A CONDITION OF BUSEMANN-FELLER TYPE FOR THE DERIVATION OF INTEGRALS OF FUNCTIONS IN THE CLASS $\phi(L)$

by

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1. — INTRODUCTION.

In this paper we give an extensión of results of Busemann-Feller [1934] and Rubio [1971] obtaining a caracterization of the derivation for integrals of functions belonging to $\phi(L)$, we make use of the behavior of the Hardy-Littlewood maximal function.

More details can be seen in Guzmán [1975].

For each $x \in R^n$ let $\mathcal{B}(x)$ be a family of open bounded sets such that for $B \in \mathcal{B}(x)$ we have $x \in B$ and there exists a sequence $\{B_k\}_{k \in N} \subset \subset \mathcal{B}(x)$ such that

$$\delta(B_k) \underset{k \to \infty}{\to} 0$$

where $\delta(B_k)$ is the diameter of B_k. We say then that $\{B_k\}$ *contracts to* x. The family

$$\mathcal{B} = \bigcup_{x \in R^n} \mathcal{B}(x)$$

is called a *differentiation basis in* R^n.

We call

$$\mathcal{B}_r = \{B \in \mathcal{B} : \delta(B) < r\},$$

similarly

$$\mathcal{B}_r(x) = \{B \in \mathcal{B}(x) : \delta(B) < r\}.$$

We say that $\mathcal{B}$ is a Busemann-Feller's basis ($B - F$ basis) if it is a differentiation basis and if $y \in B \in \mathcal{B}$ implies $B \in \mathcal{B}(y)$.

If $A \subset R^n$ is a Lebesgue measurable set, let $|A|$ be its measure.

For a function $f \in L^1_{\text{loc}}(R^n)$ we define the upper and lower derivatives of its integral by

$$\overline{D}(\int f, x) = \sup \{\limsup_{k\to\infty} \frac{1}{|B_k|}\int_{Bk} f(y)dy : \{B_k\} \subset \mathcal{B}(x),\ B_k \to x\}$$

$$\underline{D}(\int f, x) = \inf \{\liminf_{k\to\infty} \frac{1}{|B_k|}\int_{Bk} f(y)dy : \{B_k\} \subset \mathcal{B}(x),\ B_k \to x\}$$

We say that $\mathcal{B}$ derives the integral of f if

$$\overline{D}(\int f, x) = \underline{D}(\int f, x) = f(x)$$

almost everywhere.

Let us define the maximal functions asociated to $\mathcal{B}$ and $\mathcal{B}_r$ by

$$Mf(x) = \sup_{x\in B\in \mathcal{B}} \frac{1}{|B|}\int_B |f(y)|dy$$

$$M_r f(x) = \sup_{x\in B\in \mathcal{B}_r} \frac{1}{|B|}\int_B |f(y)|dy$$

If $\mathcal{B}$ is a $(B - F)$ basis, then $Mf(x)$, $M_rf(x)$, $\overline{D}(\int f, x)$ and $\underline{D}(\int f, x)$ are measurable functions (see Guzmán [1975]).

Let us define the class $\phi(L)$. Let ϕ be a function,

$$\phi : [0, \infty) \to [0, \infty)$$

such that

1) ϕ is strictly increasing.
2) $\lim_{x\to 0} \phi(x) = \phi(0) = 0$

Then, we define

$$\phi(L) = \{f \in L^1_{\text{loc}}(R^n) : \int\phi(|f|)dx < \infty\}$$

It is clear that L^p and the Orlicz spaces are examples of classes $\phi(L)$.

2. – *Derivación of integrals of functions in the $\phi(L)$ classes.*

We will assume from now on that ϕ is a function verifying (1) and (2).

The following result extends Rubio's theorem [1971] to the $\phi(L)$ class.

THEOREM

Let $\mathcal{B}$ *be a* $(B - F)$ *differentiation basis in* R^n *Then, the following statements are equivalent:*

(A) $\mathcal{B}$ *derives integrals of functions* $f \in \phi(L)$

(B) *For each* $\lambda > 0$, *for each decreasing sequence of positive real numbers,* $\{b_k\}_{k \in N}$, *such that* $\{b_k\} \underset{k \to \infty}{\to} 0$, *and for each decreasing sequence of positive functions with compact support,* $\{f_k\}_{k \in N} \subset \phi(L)$, *such that*

$$\lim_{k \to \infty} \int \phi(f) dx = 0$$

we have

$$\lim |\{x \in R^n : M_{b_k} f_k (x) > \lambda\}| = 0$$

PROOF. —

(A) implies (B)

In fact, if we suposse that there exist λ, $\{b_k\}_{k \in N}$ and $\{f_k\}_{k \in N}$ as in (B), and such that

$$\lim |\{x \in R^n : M_{b_k} f_k (x) > \lambda\}| \neq 0$$

then for some $\eta > 0$ and some subsequence (we will denote the subsequence by the same index k) we have

i) $\lim_{k \to \infty} |\{x \in R^n : M_{b_k} f_k(x) > \lambda\}| > 2\eta$

ii) $\sum_{k=1}^{\infty} \int \phi(f_k) dx < \infty$

Define $f^*(x) = \sup_{k \in N} f_k(x)$, which is in $\phi(L)$. Consider $E_k = \{x \in R^n : M_{b_k} f_k (x) > \lambda\}$ for each $k \in N$. It is obvious that each E_k is a bounded set and that $|E_k| > 2\eta$. Since $\{E_k\}_{k \in N}$ is contractive we have that

$$E = \bigcap_{k \in N} E_k$$

has measure, bigger than 2η.

For each $x \in E$ we can construct a sequence $\{B_k(x)\}_{k \in N} \subset B(x)$ with $\delta(B_k(x)) < b_k$ and also

$$\frac{1}{|B_k(x)|} \int_{B_k(x)} f_k(y) dy > \lambda$$

for each $k \in N$.

Moreover, $\{f_k\}_{k \in N}$ and E satisfy the hypothesis of Egoroff's theorem, thus we can obtain a measurable subset, $A \subset E$, such that $|A| > \eta$ and $\{f_k\}_{k \in N}$ converges uniformly on A.

On A we have then that for some $k_0 \in N$

$$f_k(x) < \frac{\lambda}{2} \text{ if } k > k_0$$

If we define

$$f(x) = \sup_{k > k_0} f_k(x)$$

we have $f \in \phi(L)$ and for each $x \in A$

$$\overline{D}(\int f, x) > \lambda, \qquad f(x) < \frac{\lambda}{2},$$

but this is in contradiction with (A).

(B) implies (A).

Observe that according to the Busemann-Feller's result [1934], the basis B is in particular a density basis.

On the other hand it is clear that it suffices to prove the result for positive functions f with compact support.

Let f be in $\phi(L)$, $f \geqslant 0$ and let the support of f be a compact set. We will prove that for each $a > 0$ the measure of

$$B(a) = \{x \in R^n : |\, \overline{D}(\int f, x) - f(x)\,| > a\}$$

is zero; in the same way we can proceed with the lower derivative.

Let $\{g_k\}_{k \in N}$ be a monotone increasing sequence of simple functions such that

$$\lim_{k \to \infty} g_k(x) = f(x) \qquad \text{almost everywhere.}$$

Consider

$$h_k(x) = f(x) - g_k(x)$$

for each $k \in N$. From the conditions on $\{g_k\}_{k \in N}$ and ϕ and by the theorem of dominated convergence it is clear that $\{h_k\} \subset \phi(L)$ is a monotone decreasing sequence such that

$$\lim_{k \to \infty} \int \phi(h_k) dx = 0$$

Since $\mathcal{B}$ is a density basis we have

$$| B(a) | = | \{x \in R^n : | \overline{D}(\int h_k, x) - h_k(x) | > a\} |$$

for each $k \in N$.

Let $\{b_k\}_{k \in N}$ be a decreasing sequence of positive real numbers with $\lim_{k \to \infty} b_k = 0$

then

$$| B(a) | = | \{x \in R^n : | \overline{D}(\int h_k, x) - h_k(x) | > a\} | \leqslant$$

$$\leqslant | \{x \in R^n : M_{b_k} h_k(x) > \frac{a}{2}\} | + \frac{1}{\phi(a/2)} \int \phi(h_k(x)) dx$$

for each $k \in N$.

The proof is finished by observing that both terms tend to zero as k tend to infinite, the first one, by the hypothesis and the second one, by the construction of the h_k.

Observe also that the part $(A) => (B)$ is true without the condition

$$\lim_{x \to 0} \phi(x) = \phi(0)$$

For aplications of this result see Peral [1974].

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