SIZE OF GAPS AND REGION OF OVERCONVERGENCE

BY

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1. Let

$$f(z) = \sum_{0}^{\infty} c_n z^n$$

be regular in a simply connected schlicht domain $\mathfrak{D}$ properly containing $|z| < 1$ but not the whole of $|z| = 1$. Suppose also the power series is lacunary with $c_n = 0$ for the sequence of intervals $n_k \leqslant n \leqslant N_k$. Then if $N_k/n_k \geqslant \lambda > 1$ there is overconvergence at each point of $|z| = 1$ in $\mathfrak{D}$, extending to the whole of $\mathfrak{D}$ if N_k/n_k tends to infinity. [Ostrowski's theorems **2**, 12-13]. It is natural to conjecture the existence of a region of overconvergence $\mathfrak{D}(\lambda)$ corresponding to a given value of λ independent of the particular $f(z)$ and further that this $\mathfrak{D}(\lambda)$ increases with λ and has $\mathfrak{D}$ as limit as λ tends to infinity. These conjectures are established below in the case that $\mathfrak{D}$ is the whole plane cut radially from a point on $|z| = 1$ to infinity, or one of a family of domains bounded by logarithmic spirals and containing $|z| = 1$ apart from a single point. The method of interpolating the c_n by means of an entire function is used. With $c_n = G(n)$ the representation [**3**, 53]

$$f(-z) = \sum_{0}^{\infty} (-)^n G(n) z^n = \frac{1}{2i} \int \frac{G(\zeta) z^{\zeta} d\zeta}{\sin \pi \zeta}$$

establishes overconvergence provided $G(\zeta)$ is exceptionally small on a particular sequence of arcs $|\zeta| = n + \frac{1}{2}$. This property of $G(\zeta)$ will follow from the existence of sufficiently numerous groups of zeros of $G(\zeta)$ corresponding to the vanishing c_n. The phenomenon of overconvergence is thus related to the general theory of entire functions and a generalization is immediately suggested. Our hypothesis need not demand $c_n = 0$ for all n in $n_k \leqslant n \leqslant N_n$ but only for sufficiently many such values of n. It is convenient to define « occasional density » for a sequence of integers. In the special case of the regions just described we then have the generalised assertions (i) *If the occasional density of zero coefficients is positive there is overconvergence to $f(z)$ at regular points of $|z| = 1$* (ii) *If λ is sufficiently large and the occasional density of zero coefficients in the gaps is sufficiently near unity then overconvergence extends to a prescribed point in the region of regularity and* (iii) *If N_k/n_k tends to infinity and almost all c_n vanish in the gaps then overconvergence extends to the whole interior of $\mathfrak{D}$*. These results extend Theorem E of **4**.

The methods used do not conveniently extend to general domains but other special cases can be handled. We can prove for example the following theorem.

If
$$f(z) = \sum_0^{\infty} c_n z^n$$
is regular on an arc C of length γ on its circle of regularity $|z| = 1$ and if the occasional density of zero c_n exceeds $1 - \gamma/2\pi$ then there is overconvergence on C. From this theorem follow the gap theorems of FABRY and POLYA [**5,** 626] just as Hadamard's gap theorem is commonly obtained from Ostrowski's overconvergence theorem [**2,** 13].

2. In this paragraph $\mathfrak{D}$ is one of a family of domains bounded by spirals and containing the whole of $|z| = 1$ except for a single point. It is evident that this point can be taken as $z = 1$ without loss of generality. The statements of 1 are proved by a series of lemmas most of which are well known.

LEMMA 1. *If $f(z)$ is regular in the whole plane less the segment $z \geqslant 1$ of the positive real axis then*
$$f(z) = \sum_0^{\infty} G(n) z^n$$
where $G(z)$ is an entire function of exponential type and is moreover of zero type in the angle $|\arg z| \leqslant \alpha$ where $\alpha < \frac{1}{2}\pi$ may be arbitrarily near $\frac{1}{2}\pi$. $G(z)$ will depend on the choice of α.

This lemma is a straightforward exercise in the interpolation of the coefficients of power series as developed in particular by PÓLYA [**5,** 623]. $G(z)$ is defined by

$$G(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{-z\zeta} f(e^{-\zeta})\, d\zeta.$$

The contour Γ joins the points $-R-\pi i$ and $-R+\pi i$ without intersecting the negative real axis. R can be chosen as large as we please. The choice of R affects $G(z)$ although not the values $G(n)$ for positive integers. α in the lemma can evidently be taken as $tan^{-1}(R/\pi)$. The properties stated in the lemma are readily obtained by taking Γ close to the contour formed by the two radii joining the origin to the points $-R \pm \pi i$.

PÓLYA discusses [5, 598 — 610] the converse theory. Our results amount to modified converses of Lemma 1. To reach them it seems necessary to depart from PÓLYA's methods and discuss the definition of $f(z)$ in terms of $G(z)$ more directly [**3,** 52] by the integral

$$f(-z) = \frac{1}{2i} \int_{\Gamma} \frac{G(\zeta)\, z^{\zeta}\, d\zeta}{\sin \pi\zeta} \tag{1}$$

where Γ encloses the points $\zeta = 0, 1, 2, \ldots$ and apart from a necessary modification near the origin consists of the radii $\arg \zeta = \pm\alpha$.

LEMMA 2. *If $G(z)$ is an entire function satisfying for a fixed α in $0 < \alpha < \frac{1}{2}\pi$ the condition*

$$\limsup_{r\to\infty} r^{-1} \log |G(re^{i\theta})| \leqslant 0, \qquad \theta = \pm\alpha \tag{2}$$

then $f(z)$ as defined by (1) *is regular in the region of the z plane defined by the inequalities*

$$\begin{aligned} R\left\{e^{i\alpha} \log(-z)\right\} &< \pi \sin\alpha \\ R\left\{e^{-i\alpha} \log(-z)\right\} &< \pi \sin\alpha. \end{aligned} \tag{3}$$

The inequalities (3) are evidently sufficient to ensure convergence of the integral (1). With $w = \log(-z)$ they require w to be confined to the acute angle determined by the lines joining $\pm\pi i$ to $\pi \tan\alpha$ and bisected by the segment $w < \pi \tan\alpha$ of the real axis (Fig. 2).

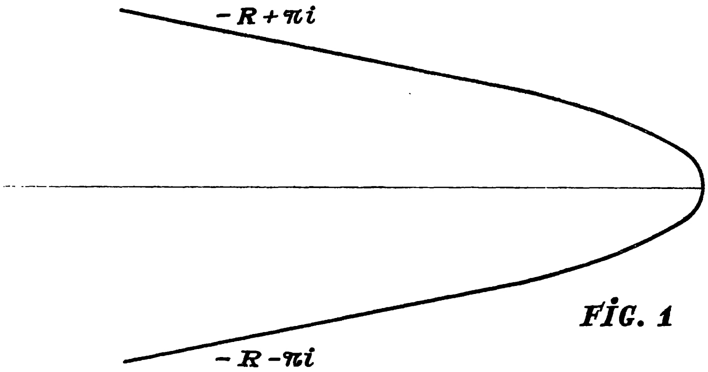


FIG. 1

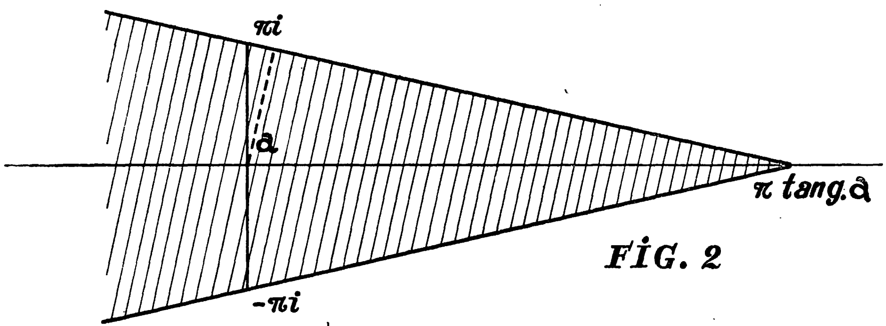
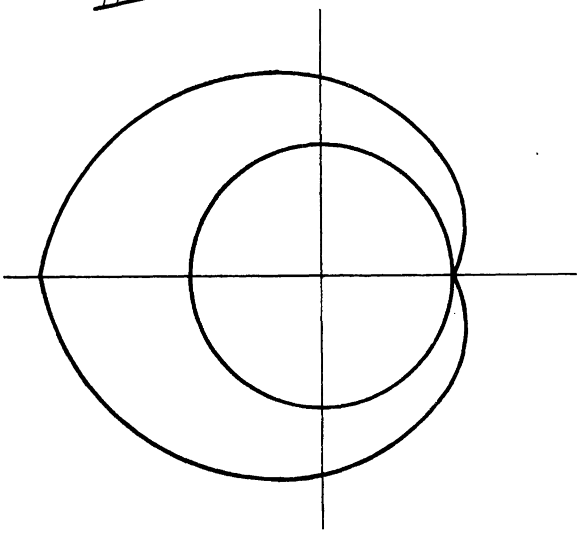


FIG. 2



In the z plane itself the inequalities (3) require z to lie in the region $\mathfrak{D}_\alpha$ between two equiangular spirals (Fig. 3). Outside $|z| = 1$ the region is schlicht but inside has to be regarded as a multiply covered RIEMANN surface. It may be noted that for $|z| > 1$ this region increases as α increases and its limit as α tends to $\frac{1}{2}\pi$ is the whole of $|z| > 1$ except for the segment $z > 1$ of the real axis. For $|z| < 1$ the region of regularity decreases as α increases. However if $G(z)$ is of minimum type throughout the angle $|\arg z| \leqslant \alpha$ it can be shown [**3,** 133-138] that $f(z)$ is regular and one valued throughout $|z| < 1$. This is done by noting that

$$|G(\zeta) z^\zeta| < K \exp\{\varepsilon r + r \cos\theta \log|z| + \pi r |\sin\theta|\} \tag{4}$$

for $\zeta = re^{i\theta}$ with $|\theta| \leqslant \alpha$ and $|\mathfrak{I}(z)| < \pi$. From this it follows that the integral

$$\int G(\zeta) z^\zeta d\zeta / \sin \pi\zeta$$

taken along $|\zeta| = \left(n + \frac{1}{2}\right)\pi$ from $\arg\zeta = -\alpha$ to $\arg\zeta = \alpha$ will tend to zero as n tends to infinity if $|z| < 1$.

By calculation of residues it follows that

$$f(-z) = \sum_0^\infty (-)^n G(n) z^n. \tag{5}$$

It is evident that Lemma 1 can be extended to include a converse of Lemma 2.

If $G(\zeta)$ was substantially smaller on a particular sequence of arcs $|\zeta| = n + \frac{1}{2}$, $|\arg\zeta| \leqslant \alpha$ then evidently the same procedure will establish convergence of a subsequence of partial sums of (5) into a wider region of variation of $\log z$.

Suppose, for example, that the inequality

$$|G(re^{i\theta})| \leqslant K \exp\{(-h(\theta) + \varepsilon) r\}$$

is satisfied for an infinity of values $r = m_k + \frac{1}{2}$ where $h(\theta)$ is some continuous non-negative function and ε an arbitrary positive number. Then for such values of r, (4) can be replaced by

$$\left|\frac{G(\zeta)\, z^{\zeta}}{\sin \pi \zeta}\right| \leqslant K \exp\left\{(-h(\theta) + \varepsilon + \Re(e^{i\theta}\log z) - \pi|\sin\theta|)\, r\right\}, \quad \zeta = re^{i\theta},$$

and it is evident that as k tends to infinity

$$\sum_{0}^{m_k} (-)^n G(n)\, z^n$$

converges to $f(-z)$ throughout the region defined by the family of inequalities

$$\Re\left\{e^{i\theta}\log z\right\} < h(\theta) + \pi\,|\sin\theta|.$$

The case $h(\theta) = 0$ gives ordinary convergence in $\log|z| < 0$, $|\arg z| < \pi$. These remarks can be summarized in the following two statements.

LEMMA 3. *If $G(z)$ is an entire function of exponential type satisfying* (2) *for $|\theta| \leqslant \alpha$ then the function $f(z)$ defined by* (1) *is regular and one valued for $|z| < 1$ and has the* Taylor *series* (5).

LEMMA 4. *If $G(z)$ satisfies the conditions of Lemma 3 and if also there exists a sequence of integers m_k such that*

$$|G(re^{i\theta})| < K \exp\left\{(-h(\theta) + \varepsilon)\, r\right\}, \quad |\theta| \leq \alpha, \quad K = K(\varepsilon)$$

for all sufficiently large $r = m_k + \frac{1}{2}$ then

$$S_k(z) = \sum_{0}^{m_k} (-)^n\, G(n)\, z^n$$

converges to $f(-z)$ throughout the region defined by the family of inequalities

$$\Re\left\{e^{i\theta}\log z\right\} < h(\theta) + \pi\,|\sin\theta|. \tag{6}$$

The region in the $\log z$ plane defined by (6) will clearly be convex.

Suppose now $G(z)$ has groups of zeros $G(n) = 0$ for $n_k \leqslant n \leqslant \lambda n_k$ where $\lambda > 1$ and n_k tends to infinity. Then with arbitrary $\varepsilon > 0$ and k sufficiently large $G(\zeta)/\sin \pi\zeta$ is regular in the part sectors S_k

$$n_k \leqslant |\zeta| \leqslant \lambda n_k, \qquad |\arg \zeta| \leqslant \alpha$$

and satisfies there the inequality

$$\left|\frac{G\left(re^{i\theta}\right)}{\sin\left(\pi re^{i\theta}\right)}\right| < \exp\left\{(\varepsilon - \pi\,|\sin\theta|)\,r\right\}. \tag{7}$$

Now this inequality is strongest on the arms $\theta = \pm\alpha$ of the sector and can therefore be improved upon by use of a harmonic majorant based on the upper bound on the boundary. The resulting harmonic majorant could indeed be given an explicit form since the interior of the sector may be conformally represented on the interior of a circle by means of elementary and elliptic functions.

On or near the median circle $r = \lambda^{\frac{1}{2}} n_k$ we shall have indeed from (7) and the associated harmonic majorant the inequality

$$\left|\frac{G\left(re^{i\theta}\right)}{\sin\left(\pi re^{i\theta}\right)}\right| < \exp\left\{[\varepsilon - H(\theta)]\,r\right\}$$

where $H(\theta)$ is a continuous function depending on α and λ.

Overconvergence will hold in the region satisfying

$$\Re\left\{e^{i\theta}\log(-z)\right\} < H(\theta), \qquad |\theta| \leqslant \alpha. \tag{8}$$

This region would coincide with the region of regularity prescribed by Lemma 2 if

$$H(\theta) = \pi\tan\alpha\cos\theta.$$

No overconvergence would be established if for example $H(\theta) = 0$. The whole of $|z| = 1$, $|\arg z| < \pi$ will be within the region of overconvergence if $H(\theta) > \pi\,|\sin\theta|$ for $|\theta| < \alpha$. To obtain approximations to $H(\theta)$ consider the modification of (7) obtained by multiplication by a suitable exponential factor. On the boundary of S_k

$$\left|\frac{e^{\pi\zeta\tan\alpha}G(\zeta)}{\sin\pi\zeta}\right| < \begin{matrix}\exp(\varepsilon r), \qquad \zeta = re^{\pm i\alpha} \\ \exp(\varepsilon r + \pi r\cos\theta\tan\alpha - \pi r\,|\sin\theta|),\end{matrix} \tag{9}$$

$$r = n_k + \frac{1}{2}, \qquad r = N_k - \frac{1}{2}.$$

Now using only the «two constants» theorem and noting that the harmonic measure of the circular arcs of the boundary of S_k is small when λ is large we see that given $\delta > 0$ we shall have on and near the median arc

$$\left|\frac{e^{\pi\zeta \tan\alpha} G(\zeta)}{\sin \pi \zeta}\right| < e^{\delta r}, \qquad |\zeta| = r, \qquad |r - n_k^{\frac{1}{2}} N_k^{\frac{1}{2}}| < 1$$

provided λ is sufficiently large. This ensures overconvergence in the region defined by (8) with

$$H(\theta) = \pi \tan \alpha \cos \theta + \delta.$$

With δ sufficiently small any prescribed point in $\mathfrak{D}_\alpha$ is included in the overconvergence region. This establishes the first two conjectures stated in the introduction. We may note in passing that by using the estimate (9) for $|\theta| \leqslant \alpha' < \alpha$ and the original approximation (7) in the rest of the range a proof of BOURION's theorem [**2,** 33] in this special case would result.

3. The procedure of comparing $G(z)$ with $\sin \pi z$ in a region where the zeros of $G(z)$ include those of $\sin \pi z$ can be modified by using a different comparison function. In order to exploit known results on entire functions with measurable sequences of zeros it seems convenient to introduce a new density measure for sequences of integers. We compare a given sequence $\{\lambda_n\}$ with a measurable sequence $\{\lambda_n^*\}$ of density d. If in some arbitrarily large intervals $r \leqslant \lambda_n \leqslant \lambda r$ the sequence $\{\lambda_n\}$ includes $\{\lambda_n^*\}$ we say that $\{\lambda_n\}$ is of occasional density to the basis $(1, \lambda)$ at least d. The occasional density Δ to the basis $(1,\lambda)$ is the upper bound of such d. The occasional density without qualification is the limit of Δ as λ tends to unity from above. If $\{\lambda_n\}$ includes $\{\lambda_n^*\}$ in a sequence of intervals $n_k \leqslant \lambda_n \leqslant N_k$ with N_k/n_k tending to infinity, we say that the occasional density of $\{\lambda_n\}$ on an infinite basis is at least d. If Δ is the upper bound of such d, we call Δ the upper density on an infinite basis. The comparison function to replace $\sin \pi z$ will be

$$G^*(z) = \prod_1^\infty (1 - z^2/\lambda_n^{*2}).$$

If the occasional density of the sequence $\{\lambda_n\}$ is unity, [**5,** 571 **1,** 281] there will exist a $G^*(z)$ possibly depending on $\varepsilon > 0$ such that for sufficiently large $|z| = n + \frac{1}{2}$ and sufficiently large $|z|$ with $\arg z = \pm \alpha$

$$e^{-\varepsilon |z|} < |G^*(z)/\sin \pi z| < e^{\varepsilon |z|}.$$

Given ε in advance the same inequality will be available associated with a sequence $\{\lambda_n\}$ if its occasional density is sufficiently near unity. These remarks suffice for the extensions of the results of 2 as stated in (ii) and (iii) of 1. To establish (i) note that if the occasional density of $\{\lambda_n\}$ is positive there will exist a $G^*(z)$ with a positive density h of zeros. This function will satisfy the inequality

$$\exp\{(h|\sin\theta| - \varepsilon)r\} < |G^*(re^{i\theta})| < \exp\{(h|\sin\theta| + \varepsilon)\}$$

for $\theta = \pm\alpha$ or $r = n + \frac{1}{2}$ and r sufficiently large.

$G(z)/G^*(z)$ will be regular in a sequence of part sectors S_k and will satisfy the inequality

$$|G(re^{i\theta})/G^*(re^{i\theta})| < \exp\{(\varepsilon - h|\sin\theta|r\}$$

on the boundary. Denote by S'_k a part of S_k defined as follows. If S_k is defined by $r_k \leqslant |z| \leqslant \lambda r_k$, $|\arg z| \leqslant \alpha$ then S'_k is defined by $\lambda^{\frac{1}{4}} r_k \leqslant |z| \leqslant \lambda^{\frac{3}{4}} r_k$, $|\arg z| \leqslant \frac{1}{2}\alpha$. It is evident that there exists a positive δ such that for $\zeta = re^{i\theta}$ in S'_k and k sufficiently large then

$$|G(re^{i\theta})/G^*(re^{i\theta})| < e^{-\delta r}.$$

The integrand of (1) will now satisfy in S'_k with some positive δ' the inequality

$$|G(\zeta)z^\zeta/\sin\pi\zeta| < \exp\{-\delta' r + (h-\pi)|\sin\theta|r + \Re(\zeta\log z)\}, \zeta = re^{i\theta}$$

if k is sufficiently large. This is sufficient to establish the overconvergence on $|z| = 1$. The conclusion follows also from Ostrowski's original theorem on observing that $|G(n)| < \exp(-\delta' n)$ for the integers in S'_k.

4. Suppose $f(z) = \sum_0^\infty c_n z^n$ is regular for $|z| < 1$ and that its singularities on $|z| = 1$ are within the arc $|\arg z| < \eta$. Then the coefficients c_n can be interpolated by an entire function $G(z)$ of exponential type 2π and satisfying the inequality

$$|G(re^{i\theta})| < \exp\{(\eta|\sin\theta| + \varepsilon)r\} \tag{10}$$

for sufficiently large r and $|\theta| \leqslant \alpha$ with a certain positive value of α. Conversely if $G(z)$ is regular in $|\arg z| \leqslant \alpha$ and satisfies (10) the integral (1) defines a function regular in the region defined by the inequalities

$$R\left\{e^{i\alpha} \log(-z)\right\} < (\pi - \eta) \sin \alpha$$

$$R\left\{e^{-i\alpha} \log(-z)\right\} < (\pi - \eta) \sin \alpha.$$

The representation (5) follows as before. The considerations developed in 3 will also continue to apply provided $h > \eta$. This establishes the statement (i).

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