AN INTRINSIC DEFINITION OF THE DIRAC OPERATOR

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Dedicated to K. Bleuler on his 60th birthday

Abstract:

A coordinate free intrinsic definition of spinor fields and the Dirac operator is presented for n-dimensional spin manifolds. Use is made of the modern mathematical concepts of vector bundles, principal bundles, linear and principal connections.

Introduction: This note presents the elementary theory of spinor fields and the Dirac operator for n-dimensional spin manifolds. In the case of Lorentz manifolds and the standard spin representation this material has been extensively treated by A. Lichnerowicz (10). Our exposition involves no new ideas, but states the definitions and results in a coordinate free intrinsic form.

The basic definitions (eg. vector bundles, principal bundles, linear and principal connections) are taken from (6) and (12).

§ 1. Clifford Algebras

1. *The Clifford algebra over an inner product space.* Let $\mathbf{R}^n$ be a real n-dimensional real vector space with a non-degenerate inner product, $< , >$, (not necessarily positive definite). The group of proper isometries of $\mathbf{R}^n$ will be denoted by $S0(n)$. Consider the Clifford algebra, $\mathcal{C}(\mathbf{R}^n)$, over $\mathbf{R}^n$. We shall use the canonical inclusion map $\mathbf{R}^n \to \mathcal{C}(\mathbf{R}^n)$ to identify $\mathbf{R}^n$ with a subspace of $\mathcal{C}(\mathbf{R}^n)$. Then we have the relation

$$xy + yx = 2 < x, y > e \qquad x, y \in \mathbf{R}^n,$$

where e denotes the unit element of $\mathcal{C}(\mathbf{R}^n)$. For every element $u \in \mathcal{C}(\mathbf{R}^n)$ denote by $L(u)$ the linear transformation of $\mathcal{C}(\mathbf{R}^n)$ given by

$$\mathbf{L}(u)\, v = u\, v \qquad v \in \mathcal{C}(\mathbf{R}^n)$$

Let $\mathcal{C}^*(\mathbf{R}^n)$ be the multiplicative group of $\mathcal{C}(\mathbf{R}^n)$. Then the *adjoint representation*, Ad, of $\mathcal{C}^*(\mathbf{R}^n)$ in $\mathcal{C}(\mathbf{R}^n)$ is defined by

$$Ad\,(g)\, v = g^{-1} vg \qquad g \in \mathcal{C}^*(\mathbf{R}^n) \qquad v \in \mathcal{C}(\mathbf{R}^n)$$

2. *The group Spin* (n). Let G be the subgroup of $\mathcal{C}^*(\mathbf{R}^n)$ consisting of the elements, g, which satisfy

i) $\mathbf{R}^n$ is stable under $Ad\,(g)$

ii) det $L\,(g) = 1$.

The 1 — component of G is called the *spinor group and* is denoted by Spin (n). For every element $g \in$ Spin (n), $Ad\,(g)$ is a proper isometry of $\mathbf{R}^n$ and so we have a natural homomorphism

$$Ad\colon\ Spin\,(n) \to S0\,(n).$$

This homomorphism is surjective and its kernel consists of the elements e and $-e$. Thus $Spin(n)$ is a double covering group of $S0(n)$. (For details, cf. [4]).

§ 2. Vector Bundles

3. *Bundle valued tensor fields.* Let $\xi = (E, \pi, B, F)$ be a vector bundle (real or complex). A *covariant tensor field of degree* p *on* B *with values in* ξ is a smooth assignment $x \to \Omega_x$ of p-linear maps

$$\Omega_x\colon T_x\,(B) \times \ldots \times T_x\,(B) \to F_x$$

($T_x(B)$ denotes the tangent space of B at x and F_x. denotes the fibre of ξ at x). Equivalently, Ω is a cross-section in the vector bundle $\overset{p}{\oplus}\, \tau_B{}^* \otimes \xi$. The ξ-valued tensor fields of degree p form a module over the function ring, $S\,(B)$, denoted by $X^p\,(B; \xi)$.

If $\xi = B \times \mathbf{R}$ is the product bundle then $X^p\,(B; \xi)$ is just the module, $X^p\,(B)$, of ordinary covariant tensor fields on B.

Now observe that $S(B)$-bilinear maps $X^p\,(B) \times X^q\,(B; \xi) \to X^{p+q}\,(B; \xi)$ are given by

$$(\Phi \cdot \Omega)(x; h_1 \dots h_{p+q}) = \Phi(x; h_1 \dots h_p)\, \Omega(x; h_{p+1} \dots h_{p+q}).$$
$$x \in B,\ h_j \in T_x(B).$$

These induce isomorphisms

$$X^p(B) \otimes X^q(B; \xi) \xrightarrow{\cong} X^{p+q}(B; \xi)$$

(where the tensor product is over $S(B)$). In particular for $q = 0$ $X^q(B; \xi) = \text{Sec}\ \xi$ and we obtain isomorphisms

$$X^p(B) \otimes \text{Sec}\ \xi \xrightarrow{\cong} X^p(B; \xi).$$

On the other hand, there is an obvious canonical isomorphism

$$X^p(B; \overset{q}{\otimes} \tau_B{}^* \otimes \xi) \to X^{p+q}(B; \xi).$$

Finally, note that we may interpret $X^p(B; \xi)$ as the module of p-linear (over the ring $S(B)$) maps

$$X(B) \times \dots \times X(B) \to \text{Sec}\ \xi$$

where $X(B)$ denotes the $S(B)$-module of vector. fields on B.

Next consider the space, $X_r(B)$ of contravariant tensor fields of degree r on B. There is a unique bilinear map

$$i\colon X_r(B) \times X^p(B; \xi) \to X^{p-r}(B; \xi) \qquad p \geqq r$$

such that

$$(i(X_r \otimes \dots \otimes X_1)\Phi)(X_{r+1}, \dots, X_p) = \Phi(X_1, \dots X_r, \dots X_p)$$
$$X_j \in X(B),\ \Phi \in X^p(B\colon \xi).$$

If a is a fixed contravariant tensor field on B we shall write

$$i(a, \Phi) = i(a)\Phi \qquad \Phi \in X^r(B; \xi) \qquad p \geqq r$$

Observe that $i(X)$ is simply the substitution operator.

Next, let $\xi = (E_\xi, \pi_\xi, B_\xi, F)$ and $\eta = (E_\eta, \pi_\eta, B_\eta, F)$ be vector bundles with the same typical fibre and let $\varphi\colon \xi \to \eta$ be a bundle map restricting to isomorphisms, φ_x, in the fibres. Let $\psi\colon B_\xi \to B_\eta$ be the induced map between the base manifolds. Then φ determines for each $p \geqq 0$ a linear map

$$\varphi^{\#}\colon X^p(B_\xi; \xi) \leftarrow X^p(B_\eta; \eta)$$

given by

$$(\varphi^{\#} \Phi)(x; h_1, \ldots, h_p) = \varphi_x^{-1} \Phi(\psi x; (d\psi)_x h_1, \ldots (d\psi)_x h_p).$$
$$x \in B_\xi, \ h_j \in T_x(B_\xi).$$

On the other hand, let $\xi = (E_\xi, \pi_\xi, B, F_\xi)$ and $\eta = (E_\eta, \pi_\eta, B, F_\eta)$ be vector bundles over the same base and let $\varphi: \xi \to \eta$ be a strong bundle map; i.e., $\varphi: E_\xi \to E_\eta$ is a bundle map satisfying $\pi_\eta \circ \varphi = \pi_\xi$. Then φ induces linear maps

$$\varphi_*: X^p(B; \xi) \to X^p(B; \eta)$$

given by

$$(\varphi_* \Phi)(x; h_1, \ldots h_p) = \varphi_x \Phi(x; h_1, \ldots, h_p) \qquad p \geqq 0$$
$$x \in B, h_j \in T_x(B).$$

Finally, let ξ be any vector bundle and denote by L_ξ the vector bundle over the same base, B, whose fibre at X is the space of linear transformations $F_x \to F_x$. Then every pair $\Phi \in X^p(B; L_\xi)$, $\Psi \in X^q(B; L_\eta)$ determines the L_η-valued tensor field of degree $p + q$ given by

$$(\Phi \circ \Psi)(x; h_1, \ldots, h_{p+q}) = \Phi(x; h_1, \ldots h_p) \circ \Psi(x; h_{p+1}, \ldots h_{p+q})$$
$$x \in B, h_j \in T_x(B).$$

On the other hand, if $\Phi \in X^p(B; L_\xi)$ and $\Psi \in X^q(B; \xi)$ then an element $\Phi(\Psi) \in X^{p+q}(\xi)$ is determined by

$$\Phi(\Psi)(x; h_1, \ldots, h_{p+q}) = \Phi(x; h_1, \ldots h_p)(\Psi(x; h_{p+1}, \ldots h_{p+q}))$$
$$x \in B, h_j \in T_x(B).$$

The substitution operator satisfies the relation

$$(0) \qquad i(X \otimes Y)\Phi(\Psi) = i(X)\Phi(i(Y)\Psi) \qquad X, Y \in X(B)$$
$$\Phi \in X^p(B; \mathrm{Sec}\ L_\xi), \ \Psi \in X^q(B; \xi)$$

4. *Linear connections.* A *linear connection* in a real (complex) vector bundle ξ is a real (complex) linear map

$$\nabla: \mathrm{Sec}\ \xi \to X^1(B; \xi)$$

which satisfies

$$\nabla(f \cdot \sigma) = \delta f \cdot \sigma + f \cdot \nabla \sigma \qquad f \in S(B) \qquad \sigma \in \operatorname{Sec} \xi$$

where δf deuotes the gradient of f.

EXAMPLE: If ξ is the trivial bundle $B \times F$, the cross-sections in ξ can be identified with the smooth functions $B \to F$. Then a linear connection is defined by $\nabla = \delta$. If is called the *standard linear connection.*

Next, let ξ and η be vector bundles with the same typical fibre and let $\varphi: \xi \to \eta$ be a bundle map which restricts to linear isomorphisms on the fibres. Then φ is called *connection preserving* (with respect to linear connections ∇_ξ and ∇_η) if

$$\varphi^{\#} \circ \nabla_\eta = \nabla_\xi \circ \varphi^{\#}$$

On the other hand a strong bundle map $\varphi: \xi \to \eta$ between vector bundles over the same base is called *connection preserving,* if

$$\nabla_\eta \circ \varphi_* = \varphi_* \circ \nabla_\xi.$$

If ∇ is a linear connection in ξ the *covariant derivative of a cross-section σ with respect to a vector field X on B* is the cross-section, $\nabla_X \sigma$, defined by

$$\nabla_X \sigma = i(X) \nabla \sigma.$$

The *curvature* of a linear connection is the skew symmetric tensor field, R, on B with values in the vector bundle L_ξ given by

$$R(X, Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]} \sigma \quad X, Y \in X(B) \quad \sigma \in \operatorname{Sec} \xi.$$

Next assume that ξ is a vector bundle over a manifold B and that linear connections, ∇_ξ and ∇_B, are defined in ξ and in the tangent bundle τ_B. Then there are unique linear connections, ∇, in the bundles $\overset{p}{\otimes} \tau_B{}^* \otimes \xi$ which satisfy

$$\nabla_X (\omega_1 \otimes \ldots \otimes \omega_p \otimes \sigma) = \sum_{i=1}^{p} \omega_1 \otimes \ldots \otimes (\nabla_B{}^*)_X\, \omega_i \otimes \ldots \otimes \omega_p \otimes \sigma +$$
$$+ \omega_1 \otimes \ldots \otimes \omega_p \oplus (\nabla_\xi)_X\, \sigma \qquad p \geqq 0.$$

Since

$$\operatorname{Sec}(\otimes^p \tau_B{}^* \otimes \xi) = X^p(B, \xi)$$

we have

$$X^1(B; \otimes^p \tau_B{}^* \otimes \xi) = \operatorname{Sec}(\otimes^{p+1} \tau_B{}^* \otimes \xi) = X^{p+1}(B; \xi).$$

Thus we may regard ∇ as an operator

$$\nabla : X^p (B; \xi) \to X^{p+1} (B; \xi).$$

This operator is called the *covariant derivative* (with respect to the linear connections ∇_ξ and ∇_B).

Note that in the case $\xi = B \times \mathbf{R}$, $\nabla_\xi = \delta$, the operator ∇ reduces to the classical covariant derivative, Finally, if a is a contravariant tensor field on B which satisfies $\nabla_B a = 0$ we have the formula

$$(1) \quad \nabla_X i (a) \Phi = i(a) \nabla_X \Phi \qquad \Phi \in X^p (B; \operatorname{Sec} \xi) \qquad X \in X (B).$$

Now assume that a pseudo-Riemannian metric on B is defined and denote the metric tensor by γ. Then there is precisely one linear connection, ∇_B, in the tangent bundle τ_B such that

1) $\nabla_B \gamma = 0$.
2) $(\nabla_B)_X Y - (\nabla_B)_Y X - [X, Y] = 0 \qquad X, Y \in X (B)$.

It is called the *Levi-Civita connection* induced by the metric tensor γ.

Finally, if ξ is a vector bundle over B with a linear connection, ∇_ξ, and if τ_B is given the Levi-Civita connection, the second covariant derivative of a cross-section satisfies

$$(2) \quad \nabla^2 \sigma (X, Y) - \nabla^2 \sigma (Y, X) = R (X, Y) \sigma \quad X, Y \in X (B) \quad \sigma \in \operatorname{Sec} \xi$$

where R denotes the curvature for ∇_ξ.

The *Laplacian* of a cross-section, σ, is the cross-section, $\triangle\sigma$, defined by

$$\boxed{\triangle \sigma = i (g) \nabla^2 \sigma}$$

where g is the contravariant metric tensor on B.

Note that if ξ is the trivial bundle $B \times \mathbf{R}$ and if ∇_ξ is the standard connection, this reduces to the classical Laplacian of a function.

§ 3. Principal bundles and associated vector bundles

5. *Principal bundles.* Let $\mathcal{P} = (P, \pi, B, G)$ be a principal bundle with structure group a Lie group G. Then G acts without fixed

points from the right on P. We shall denote this action by $T\colon P \times G \to P$ and write

$$T(z, g) = z \cdot g \qquad z \in \mathcal{P},\ g \in G.$$

Since every fibre $G_x = \pi^{-1}(x)$, $x \in B$, is a submanifold of P, the tangent space $T_z(G_x)$ is a subspace of $T_z(P)$. It is called the *vertical subspace* and is denoted by $V_z(P)$. A vector field, Z, on P is called *vertical*, if every vector $Z(z)$, $z \in P$ is contained in the vertical subspace $V_z(P)$.

A covariant tensor field, Φ, of degree p on B with values in a vector space, W, is called *horizontal*, if $\Phi(Z_1, \ldots, Z_p) = 0$ whenever one of the vector fields Z_j $(j = 1 \ldots p)$ is vertical.

Next, let Q be a representation of G in W. Then $\Phi \in X^p(P; W)$ is called *equivariant*, if

$$T_g^* \Phi = Q(g^{-1})\, \Phi \qquad g \in G.$$

A tensor field which is both horizontal and equivariant is called *basic*. The spaces of horizontal (equivariant, basic) tensor fields of degree p are respectively denoted by $X^p{}_H(P; W)$, $X^p{}_I(P; W)$ and $X^p{}_B(P; W)$. Note that $X^0{}_H(P; W) = S(P; W)$ and thus $X^0{}_B(P; W) = S_I(P; W)$.

Next, let $\mathcal{P} = (P, \pi, B, G)$ and $\hat{\mathcal{P}} = (\hat{\varrho}, \hat{\pi}, \hat{B}, \hat{G})$ be principal bundles. A *homomorphism* $\mathcal{P} \to \hat{\mathcal{P}}$ is a pair (φ, ϱ) where

1) $\varrho\colon G \to \hat{G}$ is a homomorphism of Lie groups.

2) $\varphi\colon P \to \hat{P}$ is a smooth map satisfying

$$\varphi(z \cdot g) = \varphi(z) \cdot \varrho(g) \qquad z \in P,\ g \in G.$$

Then φ is fibre preserving and hence it induces a map $\psi\colon B \to \hat{B}$.

6. *Principal connections. A principal connection* in a principal bundle P, is smooth assignment of projection operators

$$V(z)\colon T_z(P) \to V_z(P)$$

in such a way that the diagrams

$$\begin{tikzcd} T_z(P) \arrow[r, "dT_g", "\cong"'] \arrow[d, "V(z)"] & T_{zg}(P) \arrow[d, "V(z \cdot g)"] \\ V_z(P) \arrow[r, "dT_g", "\cong"'] & V_{zg}(P) \end{tikzcd}$$

commute. The operators $V(z)$ and $H(z) = \iota - V(z)$ (ι the identity map) are called the *vertical* and *horizontal projections*.

Suppose a principal connection is defined in P and let Φ be a tensor field on P with values in a vector space W. Then the *horizontal part of* Φ, $H^* \Phi$, is the tensor field given by

$$(H^* \Phi)(z; \zeta_1, \ldots \zeta_p) = \Phi(z; H(z)\zeta_1, \ldots H(z)\zeta_p)$$
$$z \in P, \zeta_j \in T_z(P).$$

If f is a smooth function on P with values in W, then $H^* \delta f$ is a horizontal tensor field of degree 1 on P. It is called the *covariant exterior derivative of* f. If f is equivariant (with respect to a representation of G in W), then $H^* \delta f$ is also equivariant and hence basic. Thus $H^* \delta$ restricts to an operator

$$H^* \delta \colon S(P; W)_I \to X_B^1(P; W).$$

Next, let $\varphi\colon P \to \hat{P}$, $\varrho\colon G \to \hat{G}$ be a homomorphism of principal bundles P and $\hat{P}$ over the same base B and assume that principal connections, V and $\hat{V}$ are defined in P and $\hat{P}$. Then φ is called *connection preserving*, if the diagram

$$\begin{tikzcd}
T_z(P) \arrow[r, "(d\varphi)_z"] \arrow[d, "V_z"] & T_{\varphi(z)}(\hat{P}) \arrow[d, "\hat{V}_{\varphi(z)}"] \\
V_z(P) \arrow[r, "(d\varphi)_z"] & V_{\varphi(z)}(\hat{P})
\end{tikzcd}$$

commutes.

If for every $z \in P$ the linear map

$$(d\varphi)_z\colon V_z(P) \to V_{\varphi(z)}(\hat{P})$$

is an isomorphism, and $\hat{V}$ is a given principal connection in $\hat{P}$, then there is a unique principal connection, V, in P such that φ is connection preserving. V is called the *induced principal connection.*

7. *The associated vector bundle.* Let $P = (P, \pi, B, G)$ be a principal bundle and let Q be a representation of G in a vector space W (real or complex). Then an equivalence relation in the product manifold $P \times W$ is given by

$$(z, \omega) \sim (zg, Q(g)^{-1}\omega) \qquad z \in P, \omega \in W, g \in G,$$

Denote by $P \times_G W$ the quotient manifold of equivalence classes and by

$$q\colon P \times W \to P \times_G W$$

the corresponding projection.

Then a surjective smooth map $\varrho\colon P \times_G W \to B$ is determined by the commutative diagram

$$\begin{tikzcd} P \times W \arrow[r, "q"] \arrow[d, "\pi_1"] & P \times_G W \arrow[d, "\varrho"] \\ P \arrow[r, "\pi"] & B \end{tikzcd}$$

where π_1 denotes the obvious projection. This map makes $P \times_G W$ into a vector bundle over B with W as typical fibre. In fact, if we set $W_x = \varrho^{-1}(x)$, $x \in B$, then there is a unique linear structure in W_x such that q restricts to isomorphisms

$$q_z : W \to W_x \qquad x = \pi(z).$$

The vector bundle

$$\xi = (P \times {}_G W, \varrho, B, W)$$

so obtained is called the *associated vector bundle* of P (with respect to the representation Q) and q is called the ***principal map***. Note that the linear isomorphisms q_z satisfy the relations

3) $$q_{zg} = q_z \circ Q(g) \qquad z \in P, g \in G.$$

Since the principal map restricts to isomorphisms on the fires, it determines linear maps

$$q^{\#}: X^p(P; W) \leftarrow X^p(B; \xi) \qquad p \geqq 0$$

(cf. sec. 4). Clearly, if $\Phi \in X^p(B; \xi)$ then $q^{\#}\Phi$ is a horizontal tensor field. Moreover, relation (3) implies that $q^{\#}\Phi$ is equivariant with respect to the representation Q. Hence $q^{\#}$ can be regarded as a map into $X^p{}_B(P; W)$. It is easy to verify that this map is an isomorphism,

$$q^{\#}: \ X^p{}_B(P; W) \xleftarrow{\cong} X^p(B; \xi) \qquad p \geqq 0.$$

In particular, for $p = 0$, we have an isomorphism

$$q^{\#}\colon\ S_I\,(P;\,W) \xleftarrow{\cong} \mathrm{Sec}\ \xi.$$

8. *Associated connections.* Assume now that a principal connection is defined in the principal bundle P and let ξ denote the associated vector bundle with respect to a representation of G. Then there is precisely one linear connection, ∇_ξ, in ξ such that the diagram

$$\begin{tikzcd}
\mathrm{Sec}\ \xi \arrow[r, "q^{\#}"] \arrow[d, "\nabla_\xi"'] & S_I\,(P;W) \arrow[d, "H^*\delta"] \\
X^1\,(B;\,\xi) \arrow[r, "q^{\#}"] & X^1{}_B(P;W)
\end{tikzcd}$$

commutes. ∇_ξ is called the *associated linear connection.*

Next, let $\varphi\colon P \to \hat{P}$, $\varrho\colon G \to \hat{G}$ be a homomorphism of principal bundles over the same base. Assume that representations

$$Q\colon G \to GL\,(W), \qquad \hat{Q}\colon \hat{G} \to GL\,(W)$$

are given such that

$$\hat{Q}\,(\varrho g) = Q\,(g) \qquad g \in G.$$

Then there is a unique strong bundle map, ψ, between the associated vector bundles such that the diagram

$$\begin{tikzcd}
P \times W \arrow[rr, "\varphi \times \iota"] \arrow[d, "q"'] & & \hat{P} \times W \arrow[d, "\hat{q}"] \\
P \times_G W \arrow[rr, "\psi"] \arrow[dr, "\varrho"'] & & \hat{P} \times_G W \arrow[dl, "\hat{\varrho}"] \\
& B &
\end{tikzcd}$$

commutes. Moreover, if φ is connection preserving with respect to principal connections in P and $\hat{P}$, then ψ preserves the associated linear connections.

Finally, let $Q: G \to GL(W)$ and $R: G \to GL(V)$ be representations of G and consider the induced representation, S, in the space $L(V; W)$,

$$S(g)\,\varphi = Q(g) \circ \varphi \circ R(g)^{-1} \qquad g \in G \qquad \varphi \in L(V; W).$$

Then there is a natural strong bundle isomorphism

$$\varkappa: L(P \times_G V, P \times_G W) \xrightarrow{\cong} P \times_G L(V; W)$$

(here $L(P \times_G V; P \times_G W)$ denotes is the vector bundle over B whose fibre at x is the space of linear transformations maps from V_x to W_x).

Moreover, if a principal connection is defined in P and if the associated vector bundles are given the associated connections, then $\varkappa$ is connection preserving.

9. *G-structures*. Let M be an n-manifold with tangent bundle $\tau_M = (T_M, \pi_M, M, \mathbf{R}^n)$ and let G be a Lie group. A *G-structure* on M consists of

1) A principal bundle $\mathcal{P} = (P, \pi, M, G)$.
2) A representation $R: G \to GL(\mathbf{R}^n)$.
3) A strong bundle isomorphism $\alpha: P \times_G \mathbf{R}^n \xrightarrow{\cong} \to \tau_M$.

Assume that a G-structure is defined on M and let Q be a representation of G in a vector space W (real or complex). The representations R and Q induce a representation, S, of G in the space $L(\mathbf{R}^n; W)$. Consider the associated vector bundles

$$\begin{aligned} \xi &= (P \times_G \mathbf{R}^n, \varrho_\xi, M, \mathbf{R}^n) \\ \eta &= (P \times_G W, \varrho_\eta, M, W) \\ \zeta &= (P \times_G L(\mathbf{R}^n, W), \varrho_\zeta, M, L(\mathbf{R}^n; W)). \end{aligned}$$

Let $f: \mathbf{R}^n \to W$ be a linear map invariant under the representation S; i.e.,

$$S(g)\,f = f \qquad g \in G.$$

Then f determines a tensor field $\Gamma_f \in X^1(M; \eta)$ given by

$$\Gamma_f(x; h) = ((q_\eta)_z \circ f \circ (q_\xi)_z^{-1} \circ \alpha_x^{-1})\, h \quad x \in M \quad h \in T_x(M) \quad z \in \pi^{-1}(x).$$

(Note that, since f is invariant, the right handsi de of the equetion above is independent of the choice of $z \in \pi^{-1}(x)$.

PROPOSITION I: Assume that a principal connection is defined in P and let ξ, η and ζ have the associated connections. Give τ_M the linear connection induced by α. Then the covariant derivative of Γ_f is zero,

$$\boxed{\nabla_\eta \Gamma_f = 0}$$

PROOF: Let

$$\beta\colon L(\tau_M; \eta) \xrightarrow{\cong} L(\xi; \eta)$$

be the bundle map induced by α,

$$\beta(\varphi_x) = \varphi_x \circ \alpha_x \qquad x \in B.$$

Consider the sequence of isomorphisms

$$\operatorname{Sec} L(\tau_M; \eta) \underset{\cong}{\xrightarrow{\beta_*}} \operatorname{Sec} L(\xi; \eta) \underset{\cong}{\xrightarrow{\varkappa_*}} \operatorname{Sec} \zeta \underset{\cong}{\xrightarrow{q^\#}} S_I(P; L(\mathbf{R}^n; W)).$$

Then f and Γ_f are connected by the relation

$$\boxed{q^\# \varkappa_* \beta_* \Gamma_f = f} \tag{4}$$

as follows from the definitions. (Identify f with the constant function $P \to f$).

The bundle maps β and $\varkappa$ are connection preserving and so we have the commutative diagram

$$\begin{array}{ccccccccc}
X^1(M; \eta) = \operatorname{Sec} L(\tau_M; \eta) & \underset{\cong}{\xrightarrow{\beta_*}} & \operatorname{Sec} L(\xi; \eta) & \underset{\cong}{\xrightarrow{\varkappa_*}} & \operatorname{Sec} \zeta & \underset{\cong}{\xrightarrow{q^\#}} & S_I(P; L(\mathbf{R}^n; W)) \\
\downarrow \nabla_\eta \quad \downarrow \nabla_{L(\tau_M, \eta)} & & \downarrow \nabla_{L(\xi\, \eta(} & & \downarrow \nabla_\zeta & & \downarrow H^*\delta \\
X^2(M; \eta) = X^1(M; L(\tau_m; \eta)) & \underset{\beta_*}{\xrightarrow{\cong}} & X^1(M; L(\xi; \eta)) & \underset{\varkappa_*}{\xrightarrow{\cong}} & X^1(M; \zeta) & \underset{q^\#}{\xrightarrow{\cong}} & X^1{}_B(P; L(\mathbf{R}^n; W)).
\end{array}$$

Here ∇_η denotes the covariant derivative for η, $H^*\delta$ is the covariant exterior derivative for P, and all the other operators are associated linear connections. Hence by (4), $\nabla_\eta \Gamma_f$ corresponds to $H^* \delta f (= 0)$ under the isomorphism $q^\# \varkappa_* \beta_*$.

§ 4. Spin manifolds

10. *Spin structures.* Let M be a connected oriented n-manifold and equip the tangent bundle $\tau_M = (T_M, \pi_M, M, \mathbf{R}^n)$ with a pseudo-Riemannian metric, γ. Recall from sec. 2 the homomorphism

$$\mathrm{Ad}\colon \mathrm{Spin}(n) \to SO(n).$$

A *spin structure* on M is a G-structure with $G = \mathrm{Spin}(n)$ and $\varrho = Ad$. Thus a spin structure on M consists of a principal bundle $P = (P, \pi, M, \mathrm{Spin}(n))$ and a strong bundle isomorphism

$$\alpha\colon P \times_{Spin(n)} \mathbf{R}^n \xrightarrow{\cong} T_M.$$

An n-manifold need not admit a spin structure. Indeed, if (M, γ) is an orientable *Riemannian manifold* (γ positive definite) or an orientable *Lorentz manifold* (i. e., γ has signature $n - 2$) with orientable time cone, then M admits a spin structure if and only if its second Stiefel-Whitney class is zero. (cf. [11]). If M is a non-compact Lorentz 4-manifold of this type, this condition is equivalent to the parallelizeability of M. (cf. [5]).

A manifold together with a spin structure is called a *spin manifold.* Let M be a spin manifold and consider the principal bundle of positive orthonormal n-frames, $\hat{P} = (\hat{P}, \hat{\pi}, M, SO(n))$. Then there is a natural homomorphism of principal bundles, $p\colon P \to \hat{P}$, Ad: Spin $(n) \to SO(n)$ which makes P into a double covering of $\hat{P}$. Moreover we have the commutative diagram

(5)
$$\begin{tikzcd}
P \times \mathbf{R}^n \arrow[r, "q"] \arrow[dd, "p \times L"] & P \times_{Spin(n)} \mathbf{R}^n \arrow[dr, "\alpha", "\cong"'] \arrow[dd] & \\
& & T_M \\
\hat{P} \times \mathbf{R}^n \arrow[r, "\hat{q}"] & \hat{P} \times_{SO(n)} \mathbf{R}^n \arrow[ur, "\hat{\alpha}", "\cong"'] &
\end{tikzcd}$$

where $\hat{\alpha}$ is the canonical isomorphism.

11. *Spinor fields.* Let M be a spin manifold and let Q be a representation of Spin (n) in a vector space W (real or complex). A *spinor field* on M (*with respect to* Q) is a cross-section in the associated vector bundle

$$\xi = (P \times_{Spin(n)} W, \varrho, M, W).$$

More generally, a *tensor spinor fiəld of degree p* on M is a covariant tensor field of degree p on M with values in ξ. Thus a spinor field is a tensor spinor field of degree zero.

If

$$q : P \times W \to P \times_{Spin(n)} W$$

is the principal map for the representation Q, then

$$q^{\#} : X^p{}_B (P; W) \xleftarrow{\cong} \mathbf{X}^p (M; \xi)$$

is an isomorphism between tensor spinor fields of degree p on M and basic tensor fields of the same degree on P with values in W.

12. *Connections in a spinor bundle.* Let (M, γ, α) be a spin manifold. Let ∇_M be the Levi-Civita connection in τ_M corresponding to the pseudo-Riemannian metric γ. Then there is a unique principal connection in $\hat{P}$ which induces ∇_M via $\hat{\alpha}$ (cf. sec. 10). Since the map $p; P \to \hat{P}$ is a convering projection, the connection in $\hat{P}$ induces a principal connection in P. This connection, finally, determines linear connections in the associated vector bundles. In particular, we have an induced linear connection in $P \times_{Spin(n)} \mathbf{R}^n$ and the isomorphism α is connection preserving (cf. sec. 8).

13. *The Dirac field of a representation.* Again let (M, γ, α) be a spin manifold and consider the Clifford algebra $\mathcal{C}(\mathbf{R}^n)$ over the underlying pseudo-Euclidean space $\mathbf{R}^n$. Let Φ be a representation of $\mathcal{C}(\mathbf{R}^n)$ in a vector space W (real or complex) and consider the corresponding representation (obtained by restricting Φ)

$$Q\colon \mathrm{Spin}\,(n) \to GL\,(W).$$

Then we have the associated vector bundle

$$\eta = (P \times_{Spin(n)} W, \varrho, M, W).$$

On the other hand, Φ restricts to a linear map

$$f\colon \mathbf{R}^n \to \mathbf{L}\,(W; W)$$

(where $L\,(W; W)$ deuotes the space of liuear maps $W \to W$)

and a simple computation shows that

$$(f \circ Ad\,(g))\, y = Q\,(g) \circ f\,(y) \circ Q\,(g)^{-1} \quad g \in \mathrm{Spin}\,(n) \quad y \in \mathbf{R}^n.$$

Thus we may regard f as an invariant element of $L(\mathbf{R}^n; L(W; W))$. It follows that f determines a spinor tensor field $\Gamma_f \in X^1(M; L(\eta; \eta))$ as described in sec. 9 (note that $L(\eta; \eta)$ plays the role of η in sec. 9).

DEFINITION: Γ_f is called the *Dirac field* associated with the representation Φ.

The Dirac field is given explicitly by

$$\Gamma_f(x; h) = (q_\eta)_z \circ f(q_z^{-1} \circ \alpha_x^{-1}(h)) \circ (q_\eta)_z^{-1}$$
$$x \in M,\ z \in \pi^{-1}(x),\ h \in T_x(M)$$

where $q_\eta\colon P \times W \to P \times_{Spin(n)} W$ and $q\colon P \times \mathbf{R}^n \to P \times_{Spin(n)} \mathbf{R}^n$ are the respective principal maps.

PROPOSITION II: The Dirac field has the following properties:

1) $\Gamma_f(x; h_1) \circ \Gamma_f(x; h_2) + \Gamma_f(x; h_2) \circ \Gamma_f(x; h_1) = 2 < h_1, h_2 >$

$$x \in M, \qquad h_1, h_2 \in T_x(M).$$

2) If ∇_M is the Levi-Civita connection in τ_M and P and $P \times_{Spin(n)} W$ are given the induced connections, then the covariant derivative, ∇, in $X(M; L_\eta)$ satisfies

$$\boxed{\nabla \Gamma_f = 0}$$

PROOF: 1) Set

$$y_j = (q_z^{-1} \alpha_x^{-1})\, h_j \qquad j = 1,2.$$

Then

$$< y_1, y_2 > = < h_1, h_2 >.$$

It follows that

$$\begin{aligned} &\Gamma_f(x, h_1) \circ \Gamma_f(x, h_2) + \Gamma_f(x, h_2) \circ \Gamma_f(x, h_1) = \\ &= (q_\eta)_z [f(y_1) \circ f(y_2) + f(y_2) \circ f(y_1)] \circ (q_\eta)_z^{-1} = \\ &= (q_\eta)_z \Phi(y_1 y_2 + y_2 y_1) \circ (q_\eta)_z^{-1} = \\ &= (q_\eta)_z \Phi(2 < y_1, y_2] > e) \circ (q_\eta)_z^{-1} = \\ &= 2 < y_1, y_2 > \iota = 2 < h_1, h_2 > \iota. \end{aligned}$$

2) This is an immediate consequence of Proposition I, sec.9.

14. *The Dirac operator. Let* (M, γ, α) be a spin manifold and let Φ be a representation of $\mathfrak{C}(\mathbf{R}^n)$ in a vector space W (real or complex). Consider the associated vector bundle

$$\eta = (P \times_{Spin(n)} W, \varrho, M, W).$$

Recall from sec. 12 that the Levi-Civita connection in τ_M determines a linear connection in η, with covariant derivative ∇.

We shall use Φ and ∇ to define a linear operator

$$D\colon \operatorname{Sec} \eta \to \operatorname{Sec} \eta$$

to be called the *Dirac operator.*

Let $\Gamma_f \in X^1(M; L_\eta)$ be the Dirac field associated with the representation Φ. Then, if $\sigma \in \operatorname{Sec} \eta$, we can form $\Gamma_f(\nabla\sigma) \in X^2(M; \eta)$ (cf. end of sec. 3). Applying the operator $i(g)$ (g the contravariant metric tensor) yields a cross-section in η. This cross-section is denoted by $D\sigma$,

$$\boxed{D\sigma = i(g)\, \Gamma_f(\nabla\sigma)} \tag{6}$$

The linear operator D; $\operatorname{Sec} \eta \to \operatorname{Sec} \eta$ so obtained is called the *Dirac operator* associated with the representation Φ.

REMARK: If M is a Lorentz manifold ($n = 4$) with spin structure and Φ is the Dirac representation of $\mathfrak{C}(\mathbf{R}^4)$ in the complex vector space $\mathbf{C}^4$, the Dirac operator defined above coincides with the classical Dirac operator.

15. *The square of the Dirac operator.* Consider the operator

$$D^2\colon \operatorname{Sec} \eta \to \operatorname{Sec} \eta.$$

To obtain an explicit expression for this operator, define the *commutator* $[\Gamma_f, \Gamma_f] \in X^2(M; L_\eta)$ by

$$[\Gamma_f, \Gamma_f](x; h_1, h_2) = \Gamma_f(x; h_1) \circ \Gamma_f(x; h_2) - \Gamma_f(x; h_2) \circ \Gamma_f(x, h_1)$$

$$x \in M, \qquad h_1, h_2 \in T_x(M).$$

On the other hand, the curvature tensor, R, for the induced connection in η is also an element of $X^2(M; L_\eta)$. Hence we can form $[\Gamma_f, \Gamma_f] \circ R \in X^4(M; L_\eta)$ (cf. sec 3). Finally, let $\tilde{g}$ denote the contravariant tensor field on M of degree four given by

$$\tilde{g}(x; h_1, h_2, h_3, h_4) = g(x; h_1, h_3)\, g(x; h_2, h_4).$$

$$x \in M,\ h_j \in T_x(M).$$

PROPOSITION III: The square of the Dirac operator is given by

$$\boxed{D^2 = \triangle - 1/4\, i(\tilde{g})\, ([\Gamma_f, \Gamma_f] \circ R)}$$

PROOF: Let $e_1, \ldots, e_n$ be a local n-frame on M and let $e^1, \ldots, e^n$ be the n-frame. Determined by

$$\langle e^v, e_p \rangle = \delta^v_\mu \qquad (\nu, \mu = 1 \ldots n).$$

Then

$$g = \sum_v e_v \otimes e^v$$

and

$$\tilde{g} = \sum_{\mu, v} e_\mu \otimes e_v \otimes e^\mu \otimes e^v.$$

Set

$$\nabla_\nu = i(e_v) \circ \nabla$$

and

$$\Gamma_v = i(e^v)\, \Gamma_f.$$

Now formulae (1), (0) and Proposition II, 2) imply that

$$\begin{aligned} D^2 \sigma\, i(g)\, \Gamma_f \nabla (i(g)\, \Gamma_f \nabla \sigma) &= \\ = \sum_v \Gamma_v (\nabla_v (i(g)\, \Gamma_f \nabla \sigma)) &= \\ = \sum_v \Gamma_v (i(g)\, \Gamma_f \nabla_v \nabla \sigma)). \end{aligned}$$

This yields

$$D^2 \sigma = \sum_{v, \mu} \Gamma_v \Gamma_\mu\, i(e_\mu)\, \nabla_v \nabla \sigma = \sum_{v, \mu} \Gamma_v\, \Gamma_\mu\, i(e_\mu)\, i(e_v)\, \nabla^2 \sigma \tag{7}$$

Now write

$$\Gamma_\nu \Gamma_\mu = 1/2\,(\Gamma_\nu \circ \Gamma_\mu + \Gamma_\mu \dot{\circ} \Gamma_\nu) + 1/2\,[\Gamma_\nu, \Gamma_\mu]$$

and observe that, in view of Proposition II, 1),

$$\Gamma_\nu \circ \Gamma_\mu + \Gamma_\nu \dot{\circ} \Gamma_\mu = 2 \langle e^\nu, e^\mu \rangle\, \iota.$$

It follows that

$$\Gamma_\nu \circ \Gamma_\mu = \langle e^\nu, e^\mu \rangle\, \iota + 1/2\,[\Gamma_\nu, \Gamma_\mu]. \tag{8}$$

Combining formulae (7) and (8) yields

$$\begin{aligned} D^2 \sigma &= \sum_{\nu,\mu} \langle e^\nu, e^\mu \rangle\, i(e_\mu)\, i(e_\nu) \nabla \sigma + \\ &\quad + 1/2\,[\Gamma_\nu, \Gamma_\mu]\, i(e_\mu)\, i(e_\nu) \nabla^2 \sigma = \\ &= \triangle \sigma - 1/4\,[\Gamma_\nu, \Gamma_\mu]\,((i(e_\nu)\, i(e_\mu) - i(e_\mu)\, i(e_\nu)) \nabla^2 \sigma) \end{aligned}$$

Finally, applying formula (2) we obtain

$$\begin{aligned} D^2 \sigma &= \triangle \sigma - 1/4\,[\Gamma_\nu, \Gamma_\mu]\, R(e_\nu, e_\mu)\, \sigma = \\ &= \triangle \sigma - \frac{1}{4} \sum_{\nu,\mu} i(e_\mu \otimes e_\nu \otimes e^\mu \otimes e^\nu)\,([\Gamma_f, \Gamma_f] \circ R)\, \sigma = \\ &= \triangle \sigma - \frac{1}{4}\, i(\tilde{g})\,([\Gamma_f, \Gamma_f] \circ R)\, \sigma \end{aligned}$$

which completes the proof.

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