A SPECIAL CLASS OF TRIANGULAR ARRAYS

by

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1. Introduction. In various combinatorial problems one encounters triangular arrays

(1.1) $A = (A_{nk}) \qquad (n = 0, 1, 2, \dots ; k = 0, 1, \dots, n)$

such that

(1.2) $A_{nn} \neq 0 \qquad (n = 0, 1, 2, \dots).$

It follows from (1.2) that there exists an array

(1.3) $B = (B_{nk}) \qquad (n = 0, 1, 2, \dots; k = 0, 1, \dots, n)$

such that

$$\sum_{k=j}^{n} A_{nk} B_{kj} = \delta_{nj} = \sum_{k=j}^{n} B_{nk} A_{kj} . \tag{1.4}$$

Moreover (B_{nk}) is uniquely determined by (A_{nk}).

A familiar example is

$$A = \left[\binom{n}{k}\right], \quad B = \left[(-1)^{n-k}\binom{n}{k}\right]. \tag{1.5}$$

Indeed

$$\sum_{k=j}^{n} \binom{n}{k} \cdot (-1)^{k-j} \binom{k}{j} = \binom{n}{j} \sum_{k=j}^{n} (-1)^{k-j} \binom{n-j}{k-j} = \delta_{n,j}$$

and similarly for the second part of (1.4). Alternatively we may replace (1.5) by

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$$(1.6)\qquad A = \left[(-1)^k \binom{n}{k}\right] = B,$$

which is not essentially different.

We shall consider the special class of arrays (1.1) defined by

$$(1.7)\qquad \Phi\,(xf(z)) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} A_{kn}\, x^k,$$

where

$$(1.8)\qquad \Phi\,(z) = 1 + \sum_{n=1}^{\infty} a_n \frac{z_n}{n!}$$

and

$$(1.9)\qquad f(z) = \sum_{n=1}^{\infty} b_n \frac{z_n}{n!} \qquad (b_1 = 1).$$

The condition $b_1 = 1$ can be replaced by $b_1 \neq 0$; however for simplicity we take $b_1 = 1$. Note that in (1.7) we have

$$A_{n0} = 0 \qquad (n = 1, 2, 3, \ldots)\,.$$

It follows from (1.9) that there exists a unique

$$(1.10)\qquad g(z) = \sum_{n=1}^{\infty} c_n \frac{z^n}{n!} \qquad (c_1 = 1)$$

such that

$$f(g(z)) = g(f(z)) = z,$$

that is, $g(z)$ is the inverse of $f(z)$. All series may be thought of as formal power series.

We may put

$$(1.11)\qquad \Phi\,(x(g(z)) = 1 + \sum_{n=1}^{\infty} \frac{z}{n!} \sum_{k=1}^{n} B_{nk}\, x^k.$$

It is then easy to show that the arrays (A_{nk}), (B_{nk}), where $B_{n0} = 0$ $(n = 1, 2, 3, \ldots)$, satisfy (1.4). Thus in the case of arrays defined by (1.7), (1.8) and (1.9) we have a simple method for constructing the inverse array.

It will also be seen that there is no essential loss of generality in assuming that $\Phi(z) = e^z$.

A simple example is furnished by

$$\Phi(z) = e^z,\ f(z) = \frac{z}{1-z},\ g(z) = \frac{z}{1+z}.$$

Thus

$$\begin{aligned} e^{xz/(1-z)} &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!}(1-z)^{-k} \\ &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} \sum_{j=0}^{\infty} \binom{j+k-1}{j} z^j \\ &= 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \frac{x^k}{k!}\binom{n-1}{k-1}, \\ e^{xz/(z+1)} &= 1 + \sum_{n=1}^{\infty} (-1)^n z^n \sum_{k=1}^{n} (-1)^k \frac{x^k}{k!}\binom{n-1}{k-1}. \end{aligned}$$

Hence we have

$$A_{n,k} = \frac{n!}{k!}\binom{n-1}{k-1},\ B_{n,k} = (-1)^{n-k}\frac{n!}{k!}\binom{n-1}{k-1}. \tag{1.12}$$

It is easy to verify that $(A_{n,k})$, $(B_{n,k})$ are an inverse pair.

The basic properties of inverse pairs of arrays are gven in § 2. In § 3 we briefly recount the properties of the Stirling arrays $(S(n,k))$, $((-1)^{n-k}S_1(n,k))$, where $S_1(n,k)$, $S(n,k)$ denote the Stirling numbers of the first and second kind, respectively. They may be defineed by means of

$$\begin{cases} x(x-1)\dots(x-n+1) = \sum_{k=0}^{n} (-1)^{n-k} S_1(n,k)\, x^k \\ x^n = \sum_{k=0}^{n} S(n,k)\, x(x-1)\dots(x-k+1); \end{cases}$$

in this notation both $S_1(n,k)$ and $S(n,k)$ are non-negative integers. Since $S_1(n, n-k)$ and $S(n, n-2k)$ are polynomials in n of degree $2k$, we have the associated arrays $(S'_1(n,k))$, $(S'(n,k))$ defined by [1]

$$(1.13)\quad \begin{cases} S_1(n, n-k) = \sum_{j=0}^{k=1} S'_1(k, j)\binom{n}{2k-j} \\ S(n, n-k) = \sum_{j=0}^{k-1} S'(k, j)\binom{u}{2k-j} \end{cases}$$

It is well known that $S_1(n,k)$ is equal to the number of permutations of $Z_n = \{1,2,\ldots, n\}$ with k cycles while $S(n, k)$ is the number of partitions of the set Z_n into k blocks. This suggests the enumeration of permutations with a given number of cycles all of which are of odd cardinality or all of even cardinality. Secondly there is the corresponding problem for partition of Z_n. In § 4 we discuss in some detail the number $T_1(n, k)$, equal to the number of permutations of Z_n with k cycles all of which have odd cardinality. The generating function for $T_1(n, k)$ is

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} T_1(n, k)\, x = \exp\left\{x/2 \log \frac{1+z}{1-z}\right\} = \left(\frac{1+z}{1-z}\right)^{x/2}.$$

This implies the recurrence

$$T(n+1, k) = T(n, k-1) + n(n-1)\, T(n-1, k).$$

Analogous to the first of (1.13) we have

$$T_1(n, n-2k) = \sum_{j=0}^{k=1} T'_1(k, j)\binom{n}{3k-j};$$

the $T'_1(k, j)$ have a simple combinatorial meaning (§ 6).

In § 5 we discuss the inverse of the array $(T_1(n, k))$. This is defined by means of

$$\exp(x \tanh z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} T(n, k)\, x^k.$$

The $T(n, k)$ satisfy

$$T(n+1, k) = T(n, k-1) - k(k+1)\, T(n, k+1).$$

Also we have

$$T(n, n-2k) = \sum_{j=0}^{k-1} T'(k, j)\binom{n}{3k-j}.$$

A combinatorial interpretation of $T(n, k)$ is not known. However if we put

$$(-1)^{1/2(n-k)}\, T(n,k) = \frac{1}{k!}\, T_n^{(k)},$$

it is known [3] that $T_n^{(k)}$ is equal to the number of up-down sequences of length $n + k$ with k «infinite» elements.

In § 7 we discuss the array $(U(n, k))$, where $U(n, k)$ is equal to the number of partitions of Z_n into k blocks of odd cardinality:

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!}\, U(n, k)\, x^k = \exp\,(x \sinh z).$$

This enumerant was introduced in [2]. It satisfies the recurrence

$$U(n + 2, k) = U(n, k - 2) + k^2\, U(n, k)$$

or, preferably,

$$\begin{cases} U(2n + 2, 2k) = U(2n, 2k - 2) + (2k)^2\, U(2n, 2k) \\ U(2n + 1, 2k + 1) = U(2n - 1, 2k - 1) + (2k + 1)^2\, U(2n - 1, 2k + 1). \end{cases}$$

Moreover

$$U(n, n - 2k) = \sum_{j=0}^{k-1} U'(k, j) \binom{n}{3k - j}.$$

In § 8 we define the array $(V(n, k))$ by means of

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k V(n, k)\, x^k = \exp\,(x(\cosh z - 1)).$$

Clearly

$$V(2n + 1, k) = 0 \quad (k = 0, 1, 2, \ldots)\,.$$

Also it is proved in [2] that

$$V(2n, k) = \frac{(2k)!}{2^{2n-k}\, k!}\, U(2n, 2k),$$

so that a detailed discussion seems unnecessary. The enumerant $V(2n, k)$ is evidently equal to the number of partitions of Z_{2n} into k blocks all of even cardinality. Incidentally it follows from the generating functions that the number of permutations of Z_{2n} with k blocks all of even cardinality is equal to

$$\frac{(2n)!}{2^n\, n!} S_1(n, k).$$

In the final § 9 we discuss several additional examples of pairs of inverse arrays. The computation of the inverse arrays makes use of the Lagrange expansion formula.

2. Inverse arrays. It follows from (1.7), (1.8) and (1.9) that

$$\frac{1}{k!} a_k f^k(z) = \sum_{n=k}^{\infty} \frac{z^n}{n!} A_{nk} \quad (k = 1, 2, 3, \ldots). \tag{2.1}$$

Similarly, using (1.10) in place of (1.7), we get

$$\frac{1}{k!} a_k g^k(z) = \sum_{n=k}^{\infty} \frac{z^n}{n!} B_{nk} \quad (k = 1, 2, 3, \ldots). \tag{2.2}$$

Assume that $a_k = 0$ for some $k \geqslant 1$. Then by (2.1)

$$\sum_{n=k}^{\infty} \frac{z^n}{n!} A_{nk} = 0,$$

so that

$$A_{nk} = 0 \quad (n = k, k+1, k+2, \ldots).$$

In particular $A_{kk} = 0$ and therefore (A_{nk}) does not have an inverse. Hence a necessary condition that the array (A_{nk}) possess an inverse is

$$a_k \neq 0 \quad (k = 1, 2, 3, \ldots). \tag{2.3}$$

Now assume that (2.3) holds and replace z by $g(z)$ in (2.1). Then, by (2.2),

$$\frac{1}{k!} a_k \left(f(g(z))\right)^k = \sum_{n=k}^{\infty} \frac{1}{n!} A_{nk} \cdot \frac{n!}{a_n} \sum_{j=n}^{\infty} \frac{z^j}{j!} B_{jn},$$

that is,

$$\frac{1}{k!} a_k z^k = \sum_{j=k}^{\infty} \frac{z^j}{j!} \sum_{n=k}^{j} \frac{1}{a_n} B_{nj} A_{nk}.$$

This gives

$$(2.4) \qquad \sum_{n=k}^{j} \frac{1}{a_n} B_{jn} \cdot \frac{1}{a_k} A_{nk} = \delta_{jk} \quad (j, k = 1, 2, 3, \ldots).$$

Similarly we get

$$(2.5) \qquad \sum_{n=k}^{j} \frac{1}{a_n} A \cdot \frac{1}{a_k} B_{nk} = \delta_{jk} \quad (j, k = 1, 2, 3, \ldots).$$

Hence if we put

$$A'_{nk} = \frac{1}{a_k} A_{nk}, \; B'_{nk} = \frac{1}{a_k} B_{nk},$$

it follows that

$$(2.6) \qquad \sum_{k=j}^{n} A'_{nk} B'_{kj} = \sum_{k=j}^{n} B'_{nk} A'_{kj} = \delta_{nj} \quad (n, j = 1, 2, 3, \ldots),$$

that is, (A'_{nk}), (B'_{nk}) are an inverse pair.

Thus we have proved that for arrays (A_{nk}) defined by (1.7), (1.8) and (1.9), a necessary and sufficient condition for the existence of an inverse is given by (2.3). Moreover the inverse is given by (1.1) or, equivalently, (2.2).

If we take $\phi(z) = e^z$, then (1.7) and (1.1) become

$$(2.7) \qquad e^{xf(z)} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} A_{nk} x^k$$

and

$$(2.8) \qquad e^{xg(z)} = 1 + \sum_{n=1}^{\infty} \frac{z_n}{n!} \sum_{k=1}^{n} B_{nk} x^k,$$

respectively. It is clear from (2.5) and (2.6) that there is no essential loss in generality in restricting ourselves in the remainder of the paper to $\phi(z) = e^z$.

3. Stirling arrays. For

$$(3.1) \qquad f(z) = e^z - 1,$$

(2.7) becomes

$$\begin{aligned} e^{x(e^z-1)} &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} e^{sz} \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \frac{x^k}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} s^n. \end{aligned}$$

Thus we have

$$A_{nk} = \frac{1}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} s^n \equiv S(n, k), \tag{3.2}$$

the Stirling number of the second kind.

The inverse of (3.1) is given by

$$g(z) = \log(1+z). \tag{3.3}$$

Hence

$$e^{xg(z)} = e^{v \log(1+z)} = (1+z)^x = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} x(x-1) \dots (x-n+1).$$

Thus

$$\sum_{k=1}^{n} B_{nk} x^k = x(x-1) \dots (x-n+1) \qquad (n \geqslant 1)$$

and therefore

$$B_{nk} = (-1)^{n-k} S_1(n, k), \tag{3.4}$$

where $S_1(n, k)$ denotes the Stirling number of the first kind.

We recall that [7, Ch. 4] that $S_1(n, k)$ enumerates the number of permutations of $Z_n = \{1, 2, \dots, n\}$ with k cycles while $S(n, k)$ enumerates the number of partitions of Z_n into k blocks.

The Stirling number $S(n, k)$ satisfies the recurrence

$$S(n+1, k) = S(n, k-1) + k S(n, k). \tag{3.5}$$

It follows from (3.5) that

$$S(n, n-k) = \sum_{j=0}^{k-1} S'(k, j) \binom{n}{2k-j} \qquad (k > 0), \tag{3.6}$$

where

$$S'(k+1, j) = (k-j+1) S'(k, j-1) + (2k-j+1) S'(k, j). \tag{3.7}$$

On the other hand [7, Ch. 4]

$$S(n, k) = \sum_{j=0}^{n} \binom{n}{j} b(n-j, k-j), \tag{3.7}$$

where $b(n, k)$ is the number of partitons of Z_n into k blocks, each of cardinality > 1. Moreover

$$S'(k+j, j) = b(2k+j, k). \tag{3.8}$$

As for the Stirling number $S_1(n, k)$ of the first kind, we have first the recurrence

$$S_1(n+1, k) = S_1(n, k-1) + n\,S_1(n, k), \tag{3.9}$$

which implies

$$S_1(n, n-k) = \sum_{j=0}^{k-1} S_1'(k, j)\binom{n}{2k-j} \qquad (k > 0), \tag{3.10}$$

where

$$S_1'(k+1, j) = (2k-j+1)\,(S_1'(k, j-1) + S_1'(k, j)). \tag{3.11}$$

We have also

$$S_1(n, k) = \sum_{j=0}^{n}\binom{n}{j} d(n-j, k-j), \tag{3.12}$$

where $d(n, k)$ is the number of permutations of Z_n with k cycles, each of legth > 1. Moreover

$$S_1'(k+j, j) = d(2k+j, k). \tag{3.13}$$

For references to $S'(n, k)$ and $S_1'(n, k)$ see [1], [4], [8].
We remark that

$$\exp\{x(e^z - z - 1)\} = 1 + \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_k b(n, k)\,x^k$$

and

$$\exp\left\{x\left(\log\frac{1}{1-z} - z\right)\right\} = 1 + \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_k d(n, k)\,x^k.$$

It follows from (3.8) and (3.13) that

$$\exp\{x(e^z - z - 1)\} = 1 + \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{2k<n} S'(n-k, n-2k)\,x^k \tag{3.14}$$

and

$$(3.15) \quad \exp\left\{x\left(\log\frac{1}{1-z}-z\right)\right\}=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{2k<n}S_1(n-k,\,n-2k)\,x^k.$$

The generating functions (3.14), (3.15) are not quite of the kind we have been discussing. However if we replace x by xz^{-1} we get

$$(3.16) \quad \exp\{xz^{-1}(e^z-z-1)\}=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n-1}\frac{n!}{(n+k)!}S'(n,\,n-k)\,x^k$$

and

(3.17)

$$\exp\left\{xz^{-1}\left(\log\frac{1}{1-z}-z\right)\right\}=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n-1}\frac{n!}{(n+k)!}S_1'(n,\,n-k)\,x^k,$$

respectively; (3.16) and (3.17) are of the desired type. This suggests that we put

(3.18)

$$A_{nk}=\frac{n!}{(n+k)!}S'(n,\,n-k),\quad B_{nk}=\frac{n!}{(n+k)!}S_1'(n,\,n-k)\qquad (1\leqslant k\leqslant n).$$

However

$$z^{-1}(e^z-z-1)\quad\text{and}\quad z^{-1}\left(\log\frac{1}{1-z}-z\right)$$

are not a pair of inverse functions and (A_{nk}), (B_{nk}) are therefore not an inverse pair.

The generating functions (3.16), (3.17) are presumably new.

4. The numbers $T_1(n, k)$. From the general results on the cycle index of the symmetric group [7, Ch. 4], if we put

$$(4.1) \quad \exp\left\{x\left(z+\frac{1}{3}z^3+\frac{1}{5}z^5+\ldots\right)\right\}=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{k=1}^{n}T_1(n,\,k)\,x^k,$$

then clearly $T_1(n, k)$ is equal to the number of permutations of Z_n with k cycles all of which have odd cardinality.

Since

$$z + \frac{1}{3} z^3 + \frac{1}{5} z^3 + \dots = \frac{1}{2} \log (1 + z) - \frac{1}{2} \log (1 - z) = \frac{1}{2} \log \frac{1+z}{1-z},$$

we may replace (4.1) by

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} T_1 (n, k)\, x^k = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}x}. \tag{4.2}$$

Put

$$T_{1.n} (x) = \sum_{k=1}^{n} T_1 (n, k)\, x^k, \qquad T_{1,0} (x) = 1 \tag{4.3}$$

and (4.2) becomes

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} T_{1,n} (x) = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}x}. \tag{4.4}$$

Differentiation with respect to z gives

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} T_{1.n+1} (x) = \frac{x}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}x-1} = \frac{x}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}x}.$$

Hence

$$(1 - z^2) \sum_{n=0}^{\infty} \frac{z^n}{n!} T_{1,n+1} (x) = x \sum_{n=0}^{\infty} \frac{z^n}{n!} T_{1,n} (x),$$

so that

$$T_{1.n+1} (x) = x T_{1,n} (x) + n (n-1)\, T_{1,n-1} (x) \qquad (n \geqslant 1). \tag{4.5}$$

Therefore, by (4.3), we get

$$T_1 (n + 1, k) = T_1 (n, k - 1) + n (n - 1)\, T_1 (n - 1, k), \tag{4.6}$$

which may be compared with (3.9), the recurrence for the Stirling numbers of the first kind.

By means of (4,6) the following table is easily computed.

$T_1(n, k)$:

k / n	1	2	3	4	5	6	7	8
1	1							
2	0	1						
3	2	0	1					
4	0	8	0	1				
5	24	0	20	0	1			
6	0	184	0	40	0	1		
7	720	0	784	0	70	0	1	
8	0	8448	0	2464	0	112	0	1

For $x = 1$, (4.5) becomes

$$T_{1,n+1}(1) = T_{1,n}(1) + n(n-1)\, T_{1,n-1}(1).$$

If we put $t_n = T_{1,n}(1)$ we get

$$t_{n+1} = t_n + n(n-1)\, t_{n-1} \qquad (n > 1). \tag{4.7}$$

By the above table

$t_1 = t_2 = 1,\ t_3 = 3,\ t_4 = 9,\ t_5 = 45,\ t_6 = t_{225},\ t_7 = 1575,\ t_9 = 11025.$

These values suggest that

$$\begin{cases} t_{2n} = 1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2 \\ t_{2n+1} = (2n+1)\, t_{2n}. \end{cases} \tag{4.8}$$

This is easily proved by induction, using (4.7).

We note also that

$$T_1(n, k) = 0 \qquad (n \equiv k + 1 \pmod 2)), \tag{4.9}$$

$$T_1(n, n) = 1, \tag{4.10}$$

$$T_1(2n+1, 1) = (2n)! \tag{4.11}$$

For $k = n - 1$, (4.6) becomes

$$T_1(n+1, n-1) = T_1(n, n-2) + n(n-1).$$

It follows that

$$T_1(n+1, n-1) = \frac{1}{3}(n+1)n(n-1). \tag{4.12}$$

Taking $k = n - 3$ in (4.6) we get

$$T_1(n+1, n-3) = T_1(n, n-4) + n(n-1)T_1(n-1, n-3).$$

By (4.12) this becomes

$$\begin{aligned} T_1(n+1, n-3) - T_1(n, n-4) &= n(n-1)\cdot\frac{1}{3}(n-1)(n-2)(n-3) \\ &= \frac{1}{3}n(n-1)(n-2)(n-3)(n-4) \\ &\quad + n(n-1)(n-2)(n-3). \end{aligned}$$

This yields

$$T_1(n+1, n-3) = 40\binom{n+1}{6} + 24\binom{n+1}{5}. \tag{4.13}$$

For example

$$T_1(7,3) = 40\binom{7}{6} + 24\binom{7}{5} = 280 + 508 = 784.$$

Note that $T_1(n, n-2)$ is a polynomial of degree 3, $T_1(n, n-4)$ is a polynomial of degree 6. In (4.6) replace k by $n - 2k + 1$, so that

$$\begin{aligned} T_1(n+1, n-2k+1) &= T_1(n, n-2k) \\ &\quad + n(n-1)T_1(n-1, n-2k+1). \end{aligned} \tag{4.14}$$

From this it follows by induction that $T_1(n+1, n-2k+1)$ is a polynomial in n of degree $3k$.

Put (compare (3.10))

$$T_1(n, n-2k) = \sum_{j=0}^{3k} T_1'(k, j)\binom{n}{3k-j}. \tag{4.15}$$

Then by (4.14)

$$\sum_{j=0}^{3k} T'_1(k,j)\binom{n+1}{3k-j} = \sum_{j=0}^{3k} T'_1(k,j)\binom{n}{3k-j} + n(n-1)\sum_{j=0}^{3k-3} T'_1(k-1,j)\binom{n-1}{3k-j-3},$$

so that

$$\begin{aligned}\sum_{j=0}^{3k-1} T'_1(k,j)\binom{n}{3k-j-1} &= \sum_{j=0}^{3k-3} T'_1(k-1,j)\binom{n-1}{3k-j-3} n[(n-3k+j+2)+(3k-j-3)]\\ &= \sum_{j=0}^{3k-3} T'_1(k-1,j)\left\{\binom{n}{3k-j-1}(3k-j-1)(3k-j-2)\right.\\ &\quad \left. + \binom{n}{3k-j-2}(3k-j-2)(3k-j-3)\right\}\\ &= \sum_{j=0} \binom{n}{3k-j-1}\{(3k-j-1)(3k-j-2)T'_1(k-1,j)\\ &\quad + (3k-j-1)(3k-j-2)T'_1(k-1,j-1)\}\\ &= \sum_{j=0}^{3k-2}(3k-j-1)(3k-j-2)(T'_1(k-1,j-1)\\ &\quad + T'_1(k-1,j)\binom{n}{3k-j-1}.\end{aligned}$$

It follows that

$$T'_1(k,j) = (3k-j-1)(3k-j-2)(T'_1(k-1,j-1) + T'_1(k-1,j)). \tag{4.16}$$

We wish to show that, for all $k > 0$,

$$T'_1(k,j) = 0 \qquad (k \leqslant j \leqslant 3k). \tag{4.17}$$

By (4.12) and (4.13), (4.17) holds for $k = 1, 2$. Assume that it holds up to and including the value $k - 1$, so that

$$T'_1(k-1,j) = 0 \qquad (k-1 \leqslant j \leqslant 3k-3).$$

Then, for $k \leqslant j \leqslant 3k-2$, (4.16) implies $T'_1(k,j) = 0$; for $j = 3k-1$, $3k$ we clearly have the same conclusion. Thus (4.17) holds for all $k > 0$.

In view of (4.17) we may replace (4.15) by

$$T_1(n, n-2k) = \sum_{j=0}^{k-1} T_1'(k, j) \binom{n}{3k-j} \qquad (k < 0). \tag{4.18}$$

$T_1'(k, j)$:

j / k	0	1	2	3
1	2			
2	40	24		
3	$8\cdot 7\cdot 40$	$7\cdot 6\cdot 64$	$6!$	
4	$11\cdot 10\cdot 8\cdot 7\cdot 40$	$7\cdot 8^2\cdot 9\cdot 10\cdot 11$	$6\cdot 8^2\cdot 9\cdot 71$	$8!$

For $j = 0$, (4.16) becomes

$$T_1'(k, 0) = (3k-1)(3k-2)\, T_1'(k-1, 0),$$

which implies

$$T_1'(k, 0) = \frac{(3k-1)!}{3^{k-1}(k-1)!}. \tag{4.19}$$

For $j = k - 1$ we have

$$T_1'(k, k-1) = 2k(2k-1)\, T_1'(k-1, k-2),$$

so that

$$T_1'(k, k-1) = (2k)! \tag{4.20}$$

5. The inverse of $(T_1(n, k))$. To construct the inverse of $(T_1(n\ k))$ we requiere the inverse of the function

$$u = \tfrac{1}{2} \log \frac{1+z}{1-z}.$$

This is evidently

$$z = \frac{e^{2n}-1}{e^{2n}+1} = \frac{e^u - e^u}{e^u + e^{-u}} = \tanh u.$$

We accordingly put

$$(5.1) \qquad e^{x \tanh z} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} T(n, k) x^k.$$

It follows that the array $(T(n, k))$ is the inverse of $(T_1(n, k))$.
It is convenient to put

$$(5.2) \qquad T_n(x) = \sum_{k=1}^{n} T(n, k) x^k, \qquad T_0(x) = 1.$$

Thus (5.1) becomes

$$(5.3) \qquad e^{x \tanh z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} T_n(x).$$

Differentiation with respect to z gives

$$(5.4) \qquad \sum_{n=0}^{\infty} \frac{z^n}{n!} T_{n+1}(x) = x \operatorname{sech}^2 z \cdot e^{x \tanh z}.$$

Also it is clear from (5.1) that

$$(5.5) \qquad \frac{1}{k!} \tanh^k z = \sum_{n=0}^{\infty} \frac{z^n}{n!} T(n, k) \qquad (k \geqslant 0).$$

Thus (5.4) becomes

$$\sum_{n=k}^{\infty} \frac{z^n}{n!} T_{n+1}(x) = x(1 - \tanh^2 z) e^{x \tanh z},$$

so that

$$\sum_{n=k-1}^{\infty} \frac{z^n}{n!} T(n+1, k) = \frac{1}{(k-1)!} (\tanh^{k-1} z - \tanh^{k+1} z)$$

$$= \sum_{n=k+1}^{\infty} \frac{z^n}{n!} T(n, k-1) - k(k+1) \sum_{n=k+1}^{\infty} \frac{z^n}{n!} T(n, k+1).$$

Comparing coefficients of z^n, we get

$$(5.6) \qquad T(n+1, k) = T(n, k-1) - k(k+1) T(n, k+1).$$

This is equivalent to

$$(5.7) \qquad T_{n+1}(x) = x T_n(x) - k T_n'(x).$$

For $k = n + 1$, (5.6) reduces to

$$T(n+1, n+1) = T(n, n).$$

Since $T(1,1) = 1$, we gat

(5.8) $$T(n, n) = 1 \qquad (k = 0, 1, 2, \ldots).$$

It also follows from (5.5) that

(5.9) $$(-1)^n T(2n+1, 1) = T_n \quad \text{(tangent coefficient)}.$$

$T(n, k)$:

k / n	1	2	3	4	5	6
1	1					
2	0	1				
3	− 2	0	1			
4	0	− 8	0	1		
5	16	0	− 20	0	1	
6	0	136	0	− 40	0	1

If we replace z by iz, (5.5) becomes

$$\frac{1}{k!} \tan^k z = \sum_{\substack{n=k \\ n \equiv k(2)}}^{\infty} (-1)^{\frac{1}{2}(n-k)} \frac{z^n}{n!} T(n, k).$$

Since the coefficients of $\tan z$ are positive, it follows that

(5.10) $$(-1)^{\frac{1}{2}(n-k)} T(n, k) > 0 \qquad (n \geqslant k,\ n \equiv k \pmod 2).$$

For $k = n + 1$, (5.6) reduces to

$$T(n+1, n+1) = T(n, n)$$

and therefore

(5.11) $$T(n, n) = 1 \qquad (n \geqslant 0).$$

For $k = n - 1$ we get

$$T(n+1, n-1) = T(n, n-2) - n(n-1),$$

which yields

$$T(n, n-2) = -\frac{1}{3} n(n-1)(n-2). \tag{5.12}$$

For $k = n - 3$ we get

$$\begin{aligned} T(n+1, n-3) &= T(n, n-4) - (n-2)(n-3)\, T(n, n-2) \\ &= T(n, n-4) + \frac{1}{3} n(n-1)(n-2)^2 (n-3). \end{aligned}$$

It follows that

$$T(n, n-4) = 40\binom{n}{6} + 16\binom{n}{5}. \tag{5.13}$$

Similarly we find after some computation that

$$T(n, n-6) = -40 \cdot 56\binom{n}{9} - 7 \cdot 15^2\binom{n}{8} - 272\binom{n}{7}. \tag{5.14}$$

We accordingly put

$$T(n, n-2k) = \sum_{j=0}^{k-1} T'(k, j)\binom{n}{3k-j} \qquad (k > 0); \tag{5.15}$$

the upper limit of summation is justified in the induction.

Assume the truth of (5.15) up to and including the value $k - 1$. Then by (5.6)

$$\begin{gathered} T(n+1, n-2k+1) - T(n, n-2k) = \\ -(n-2k+1)(n-2k+2)\, T(n, n-2k+2), \end{gathered}$$

so that

$$\begin{gathered} \sum_{j=0}^{k-1} T'(k, j)\binom{n+1}{3k-j} - \sum_{j=0}^{k-1} T'(k, j)\binom{n}{3k-j} = \\ -(n-2k+1)(n-2k+2)\sum_{j=0}^{k-2} T'(k-1, j)\binom{n}{3k-j-3}, \end{gathered}$$

that is,

$$\sum_{j=0}^{k-1} T'(k,j)\binom{n}{3k-j-1} =$$

$$-(n-2k+1)(n-2k+2)\sum_{j=0}^{k-2} T'(k-1,j)\binom{n}{3k-j-3}.$$

Then very much as in the proof of (4.16) we find that

$$\begin{aligned}(5.16)\quad T'(k,j) = &-(3k-j-1)(3k-j-2)\,T'(k-1,j)\\ &-2(k-j)(3k-j-1)\,T'(k-1,j-1)\\ &-(k-j)(k-j+1)\,T'(k-1,j-2).\end{aligned}$$

$T'(k,j)$:

k \ j	0	1	2
1	-2		
2	40	16	
3	$-5\cdot 7\cdot 8^2$	$-7\cdot 16^2$	-272

Clearly

$$(5.17)\qquad (-1)^k T'(k,j) > 0 \qquad (0 \leqslant j \leqslant k-1).$$

Also it follows from

$$T'(k,0) = -(3k-1)(3k-2)\,T'(k-1,0)$$

that

$$(5.18)\qquad (-1)^k T'(k,0) = \frac{(3k-1)!}{3^{k-1}(k-1)!}.$$

Taking $n = 2k+1$ in (5.15), we get

$$T(2k+1,1) = T'(k,k-1).$$

Hence by (5.9)

$$(5.19)\qquad T'(k,k-1) = (-1)^k T_{2k+1}.$$

Also, by (5.16),

$$\begin{aligned}\sum_{j=0}^{k-1} (-1)^j T'(k,j) &= \sum_j (-1)^j \{(3k-j-1)(3k-j-2)T'(k-1,j)\\ &\qquad + 2(k-j)(3k-j-1)T'(k-1,j-1)\\ &\qquad + (k-j)(k-j+1)T'(k-1,j-2)\}\\ &= \sum_j (-1)T'(k-1,j)\{(3k-j-1)(3k-j-2)\\ &\qquad - 2(k-j-1)(3k-j-2)\\ &\qquad + (k-j-2)(k-j-1)\}\\ &= 2k(2k-1)\sum_{j=0}^{k-2} (-1)^j T'(k-1,j).\end{aligned}$$

It follows at once that

$$\sum_{j=0}^{k-1} (-1)^j T'(k,j) = (-1)^k (2k)!. \tag{5.20}$$

The presence of the term in $T'(k-1,j-2)$ on the right of (5.6) might lead one to expect that this would result in non-zero $T'(k,j)$ with $j \geqslant k$. That this is not the case is proved in the following way. Assume that

$$T'(k-1,j) = 0 \qquad (j \geqslant k-1).$$

Then by (5.16)

$$\begin{aligned}&-T'(k,k) = (2k-1)(2k-2)T'(k-1,k)\\ &+ 0\cdot T'(k-1,k-1) + 0\cdot T'(k-1,k-2) = 0,\\ &-T'(k,k+1) = (2k-2)(2k-3)T'(k-1,k+1)\\ &-2(2k-2)T'(k-1,k) + 0\cdot T'(k-1,k-1) = 0\end{aligned}$$

and generally

$$T'(k,j) = 0 \qquad (j \geqslant k).$$

6. Combinatorial interpretation. As noted above $T_1(n,k)$ is equal to the number of permutations of Z_n with k cycles all of which have odd cardinality. Similarly if we put

$$\exp\left\{x\left(\frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 + \dots\right)\right\} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k C_{nk} x^k, \tag{6.1}$$

then C_{nk} is equal to the number of permutations of Z_n with k cycles all of odd cardinality greater than 1. Comparing (6.1) with (4.1) we get

$$\sum_k T_1(n, k)\, x^k = \sum_j \binom{n}{j} x^j \sum_k C_{nk}\, x^k$$

so that

$$T_1(n, k) = \sum_j \binom{n}{j} C_{n-j,k-j}.$$

Replacing k by $n - 2k$ this becomes

$$T_1(n, n-2k) = \sum_{j=2k}^{n} \binom{n}{j} C_{j,j-2k}. \tag{6.2}$$

On the other hand, by (4.15),

$$T_1(n, n-2k) = \sum_j \binom{n}{3k-j} T_1'(k, j).$$

Rewriting (6.2) in the form

$$T_1(n, n-2k) = \sum_j \binom{n}{3k-j} C_{3k-j,k-j},$$

it is evident that

$$T_1'(k, j) = C_{3k-j,k-j},$$

or, if we prefer,

$$T_1'(k+j, j) = C_{3k+2j,k}. \tag{6.3}$$

Hence $T_1'(k+j, j)$ is the number of permutations of Z_{3k+2j} with k cycles, all of which have odd cardinality greater than 1.

We may rewrite (6.1) in the form

$$\exp\left\{x\left(\tfrac{1}{2}\log\frac{1+z}{1-z} - z\right)\right\} = 1 + \sum_{j,k} C_{3k+2j,k}\, x^k \frac{z^{3k+2j}}{(3k+2j)!}.$$

Replacing x by xz^{-2} and making use of (6.3), we get

$$\exp\left\{xz^{-2}\left(\tfrac{1}{2}\log\frac{1+z}{1-z} - z\right)\right\} = \tag{6.4}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k \frac{n!}{(n+2k)!} T_1'\left(\tfrac{1}{2}(n+k)\ \tfrac{1}{2}(n-k)\right) x^k.$$

This may be compared with (3.17).

Turning next to $T(n, k)$, by (5.1) we have

$$e^{x \tanh z} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} T(n, k) x^k. \tag{6.5}$$

We now put

$$e^{x(\tanh z - z)} = 1 + \sum_{n=3}^{\infty} \frac{z^n}{n!} \sum_{k} D_{nk} x^k. \tag{6.6}$$

It follows from (6.5) and (6.6) that

$$T(n, k) = \sum_{j} \binom{n}{j} D_{n-j, k-j} = \sum_{j} \binom{n}{j} D_{j, j+k-n},$$

so that

$$T(n, n-2k) = \sum_{j=2k}^{n} \binom{n}{j} D_{j, j-2k}. \tag{6.7}$$

By (5.15)

$$T(n, n-2k) = \sum_{j=0}^{k=1} T'(k, j) \binom{n}{3k-j}.$$

Rewrite (6.7) in the form

$$T(n, n-2k) = \sum_{j} \binom{n}{3k-j} D_{3k-j, k-j}$$

and it is clear that

$$T'(k, j) = D_{3k-j, k-j},$$

or, if we prefer,

$$T'(k+j, j) = D_{3k+2j, k}. \tag{6.8}$$

Rewrite (6.6) in the form

$$e^{x(\tanh z - z)} = 1 + \sum_{j, k} D_{3k+2j, k} \frac{x^k z^{3k+2j}}{(3k+2j)!}.$$

Replacing x by xz^{-2} and applying (6.8), we gat

$$\exp \{xz^{-2}(\tanh z - z)\} = 1 + \tag{6.9}$$

$$+ \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k} \frac{n!}{(n+2k)!} T'(\tfrac{1}{2}(n+k), \tfrac{1}{2}(n-k)) x^k.$$

The generating function (6.9) may be compared with (3.16).

A combinatorial interpretation of $T(n, k)$ is not known. However if we put

$$\tan^k z = \sum_{n=k}^{\infty} T_n^{(k)} z^n / n!$$

then, by (5.5),

$$(- 1)^{\frac{1}{2}(n-k)} T(n, k) = \frac{1}{k!} T_n^{(k)}. \tag{6.10}$$

It is known [3] that $T_n^{(k)}$ is equal to the number of up-down sequences of length $n + k$ with k «infinite» elements.

A combinatorial interpretation of $T'(n, k)$ is not known.

7. The array $(U(n, k))$. Put

$$\exp(x \sinh z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} U(n, k) x^k, \tag{7.1}$$

so that $U(n, k)$ is the number of partitions of Z_n into k blocks of odd cardinality. It is convenient to take $U(0, 0) = 1$. Clearly

$$U(n, k) = 0 \qquad (n \equiv k + 1 \pmod 2)). \tag{7.2}$$

We also put

$$U_n(x) = \sum_{k=1}^{n} U(n, k) x^k, \qquad U_0(x) = 1, \tag{7.2}$$

so that (7.1) becomes

$$\exp(x \sinh z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} U_n(x). \tag{7.3}$$

Put $F(x, z) = \exp(x \sinh z)$. It is easily verified that

$$\frac{\partial^2 F}{\partial z^2} = z^2 F + z \frac{\partial F}{\partial x} + z^2 \frac{\partial^2 F}{\partial^2 x^2} \tag{7.4}$$

This implies

$$U_{n-2}(x) = x^2 U_n(x) + x U_n'(x) + x^2 U_n''(x) = x^2 U_n(x) + (xD)^2 U_n(x)$$

which is equivalent to

$$U(n+2, k) = U(n, k-2) + k^2 U(n, k). \tag{7.5}$$

We replace (7.5) by the pair of recurrences

$$\begin{cases} U(2n+2, 2k) = U(2n, 2k-2) + (2k)^2 U(2n, 2k) \\ U(2n+1, 2k+1) = U(2n-1, 2k-1) \\ \qquad\qquad + (2k+1)^2 U(2n-1, 2k+1). \end{cases} \tag{7.6}$$

$U(2n, 2k)$:

k / n	1	2	3	4
1	1			
2	4	1		
3	16	20	1	
4	64	336	56	1

$U(2n+1, 2k+1)$:

k / n	0	1	2	3	4
0	1				
1	1	1			
2	1	10	1		
3	1	91	35	1	
4	1	820	966	84	1

It is clear from (7.6) that

$$U(n, n) = 1 \qquad (n = 0, 1, 2, \ldots), \tag{7.7}$$

$$U(2n, 2) = 2^{2n-2} \qquad (n = 1, 2, 3, \ldots) \tag{7.8}$$

and

$$U(2n+1, 1) = 1 \qquad (n = 0, 1, 2, \ldots). \tag{7.9}$$

By the first of (7.6)

$$U(2n+2, 2n) - U(2n, 2n-2) = 4n^2 \qquad (n = 1, 2, 3, \ldots).$$

It follows that

$$(7.10) \qquad U(2n, 2n-2) = \frac{2}{3} n(n-1)(2n-1) = \binom{2n}{3}.$$

Next taking $k = n - 1$ in the first of (7.6), we get

$$U(2n+2, 2n-2) - U(2n, 2n-4) =$$
$$= 4(n-1)^2 U(2n, 2n-2) \qquad (n = 2, 3, 4, \ldots).$$

Making use of (7.10), it follows that $U(2n+2, 2n-2)$ is a polynomial in n of degree 6. It is easily proved, by induction on k, that $U(2n, 2n-2k)$ is a polynomial in n of degree $3k$.

In the next place, taking $k = n - 1$ in the second part of (7.6), we get

$$U(2n+1, 2n-1) = U(2n-1, 2n-3) + (2n-1)^2.$$

This implies

$$(7.11) \qquad U(2n+1, 2n-1) = \binom{2n+1}{3}.$$

Again we can prove by induction that $U(2n+1, 2n-2k+1)$ is a polynomial in n of degree $3k$.

It is not immediately apparent from these results that $U(n\ n-2k)$ is a polynomial in n of degree $3k$. To see that this is the case we proceed differently. Put

$$(7.12) \qquad \exp(x(\sinh z - z)) = 1 + \sum_{n=3}^{\infty} \frac{z^n}{n!} a_n(x)$$

where

$$a_n(x) = \sum_k a(n, k) x^k.$$

Comparing this with (7.3) we get

$$U_n(x) = \sum_{j=0}^{\infty} \binom{n}{j} x^j a_{n-j}(x).$$

so that

$$(7.13)\qquad U(n, k) = \sum_{j=0}^{k} \binom{n}{j} a(n - j, k - j).$$

Replacing j by $n - j$ and k by $n - 2k$, (7.13) becomes

$$(7.14)\qquad U(n, n - 2k) = \sum_{j=2k}^{3k} \binom{n}{j} a(j, j - 2k).$$

The upper limit is precisely $3k$ in view of the results already obtained for n even or n odd. Thus $U(n, n - 2k)$ is indeed a polynomial in n of degree $3k$ but, more precisely, of the form (7.14), that is, if we take (7.14) as definition of $U(n, n - 2k)$ then it follows that

$$U(n, n - 2k) = 0 \qquad (0 \leqslant n < 2k).$$

In conformity with the notation used proviously we put

$$(7.15)\qquad U(n, n - 2k) = \sum_{j=0}^{k-1} U'(k, j) \binom{n}{3k - j}.$$

Comparing (7.15) with (7.14) we evidently have

$$(7.16)\qquad U'(k, j) = a(3k - j, j - k).$$

The upper limit in (7.15) is $k - 1$ rather than k since $a(n, 0) = 0$ for $n \geqslant 1$.

$U'(k, j)$:

k / j	0	1	2	3
1	1			
2	10	1		
3	280	56	1	
4	15400	4620	246	1

We have been unable to find a simple recurrence for $U'(k, j)$. It follows from (7.12) that

$$\frac{1}{k!} (\sinh z - z)^k = \sum_n \frac{z^n}{n!} a(n, k),$$

which yields

(7.17) $$(k+1)\,a(n, k+1) = \sum_{j \geq 1} \binom{n}{2j+1} a(n-2j-1, k).$$

By (7.16) this becomes

(7.18) $$(k-m)\,U'(k, m) = \sum_{j \geq 1} \binom{2j+1}{3k-m} U'(k-j, m-j+1) \quad (0 \leqslant m < k).$$

For example, for $m = 0$, (7.18) reduces to

$$k\,U'(k, 0) = \binom{3k}{3} U'(k-1, 0),$$

or,

(7.19) $$U'(k, 0) = \binom{3k-1}{2} U'(k-1, 0).$$

Thus

(7.20) $$U'(k, 0) = \binom{3k-1}{2}\binom{3k-3}{2} \cdots \binom{2}{2} = \frac{(3k)!}{k!\,(3!)^k}.$$

For $m = 1$, (7.18) becomes

$$(k-1)\,U'(k, 1) = \binom{3k-1}{3} U'(k-1, 1) + \binom{3k-1}{5} U'(k-2, 0),$$

from which $U'(k, 1)$ can be computed. For example

$$3\,U'(4, 1) = 165.56 + 462.10, \qquad U'(4,1) = 4620.$$

For $m = k - 1$, (7.18) reduces to

$$U'(k, k-1) = \sum_{j=1}^{k} \binom{2k+1}{2j+1} U'(k-j, k-j).$$

Since $U'(k, k) = 0$ for $k > 0$, we put

(7.21) $$U'(k, k-1) = 1 \qquad (k \geqslant 1).$$

For $m = k - 2$ we get

$$2\,U'(k, k-2) = \sum_{j=1}^{k} \binom{2k+2}{2j+1} U'(k-j, k-j-1) = \sum_{j=1}^{k-1} \binom{2k+2}{2j+1},$$

so that

$$(7.22) \qquad U'(k, k-2) = 2^{2k} - 2k - 2 \qquad (k \geqslant 2).$$

A recurrence of a different kind can be found in the following way. Put

$$F = \exp(x \sinh z), \qquad G = \exp(x(\sinh z - z)),$$

so that $F = e^{xz} G$. Then, by (7.4), G satisfies

$$\frac{\partial^2 G}{\partial z^2} + 2x \frac{\partial G}{\partial z} = x^2 \frac{\partial^2 G}{\partial x^2} + (x + 2x^2 z) \frac{\partial G}{\partial x} + (xz + x^2 z^2) G.$$

This implies

$$(7.23) \qquad a(n+2, k) + 2a(n+1, k-1) = k^2 a(n, k) + \\ + n(2k-1) a(n-1, k-1) + n(n-1) a(n-2, k-2),$$

which is equivalent to

$$(7.24) \qquad U'(k+1, j+1) + 2U'(k+1, j+2) = \\ = (j+k)^2 U'(k, j) + (3k-j)(2k-2j-1) U'(k, j+1) + \\ + (3k-j) 3k-j-1) U'(k, j+2).$$

For example, for $j = -2$, this reduces to

$$2U'(k+1, 0) = (3k+2)(3k+1) U'(k, 0),$$

in agreement with (7.19). For $j = -1$ we get

$$U'(k+1, 0) + 2U'(k+1, 1) = \\ = (3k+1)(2k+1) U'(k, 0) + 3k(3k+1) U'(k, 1),$$

and so on.

It would be desirable to find a simpler recurrence than (7.24).

8. The array $(V(n, k))$. Put

$$(8.1) \qquad \exp(x(\cosh z - 1)) = \sum_{n=0}^{\infty} \frac{z^n}{n!} V_n(x),$$

where

$$V_n(x) = \sum_k V(n, k)\, x^k.$$

Clearly

(8.2) $$V(2n + 1, k) = 0 \qquad (k = 0, 1, 2, \ldots).$$

Also it is shown in [2] that

(8.3) $$V(2n, k) = \frac{(2k)!}{2^{2n-k}\, k!}\, U(2n, 2k)$$

and

(8.4) $$V(2n + 2, k) = (2k - 1)\, V(2n, k - 1) + k^2\, V(2n, k).$$

If we put

(8.5) $$V(2n, k) = \frac{(2k)!}{2^k\, k!}\, \overline{V}(n, k),$$

then

(8.6) $$\overline{V}(n + 1, k) = \overline{V}(n, k - 1) + k^2\, \overline{V}(n, k).$$

$\overline{V}(n, k)$:

k / n	0	1	2	3	4
0	1				
1	1	1			
2	1	5	1		
3	1	21	14	1	
4	1	85	147	30	1

In view of (8.3), properties of $V(2n, k)$ can be read off from those of $U(2n, 2k)$. In addition it is clear from (8.1) that $V(2n, k)$ is the number of partitions of Z_{2n} into k blocks all of even cardinality.

9. Some additional examples. We now exhibit several additional examples of pairs of inverze arrays. I. We first take

(9.1) $$f(z) = ze^{-z}.$$

It is known [6, p. 125] that the inverse function is given by

$$g(z) = \sum_{n=1}^{\infty} \frac{u^{n-1} z^n}{n!}. \tag{9.2}$$

Moreover

$$e^{\lambda g(x)} = 1 + \sum_{n=1}^{\infty} \frac{\lambda (\lambda + n)^{n-1} z^n}{n!}. \tag{9.3}$$

Thus we have

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k A_{nk} x^k = e^{xf(z)} = \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} e^{-kz} =$$

$$= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} \sum_{j=0}^{\infty} (-1)^j \frac{(kz)^j}{j!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n-k} x^k.$$

Therefore

$$A_{nk} = (-1)^{n-k} \binom{n}{k} k^{n-n} \qquad (1 \leqslant k \leqslant n). \tag{9.4}$$

In the next place we have

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k B_{nk} x^k = e^{xg(z)}.$$

By (9.3) this is equal to

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} x (x + n)^{n-1} = 1 + x \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n=1} \binom{n-1}{k} n^{n-k-1} x^k$$

$$= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=i}^{n} \binom{n-1}{k-1} n^{n-k} x^k.$$

We have therefore

$$B_{nk} = \binom{n-1}{k-1} n^{n-k} \qquad (1 \leqslant k \leqslant n). \tag{9.5}$$

II. As a second example we take

$$f(z) = z (1 + z)^{-\lambda}. \tag{9.6}$$

It is known [6, p. 125] that the inverse function is

$$g(z) = \sum_{n=1}^{\infty} \binom{\lambda n}{n-1} \frac{z^n}{n}. \tag{9.7}$$

Moreover

$$(g(z))^k = k \sum_{n=k}^{\infty} \binom{\lambda n}{n-k} \frac{z^n}{n} \qquad (k \geqslant 1). \tag{9.8}$$

Thus

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k A_{nk} x^k &= \exp\{xz(1+z)^{-\lambda}\} \\ &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} (1+z)^{-\lambda k} \\ &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} \sum_{j=0}^{\infty} \binom{\lambda k + j - 1}{j} z^j \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\lambda k)_{n-k} x^k, \end{aligned}$$

where

$$(a)_n = a(a+1) \dots (a+n-1).$$

Hence

$$A_{n,k} = (-1)^{n-k} \binom{n}{k} (\lambda k)_{n-k}. \tag{9.9}$$

As for B_{nk}, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k B_{nk} x^k = e^{xg(z)} &= \sum_{k=0}^{\infty} \frac{x^k}{k!} (g(z))^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} \sum_{n=k}^{\infty} \binom{\lambda n}{n-k} \frac{z^n}{n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!} \binom{\lambda n}{n-k}. \end{aligned}$$

Thus

$$B_{n,k} = \frac{(n-1)!}{(k-1)!} \binom{\lambda n}{n-k}. \tag{9.10}$$

III. As an example of a different kind we take

$$f(z) = z\left(\frac{z}{e^z - 1}\right)^\lambda, \tag{9.11}$$

where λ is independent of z but otherwise arbitrary. We recall [5, Ch. 6] that Bernoulli numbers of order λ are defined by

$$\left(\frac{z}{e^z - 1}\right)^\lambda = \sum_{n=0}^{\infty} B_n^{(\lambda)} \frac{z^n}{n!}. \tag{9.12}$$

It follows from (9.12) that $B_n^{(\lambda)}$ is a polynomial in λ of degree n.

It follows from (9.11) and (9.12) that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k A_{nk} x^k &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} \left(\frac{z}{e^z - 1}\right)^{\lambda k} \\ &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} \sum_{j=0}^{\infty} B_j^{(\lambda k)} \frac{z^j}{j!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(\lambda k)} x^k \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \binom{n}{k} B_{n-k}^{(\lambda k)} x^k, \end{aligned}$$

since $B_n^{(0)} = \delta_{n0}$. We have therefore

$$A_{nk} = \binom{n}{k} B_{n-k}^{(\lambda k)} \qquad (1 \leqslant k \leqslant n). \tag{9.13}$$

For $\lambda = -1$ it is evident from (9.11) that we have

$$A_{nk} = S(n, k) \qquad (1 \leqslant k \leqslant n), \tag{9.14}$$

so that

$$\binom{n}{k} B_{n-k}^{(-k)} = S(n, k),$$

a known property of $B_n^{(\lambda)}$.

To construct the inverse array (B_{nk}) we make use of the Lagrange expansion [6, p. 125]:

Let

$$u = z / \phi(z), \tag{9.15}$$

where $\phi(z)$ is analytic about $z = 0$, $\phi(0) \neq 0$. Then

$$z = \sum_{n=1}^{\infty} \frac{u^n}{n!} \left[\frac{d^{n-1} (\phi(x))^n}{dx^{n-1}} \right]_{x=0}. \tag{9.16}$$

Moreover if $f(z)$ is analytic about the origin then

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{u^n}{n!} \left[\frac{d^{n-1} f'(x) (\phi(x))^n}{dx^{n-1}} \right]_{n=0}. \tag{9.17}$$

In particular

$$z^k = k \sum_{n=1}^{\infty} \frac{u^n}{n!} \left[\frac{d^{n-1} \{x^{k-1} (\phi(x))^n\}}{dx^{n-1}} \right]_{x=0}. \tag{9.18}$$

For the application we take

$$\phi(z) = \left(\frac{e^z - 1}{z} \right)^{\lambda}.$$

Then, by (9.18),

(9.19)

$$(g(z))^k = k \sum_{n=k}^{\infty} \frac{z^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} \left\{ x^{k-1} \left(\frac{e^x - 1}{x} \right)^{\lambda n} \right\} \right]_{x=0} \qquad (k = 1, 2, 3, \ldots).$$

Since, by (9.12),

$$\left(\frac{e^x - 1}{x} \right)^{\lambda n} = \left(\frac{x}{e^x - 1} \right)^{-\lambda n} = \sum_{j=0}^{\infty} B_j^{(-\lambda n)} \frac{x^j}{j!},$$

$$\left[\frac{d^{n-1}}{dx^{n-1}} \left\{ x^{k-1} \left(\frac{e^x - 1}{x} \right)^{\lambda n} \right\} \right]_{x=0} = (n-1)!\, B_{n-k}^{(-\lambda n)},$$

it is clear that (9.19) becomes

$$(g(z))^k = k \sum_{n=k}^{\infty} \frac{z^n}{n(n-k)!} B_{n-k}^{(-\lambda n)} \qquad (k = 1, 2, 3, \ldots). \tag{9.20}$$

It follows that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k B_{nk} x^k &= 1 + \sum_{k=1}^{\infty} \frac{x^n}{k!} (g(z))^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} \sum_{n=k}^{\infty} \frac{z^n}{n(n-k)!} B_{n-k}^{(-\lambda n)} \\ &= 1 + \sum_{n=k}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}^{(-\lambda n)} x^k \end{aligned}$$

and therefore

$$B_{nk} = \binom{n-1}{k-1} B_{n-k}^{(-\lambda n)} \qquad (1 \leqslant k \leqslant n) \tag{9.21}$$

It is interesting to compare (9.21) with (9.13). We remark that, for $\lambda = 1$, (9.21) reduce to.

$$B_{nk} = (-1)^{n-k} S_1(n, k) \qquad (1 \leqslant k \leqslant n). \tag{9.22}$$

Thus

$$\binom{n-1}{k-1} B_{n-k}^{(n)} = (-1)^{n-k} S_1(n, k),$$

which again is a known property of $B_n^{(\lambda)}$.

IN. The last example can be generalized considerably. Let

$$f(z) = z \left\{ \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \right\}^{\lambda} \qquad (a_0 = 1) \tag{9.23}$$

be analytic about $z = 0$. Define $\beta_n^{(\lambda)}$ be means of

$$\left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \right)^{\lambda} = \sum_{n=0}^{\infty} \beta_n^{(\lambda)} \frac{z^n}{n!}, \tag{9.24}$$

so that $\beta_n^{(\lambda)}$ is a polynomial in λ of degree n and $\beta_n^{(0)} = 0$ for $n > 0$.

It follows from (9.23) and (9.24) that

$$\begin{aligned} e^{xf(z)} &= \sum_{k=0}^{\infty} \frac{(xz)^n}{k!} \left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \right)^{\lambda k} \\ &= \sum_{k=0}^{\infty} \frac{(xz)^k}{k!} \sum_{j=0}^{\infty} \beta_j^{(\lambda k)} \frac{z^j}{j!} \\ &= \sum_{n=0}^{o} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{u}{k} \beta_{n-k}^{(\lambda n)} x^k \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \binom{n}{k} \beta_{n-k}^{(\lambda k)} x^{\lambda}. \end{aligned}$$

Therefore

$$A_{nk} = \binom{n}{k} \beta_{n-k}^{(\lambda k)}. \tag{9.25}$$

To construct the inverse array we again apply (9.18). We have

$$\begin{aligned} (g(z))^k &= k \sum_{n=k}^{\infty} \frac{z^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} \left\{ x^{k-1} (f(x))^{-\lambda n} \right\} \right]_{x=0} \\ &= k \sum_{n=k}^{\infty} \frac{z^n}{n(n-k)!} \beta_{n-k}^{(-\lambda n)} \qquad (k = 1, 2, 3, \ldots). \end{aligned}$$

It follows that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_k B_{nk} x^k &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} (g(z))^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} \sum_{n=k}^{\infty} \frac{z^n}{n(n-k)!} \beta_{n-k}^{(-\lambda n)} \\ &= 1 + \sum_{k=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \binom{n-1}{k-1} \beta_{n-k}^{(-\lambda n)} x^k, \end{aligned}$$

and therefo2e

$$B_{nk} = \binom{n-1}{k-1} \beta_{n-k}^{(-\lambda n)}. \tag{9.25}$$

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