A GENERALIZATION OF THE DUAL BETA-FUNCTION MODEL.

by

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I. INTRODUCTION

The motivation of this work is placed on purely physical grounds. It is known that the Veneziano amplitude is given by the Euler Beta-function

$$V(s,\ t) = B(\,-\,a(s),\ -\,a(t))$$

where $a(s)$ and $a(t)$ are linear Regge trajectories exchanged in the direct and crossed channels with the simple expressions

$$a(X) = 1 + X$$

Several conditions are essential in order that an amplitude should reproduce the physical reality: decomposition into poles, asymptotic Regge behaviour and crossing symetry. In fact the Beta-function satisfies all these conditions, and this is a familiar fact.

What we want to do here is analyse from a mathematical point of view an amplitude which generalizes the Veneziano amplitude, and which was introduced (Q) in the context of the string-model with non--constant density of mass and elasticity, namely

$$(1) \quad A(s,t,\varepsilon) = \int_0^1 X^{-s-2}(1 - X^{1+\varepsilon})^{-t-2}\, exp\{(t+2)\ (X - X^{1+\varepsilon})\}\ dX$$

$$s, t \in C \ \ \varepsilon \geqslant 0$$

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This integral is well defined for $Re(s) < -1$ and $Re(t) < -1$. For the rest of the complex s and t-planes we need an analytic continuation. By a suitable deformation of the contour of integration it is easily continued into the s-plane but it is not evident that a similar procedure would allow us to continue it into the whole complex t-plane.

In what follows we shall investigate the decomposition into poles and the asymptotic behaviour of (1) in the t-plane, which at first view is not a trivial problem.

II. THE POLES

We can study the poles of (1) corresponding to the end-point $X^{1+\varepsilon} = 1$ by expanding the exponential in a series of powers of $(t+2)\ (X - X^{1+\varepsilon})$, making the change $X \to Y = 1 - X^{1+\varepsilon}$, then expanding the function $(1 - Y)^{\alpha}$ (where $\alpha = -\dfrac{s+\varepsilon+j+2}{1+\varepsilon} + \lambda - j$ and $\lambda = 0,\ldots\ldots, \infty$, $j = 0,\ldots\ldots\ldots, \lambda$) about the point $Y = 0$. Integrating term by term we find

$$A(s, t, \varepsilon) = \sum_{\varrho=0}^{\infty} R_{\varrho}(s, \varepsilon) \frac{1}{t - \varrho + 1} \tag{2}$$

where the residues are given by

$$R_{\varrho}(s, \varepsilon) = \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\lambda} \frac{(-1)^{\lambda-\mu+\varrho}}{1+\varepsilon} \binom{\lambda}{\mu} \frac{(\varrho+1)^{\lambda}}{\varrho} \binom{\dfrac{-s-2-\varepsilon+\mu}{1+\varepsilon} + \lambda - \mu}{\varrho} \tag{3}$$

To describe physical particles without the trouble of ancestors, $R_{\varrho}(s)$ must be a polynomial of degree ϱ in s. This is the case if the series (3) is convergent. To prove its convergence we shall demonstrate the following property of binomial coefficients:

$$\sum_{\mu=0}^{\lambda} (-1)^{\mu} \mu^{k} \binom{\lambda}{\mu} = 0 \qquad \text{for} \quad 0 \leq k < \lambda \tag{4}$$

Then, if (4) is true the terms of (3) with $\lambda > \varrho$ vanish, because

$$(5) \qquad \binom{\dfrac{-s-2-\varepsilon+\mu}{1+\varepsilon}+\lambda-\mu}{\varrho} = \sum_{\tau=o}^{\varrho} f_\tau(s, \varepsilon, \lambda)\, \mu^\tau$$

where $f_\tau(s, \varepsilon, \lambda)$ are known functions of s and ε.

Now the summation in (3) goes from $\lambda = 0$ to $\lambda = \mu$, and the series is finite sum which obviously converges in the s-plane.

III. PROOF OF (4)

PROPOSITION

In the ring $Z[X]$ of polynomials in one variable with coefficients in Z, the element

$$\sum_{i=0}^{\lambda} i^k \binom{\lambda}{i} X^i \qquad \lambda > k \geq 0$$

can be expressed in the form

$$\sum_{i=1}^{\lambda} \alpha_i X^i (1+X)^{\lambda-i}$$

with $\alpha_i = \binom{\lambda}{i}\beta_i$ and $\beta_i \in Z$

PROOF

We consider the operator $\left(X\dfrac{d}{dX}\right)$. By induction on k it is easy to see that

$$\left(X\frac{d}{dX}\right)^{(k)}\left(\sum_{i=0}^{\lambda}\binom{\lambda}{i}X^i\right) = \sum_{i=1}^{\lambda} i^k \binom{\lambda}{i} X^i \quad , \quad \lambda > k > 0$$

Now take $(1+X)^\lambda$; we have

$$\left(X\frac{d}{dX}\right)(1+X)^\lambda = \lambda X (1+X)^{\lambda-1}$$

We take

$$\left(X\frac{d}{dX}\right)^{(\varrho)}(1+X)^\lambda = \sum_{i=1}^{\varrho}\binom{\lambda}{i}\gamma_i X^i (1+X)^{\lambda-i} \quad , \quad \lambda > \varrho \quad , \quad \gamma_i \in \mathbf{Z}.$$

Then

$$\left(X\frac{d}{dX}\right)^{(\varrho+1)}(1+X)^{\lambda}=\left(X\frac{d}{dX}\right)\left(\sum_{i=1}^{\varrho}\binom{\lambda}{i}\nu_i\,X^i\,(1+X)^{\lambda-i}\right)=$$

$$=\sum_{i=1}^{\varrho}\nu_i\,i\binom{\lambda}{i}X^i\,(1+X)^{\lambda-i}$$

$$+\sum_{i=1}^{\varrho}\nu_i'\binom{\lambda}{i+1}X^{i+1}\,(1+X)^{\lambda-i-1}$$

But

$$\sum_{i=1}^{\varrho}\nu_i'\binom{\lambda}{i+1}X^{i+1}\,(1+X)^{\lambda-i-1}=\sum_{j=2}^{\varrho+1}\delta_j\binom{\lambda}{j}X^j\,(1+X)^{\lambda-j}\,,\ \varrho+1<\lambda \qquad \delta_i\in\mathbf{Z}$$

And grouping

$$\left(X\frac{d}{dX}\right)^{(\varrho+1)}(1+X)^{\lambda}=\nu_1\,\lambda\,X(1+X)^{\lambda-1}+\sum_{j=2}^{\varrho}\beta_j\binom{\lambda}{j}X^j\,(1+X)^{\lambda-j}+$$

$$+\,\delta_{\varrho+1}\binom{\lambda}{\varrho+1}X^{\varrho+1}\,(1+X)^{\lambda-\varrho-1}=\sum_{j=1}^{\varrho+1}\binom{\lambda}{i}\beta_i\,X^i\,(1+X)^{\lambda-i}$$

with $\beta_i\in\mathbf{Z}$.

It is sufficient now to consider the identity

$$\sum_{i=0}^{\lambda}\binom{\lambda}{i}X^i=(1+X)^{\lambda}$$

Corollary

$$\sum_{i=0}^{\lambda}i^k\binom{\lambda}{i}(-1)^i=0 \qquad\qquad \lambda>k\geqslant 0$$

Remark

The particular case $k=1$ has been proved in (^{K})

IV. THE ASYMPTOTIC BEHAVIOUR

We are interested in the asymptotic behaviour of (1) when $t\to-\infty$.

We can write $A(s, t, \varepsilon)$ as

$$(6)\quad A(s, t, \varepsilon) = \int_0^1 X^{-s-2} \exp\{-(t+2)[\log(1 - X^{1+\varepsilon}) - X + X^{1+\varepsilon}]\}\, dX$$

Because the function in the exponent has the property

$$h(X) = \log(1 - X^{1+\varepsilon}) - X + X^{1+\varepsilon} \leq h(0) = 0$$

and

$$\begin{aligned} h(X) &= -X^{1+\varepsilon} + 0(X^{1+\varepsilon}) - X + X^{1+\varepsilon} = -X + 0(X^{1+\varepsilon}) \\ &= -X + 0(X) \end{aligned}$$

we can apply Laplaces method ([D]) which gives

$$A(s, t, \varepsilon) \sim \Gamma(-s-1)(-t)^{s+1}, \text{ as } t \to -\infty$$

which is the desired Regge-behaviour we expected

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