in I$. Because I is a prime ideal, this is a contradiction.

Therefore between I and W there must lie an ideal, distinct from both, say W_1. The same proof on I and W_1 yields an ideal W_2 with $I \subset W_2 \subset W_1$ and $I \neq W_2 \neq W_1$. By induction one can see that infinitely many different ideals lie between I and W.

Definition 4.4. We call an ideal I in H^∞ fixed, if and only if all functions of I vanish in some common point of D. Otherwise we call I a free ideal.

We note, that every fixed maximal ideal in H^∞ is principal. A free ideal of H^∞ can be finitely generated, in contrast to many other rings of holomorphic functions. However, the ideal cannot also be maximal, as the following theorem shows.

Theorem 4.5. No free maximal ideal in H^∞ is finitely generated.

Proof. Assume M is a free maximal ideal, which is finitely generated, i.e. we have $M = (f_1, \ldots, f_n)$ for $f_1, \ldots, f_n \in H^\infty$. M is a proper ideal; therefore there exists a sequence $\{z_k\}$ in D and a point α on the boundary ∂D, with $\lim_{k \to \infty} z_k = \alpha$ and $\lim_{k \to \infty} f_i(z_k) = 0$ for every

$i \in \{1, \ldots, n\}$. We can choose a subsequence $\{a_k\}$ of $\{z_k\}$ which is actually an interpolation sequence in H^∞ (see Hoffman [4], p. 204, corollary). Therefore there exists a function $f \in H^\infty$ with $f(a_{2k-1}) = 0$ and $f(a_{2k}) = 1$ $(k = 1, 2, \ldots)$. We claim $(f_1, \ldots, f_n, f) \neq H^\infty$. Otherwise by theorem 1.1 we have $|f(z)| + \sum_{i=1}^{n} |f_i(z)| \geq \delta$ in D for some $\delta > 0$. In particular it follows that $|f(a_{2k-1})| + \sum_{i=1}^{n} |f_i(a_{2k-1})| \geq \delta$, consequently $\sum_{i=1}^{n} |f_i(a_{2k-1})| \geq \delta$. This is a contradiction to $\lim_{k\to\infty} f_i(z_k) = 0$ for every $i \in \{1, \ldots, n\}$. Now we show that $f \notin (f_1, \ldots, f_n)$. Otherwise there exists some constant $C > 0$, such that $|f(z)| \leq C \sum_{i=1}^{n} |f_i(z)|$ holds in D. In particular we have $|f(a_{2k})| = 1 \leq C \sum_{i=1}^{n} |f_i(a_{2k})|$, and this is also a contradiction to $\lim_{k\to\infty} f_i(z_k) = 0$ for every $i \in \{1, \ldots, n\}$.

5. An Application to Ring Automorphisms of H^∞

Let the complex constants be identified with the constant functions in H^∞. Let $\Phi: H^\infty \to H^\infty$ be a normed, i. e. $\Phi(i) = i$, ring automorphism. We say Φ is generated by a conformal map φ, if and only if there exists a conformal map φ of D onto D with $[\Phi f](\lambda) = f[\varphi(\lambda)]$ for every $\lambda \in D$. The following theorem is a generalization of the theorem in Hoffman [4], p. 143. We should note, that the proofs there essentially use the linearity of Φ, which we do not assume.

Theorem 5.1. Every normed ring automorphism of H^∞ is generated by a conformal map.

Proof. For $\lambda \in D$ let $M_\lambda = \{f \in H^\infty : f(\lambda) = 0\}$ and $M = \{M_\lambda\}$. M_λ is a maximal ideal and also a principal ideal. Therefore so is $\Phi(M_\lambda)$, since these properties are preserved under ring automorphisms. From theorem 4.5, we deduce that $\Phi(M_\lambda)$ is a fixed ideal, i. e. there exists a point μ in D with $\Phi(M_\lambda) \subset (z - \mu)$. Since $\Phi(M_\lambda)$ is maximal, we have $\Phi(M_\lambda) = (z - \mu) = M_\mu$. It can be easily verified, that Φ is a bijection on M.

We claim that $\Phi(\lambda) = \lambda$ for every $\lambda \in \mathbf{C}$. Assume that there exists a $\lambda_0 \in \mathbf{C}$ with $|\Phi(\lambda_0) - \lambda_0| = 2\delta > 0$. We choose the polynomial $g(z) = \lambda_0 + \delta z$. Then λ_0 is in the closure of the range $R(g)$ of g, i. e. $\lambda_0 \in \overline{R(g)}$. This is equivalent to the statement, that $g - \lambda_0$ is non-invertible in H^∞. Since Φ is an automorphism, $\Phi(g) - \Phi(\lambda_0)$ is non-invertible; therefore $\Phi(\lambda_0) \in \overline{R[\Phi(g)]}$. Since Φ is normed, we have $\Phi(\lambda) = \lambda$ for every $\lambda \in \mathbf{Q}(i)$ and therefore $\mathbf{Q}(i) \cap \overline{R(g)} = \mathbf{Q}(i) \cap \overline{R[\Phi(g)]}$. That g is not constant implies that $\Phi(g)$ is not constant. Therefore the ranges $R(g)$, $R[\Phi(g)]$ are open sets. Since $\mathbf{Q}(i)$ is dense in $\mathbf{C}$, we have actually $\overline{R(g)} = \overline{R[\Phi(g)]}$. Hence $\Phi(\lambda_0) \in \overline{R(g)} = \{\lambda_0 + z : |z| \leq \delta\}$. In particular $|\Phi(\lambda_0) - \lambda_0| \leq \delta$, and this is a contradiction to the assumption.

Let us define a map $\tau: D \to M$ by means of $\tau(\lambda) := M_\lambda$. τ is clearly bijective. Let $\varphi := \tau^{-1} \circ \Phi \circ \tau$. Then, as a composition of bijective maps, φ is bijective. It remains to show, that φ is also analytic. For $\lambda \in D$, $f \in H^\infty$ we have $f - f(\lambda) \in M_\lambda$. Consequently $\Phi[f - f(\lambda)] = \Phi(f) - \Phi[f(\lambda)] \in \Phi(M_\lambda) = M_{\varphi(\lambda)}$. This implies that $\Phi(f)[\varphi(\lambda)] = \Phi[f(\lambda)] = f(\lambda)$. Now let g be the identity function in H^∞, choose $f = \Phi^{-1}(g)$. Then it follows that $\varphi(\lambda) = f(\lambda)$; since f is analytic, φ is a conformal map.

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