FISCHER CLASSES IN FINITE GROUPS

by

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ABSTRACT. — In [2] we define the concept of a «π-satured Fitting class». The present work investigates a particular type of such classes, those we have called «Fischer classes relative to π», which allow us to present a version of Fischer's theorem valid for π-solvable groups. We have followed the type of proof of the cited Hartley Theorem, which is essentially based on the behaviour of the injectors of a group G with respect to the complements of self-centralizing factors of G, with the necessary modifications not disposing of the hypothesis of solubility.

1. — NOTATION AND PRELIMINARIES. — We will represent by π a set of prime numbers and by π', the set of prime numbers which do not belong to π.

All the groups considered in the present work are assumed to be finite and π-solvable.

By the statement «χ is a class of groups», we will understand a subclass of the π-solvable groups which contains all the isomorphic copies of its members and all subgroups of order 1. In particullar $\mathcal{U}_{\pi}$, is the class of π'-groups.

A class of groups $\mathcal{F}$ is called a Fitting class if:

$$N \trianglelefteq G \in \mathcal{F} \Rightarrow N \in \mathcal{F}$$
$$G = N_1 \cdot N_2 \trianglerighteq N_1, N_2 \in \mathcal{F} \Rightarrow G \in \mathcal{F}$$

It follows that if $\mathcal{F}$ is a Fitting class, each group G possesses a single normal maximal $\mathcal{F}$-subgroup, called the $\mathcal{F}$-radical of G and which is represented by $G_{\mathcal{F}}$. Here, in speaking of an $\mathcal{F}$-subgroup, we simply mean a subgroup belonging to $\mathcal{F}$.

If $\mathcal{H}$ and $\mathcal{K}$ are Fitting classes, the class product $\mathcal{H}\cdot\mathcal{K} = \{G \mid G/G_H \in \mathcal{K}\}$ is also a Fitting class and such operation is associative.

Let $\mathcal{F}$ be a Fitting class. A subgroup V of G is said to be an $\mathcal{F}$-injector of G if $V \cap N$ is $\mathcal{F}$-maximal in N for any subnormal subgroup N of G. An $\mathcal{F}$-subgroup V of G is a Fischer $\mathcal{F}$-subgroup of G, if V contains every $\mathcal{F}$-subgroup of G wich is normalised by V

A Fitting class $\mathcal{F}$ is said to be π-saturated (cf. [2]) if $G \in \mathcal{F}$ whenever $O^{\pi'}(G) \in \mathcal{F}$, representing by $O^{\pi'}(G)$ the smallest normal subgroup N of G such that G/N is a π'-group.

Finally, as is usual in the theory of groups, for the two subgroups H, K of a group G, we represent by $[H, K]$ the subgroup conmutator of H and K, and if A is a subset of G, $\{A\}$ represents the subgroup generated by A, and $\{A^G\}$ the smallest normal subgroup of G which contains A.

2. — Fischer's theorem. —

(2.1) *Definition*. For a class $\mathcal{F}$ of groups, we say it is a «Fischer class relative to π» when:

i) is a π-saturated Fitting class

ii) $N \leq H \leq G \in \mathcal{F}$, $N \trianglelefteq G$ and H/N p-group ($p \in \pi$)

implies $H \in \mathcal{F}$.
(Note that if π is the set of all the prime numbers, the concepts of the Fischer class and the Fischer class relative to π, coincide, as do those of solvable and π-solvable)

We will inmediately recall some results concerning π-saturated Fitting classes, which we will need later on, whose proofs can be seen in [2].

(2.2) Lemma. Let $\mathcal{F}$ be a Fitting class. The following propositions are equivalent

i) $\mathcal{F}$ is π-saturated

ii) $\mathcal{F} = \mathcal{F}\cdot\mathcal{U}_\pi$.

Let $\mathcal{F}$ be a π-saturated Fitting class. It is verified:

(2.3) THEOREM. Every group G possesses $\mathcal{F}$-injectors which are of π-index and conjugate in G.

(2.4) LEMMA.

A. (1) If $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n \trianglelefteq G$ is a series of a group G such that each of its groups factors G_{i+1}/G_i $(i = 0, 1, \dots n - 1)$ is a π'-group or a nilpotent group, then, V is an $\mathcal{F}$-injector of G, if, and only if $V \cap G_i$ is a maximal $\mathcal{F}$-subgroup in G_i for $1 \leq i \leq n$.

(2) Let M be a normal subgroup of a group G and V an $\mathcal{F}$-subgroup of G, such that $M \cap V$ is an $\mathcal{F}$-injector of M and $G = M \cdot V$. Then, V is an $\mathcal{F}$-injector of G.

B. Let V be an $\mathcal{F}$-injector of a group G, it is verified

(1) if $V \leq H \leq G$, V is also an $\mathcal{F}$-injector of H

(2) V is a pronormal subgroup in G

(3) if $H \trianglelefteq G$, $V \cap H$ is an $\mathcal{F}$-injector of H. Moreover $G = H \cdot N_G(V \cap H)$

(4) V covers or avoids each chief factor of G.

The result to which we refer in the introduction is:

THEOREM 1. If $\mathcal{F}$ is a Fischer class relative to π, then the $\mathcal{F}$-injectors of any π-solvable group G and the Fischer $\mathcal{F}$-subgroups of G which contain a Hall π'-subgroup of G, are the same conjugate class.

We will not prove this directly, but will only deduce it as a consequence of another (theorem) which we will see later on.

Firstly, we will detail a series of results, proved by Hartley for solvable groups and Fischer classes, which contine being valid in the class of π-solvable groups without requiring more than that $\mathcal{F}$ be a Fischer class relative to π.

(2.5) DEFINITION. Let $\mathcal{F}$ be a π-saturated Fitting class, H/K a chief factor of a group G, and V an $\mathcal{F}$-injector of G, It is said that H/K is «$\mathcal{F}$-covered» if V covers H/K, and «$\mathcal{F}$-avoided» if $V \cap H \leq K$.

By the conjugacy of the $\mathcal{F}$-injectors of G, the preceding definition is independient of the $\mathcal{F}$-injector V considered. Moreover, as every $\mathcal{F}$-injector V of G contains a Hall π'-subgroup of G, the chief factors of G which are π'-groups are always $\mathcal{F}$-covered. The possibility of being $\mathcal{F}$-avoided remains limited therefore, by the chief π-factors of G (p-factors, $p \in \pi$).

(2.6) LEMMA. Let $\mathcal{F}$ be a Fischer class relative to π, V an $\mathcal{F}$-injector of a group G, P a Sylow p-subgroup ($p \in \pi$) of V, and H/K an $\mathcal{F}$-avoided factor of G of order a power of p. Then $[P, H] \leq K$.

Proof: cf. [1] Lemma 2, page 196.

(2.7) LEMMA. Let G be a group and V a subgroup of G such that for any normal subgroup N of G, and any $x \in G$, $N \cap V$ and $(N \cap V)^x$ are conjugate in N. Then, if P is a Sylow p-subgroup of V, the following propositions are equivalent.

i) P is a Sylow p-subgroup of $\{P^G\}$

ii) If $K \leq H \leq \{P^G\}$ and H/K is a chief p-factor of G avoided by V, then $[H, P] \leq K$

Proof: cf [1] Lemma 3 page 196.

COROLLARY. Let $\mathcal{F}$ be a Fischer class relative to π, and V an $\mathcal{F}$-injector of a group G. Then, V is p-normally embedded in G for all primes $p \in \pi$.

(2.8) LEMMA. Let G be a group and p a prime number belonging to π. Let V be a p-normally embedded subgroup of G and H/K a p-factor of G verifying

i) it is complemented

ii) it is avoided by V

iii) $C_G(H/K) \leq H$

then every complement for H/K in G contains a conjugate of V.

Proof: cf [1] Lemma 4 page 198.

(2.9) LEMMA. Let V be a pronormal subgroup of G and H/K a chief factor of G, centralised by V. Then, the subgroup $N = N_G(V)$ covers H/K.

Proof: cf [1] Lemma 5 page 198.

(2.10) DEFINITION. Let $\mathcal{F}$ be a π-saturated Fitting class and G a group. A factor H/K complemented in G, is said to be $\mathcal{F}$-complemented if every complement for H/K contains an $\mathcal{F}$-injector of G.

THEOREM 1*. Let $\mathcal{F}$ be a π-saturated Fitting class and G a group. Assume that a p-factor $(p \in \pi)$ H/K of G is $\mathcal{F}$ complemented in $L \leq G$, whenever:

i) H/K is a p-factor of $L \leq G$ and $C_L(H/K) \leq H$

ii) H/K is complemented in L

iii) H/K is $\mathcal{F}$-avoided in L

Then, every Fischer $\mathcal{F}$-subgroup of G, which contains a Hall π'-subgroup of G, is an $\mathcal{F}$-injector of G.

NOTE

Theorem 1 is an inmediate consequence of this. On account of the corollary of Lemma (2.7), if $\mathcal{F}$ is a Fischer class relative to π, every $\mathcal{F}$-injector of L is p-normally embedded in L for each prime p of π, and, by the Lemma (2.8) every factor of G of the type considered is $\mathcal{F}$-complemented in L.

Proof: We will assume that the theorem is false and consider a group G counter-exemple of minimal order. Let V be an $\mathcal{F}$-injector of G and E a Fischer $\mathcal{F}$-subgroup of G such that it contains a Hall π'-subgroup of G, but is not an $\mathcal{F}$-injector of G.

We will call the $\mathcal{F}$-radical of G, F. Necessarily F is a proper subgroup of V, whereas, in the contrary case, $E = V$. Because F is not an F-injector of G, we can consider the subgroup H of G, given by:

a) $F \trianglelefteq H \trianglelefteq G$

b) H is minimal with respect to the condition $F \leq V \cap H$

We note $V_0 = V \cap H$ and $N = N_G(V_0)$. We observe that because F is a proper subgroup of $V \cap H$, $V \leq N < G$ holds. Moreover by (2.4) $B - 3$, $G = H \cdot N$.

We will set out the separate steps of the proof.

(A) *$E \cap H$ is not an $\mathcal{F}$-injector of H*

If $E \cap H$ is an $\mathcal{F}$-injector of H, then it is conjugate to V_0 in H, and we may assume without loss of generaly that $E \cap H = V_0$ But tehn $E \leq N < G$ and by the inductive hypothesis, E and V would be conjugate in N in contradiction to G's being a minimal counter-example.

We will now consider a normal subgroup K of G such that $F \trianglelefteq K \triangleleft H$ and H/K is a chief factor of G.

(B) 1. — *H/K is a p-elemental ($p \in \pi$) abelian group.*

2. — *K is the unique maximal member of the set of normal subgroups of G lying between F and H.*

By the choice of K and G being π-solvable, H/K is either a π-group or a π'-group. We will assume that H/K is a π'-group. Since $E \cap H$ contains a Hall π'-subgroup of H, $H = (E \cap H) K$ is verified, and given that $(E \cap H) \cap K = E \cap K = F = V \cap K$ is an $\mathcal{F}$-injector of K. We coclude by (2.4) $A - 2$, that $E \cap H$ is an $\mathcal{F}$-injector of H. Then, by (A), H/K is necessarily a p-elemental abelian group ($p \in \pi$) (H/F is also a π-group, since $\mathcal{F}$ is a π-saturated Fitting class and $V \cap K = F$ is an F-injector of K).

We will now assume that at least two elements K_1, K_2, exist, maximal in the set of normal subgroups K of G such that $F \leq K < H$. Because H/K_i is an abelian group ($i = 1, 2$) and $V_0 (K_1 \cap K_2) / K_1 \cap K_2 \leq H/K_1 \cap K_2$, the subgroup $V_0 (K_1 \cap K_2)$ of G is normalised by H, and since $N \leq N_G (V_0 (K_1 \cap K_2))$; $V_0 (K_1 \cap K_2)$ is a normal subgroup of $HN = G$. By the choice of H made, $H = V_0 (K_1 \cap K_2)$ is verified and thus $K_i = K_i \cap H = K_i \cap (K_1 \cap K_2) V_0 = (K_1 \cap K_2) \cdot F = K_1 \cap K_2$ ($i = 1, 2$), that is to say, $K_1 = K_2$ as we wanted to prove.

(C) $HE = G$

If $HE < G$, then E is an $\mathcal{F}$-injector of HE and thus $E \cap H$ is an $\mathcal{F}$-injector of H, in contradiction to the result proved in (A).

(D) *H/K is unique minimal, normal subgroup of G/K*

Let J/K be a normal minimal subgroup of G/K, distinct from H/K, then, J/K. $H/K = H_1/K$ is a normal subgroup of G/K which properly contains H/K. By (C), $H_1 = H_1 \cap HE = H(E \cap H_1)$. We will consider the subgroup H_1^* of H_1, defined by $H_1^* = (E \cap H_1) \cdot K$. We will prove that H_1^* is a normal subgroup of G. Since G is π-solvable, two possibilities exist for the order of J/K.

a) If J/K is a p-elemental abelian group $(p \in \pi)$, then, H_1/K is an abelian normal subgroup of G/K, so H_1^*/K is normalised by H/K; and since H_1^*/K is also normalised by EK/K because $(E \cap H_1)$. K is by E, we conclude $H_1^*/K \trianglelefteq EK/K$. $H/K = G/K$ and therefore $H_1^* \trianglelefteq G$.

b) If J/K is a π'-group, then H_1/K is π decomposable. By the same reasoning as in the preceding case, H_1^*/K is normalised by EK/K. As E by the hypothesis, contains a Hall π'-subgroup of G H_1^*/K contains a Hall π'-subgroup of H_1/K, then $J/K \trianglelefteq H_1^*/K$, and therefore $J \trianglelefteq H_1^*$, thus verifying $H_1^*/J \leq H_1/J \cong H_1/K \,/\, J/K \cong H/K$, then $H_1^* \trianglelefteq H_1$, because H/K is abelian. The two previous assertions allow us to state that H_1^*/K is normalised by $(EK/K) \cdot (H_1/K) = G/K$ and therefore that $H_1^* \trianglelefteq G$, in this case also.

Next we will prove that $E \cap H_1^*$ is an $\mathcal{F}$-injector of H_1^* (we note in the first place that H_1^*/K is distinct from the trivial subgroup of G/K since in the contrary case $E \cap H_1 \leq K$, and thus $H_1 = H$, in contradiction to the previous observation). Due to the construction of the subgroup H_1^*, the $\mathcal{F}$-subgroup $E \cap H_1^*$ and $E \cap H_1$, coincide, thus we can state $H_1^* = (E \cap H_1^*) \cdot K$. Since $(E \cap H_1^*) \cap K = E \cap K = F$ and F is an $\mathcal{F}$-injector of K, we conclude from (2.4) $A - 2$ that $E \cap H_1^*$ is effectlively an $\mathcal{F}$-injector of H_1^*.

Due to the conjugacy of $\mathcal{F}$-injectors we can assume $V_1^* = V \cap H_1^* = E \cap H_1^*$, thus disposing of the following sequence of $\mathcal{F}$-subgroups.

$F = E \cap K \leq E \cap H \leq E \cap H_1 = E \cap H_1^* = V_1^*$, being some of the preceding contents necessarily strict because H is a proper subgroup of H_1, and therefore $N_1^* = N_G(V_1^*)$ is also a proper subgroup of G. Because H_1^*, is a subgroup normal in G, E and V are subgroups of N_1^*, then by the inductive hypothesis they are conjugate in N_1^*, in contradiction to the fact that G is a minimal counter-example, in conclusión, the affirmation (D) remains proved.

(E) $G = KN$, *equivalently* N *covers* G/K

According to the choice of H made, V doesn't avoid the thief factor H/K of G, then necessarily $(V \cap H)K = H$ and therefore $G = HN = (V \cap H)KN = KN$.

As $F < N < G$; the previous result allows us to assert that the set of thief factors of G, between F and K, which are not covered by N, is not empty. Let A/B be one of these factors, with A of maximal order. ($\mathcal{F}$ π-saturated Fitting class implies that A/B is a p-elemental abelian group ($p \in \pi$). Because $G = AN$, the subgroup $B(A \cap N)$ is normal in G, then by the definition of A/B $A \cap N \leq B$, and therefore the subgroup BN complements A/B in G.

(F) $A = C_G(A/B)$

Let C be the normal subgroup of G given by $C = C_G(A/B)$. Necessarily $C \cap H < H$, since if $H \leq C$ is verified, V_0 would centralise A/B and as V_0 is pronormal in G, then, from the Lemma (2.9) N would cover A/B in contradiction to the choice of such a chief factor of G. Because A/B is abelian, $F \leq A \leq C$, then $C \cap H$ is a normal subgroup of G lying between F and H, which by (B) verifies $C \cap H \leq K$, that is to say, $H/K \cap CK/K = \{1\}$ and by (D), is now $C \leq K$.

Then $F \leq A \leq C \leq K$ and $C \cap D \trianglelefteq DA = G$, that is, $C/C \cap D$ is a factor of G between F and K which is trivially avoided by D, so, a fortiori by N. Due to the choice of A/B made, we conclude $C = C_G(A/B) = A$.

We will now consider the subgroup L of G, given by $L = K \cdot E$.

As $K \trianglelefteq G$ and $F = E \cap K$ is an $\mathcal{F}$-injector of K, it holds from (2.4) A, 2 that E is an $\mathcal{F}$-injector of L, then by the inductive hypothesis, L is a proper subgroup of G. In such subgroup L of G, A/B is a self-centralizing, $\mathcal{F}$-avoided factor of the power order of a prime number p ($p \in \pi$), complemented by $D \cap L$, hence, following the Lemma (2.8) $D \cap L$ contains an $\mathcal{F}$-injector of L. Without loss of generality we can assume following the inductive hypothesis that $E \leq D \cap L$.

Since $V \leq N \leq BN = D$, it holds that E and V are F-injectors of $D < G$, and are then conjugate in D, the final contradiction which completes the proof,

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