THE LOCATION OF SINGULARITIES ON THE CIRCLE OF CONVERGENCE OF GAP SERIES, II

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1. We consider the functions

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

which have exactly k different singularities (all isolated non-critical and of finite exponential order) on their circles of convergence. We shall prove the following theorem for any even k, and for convenience we write $k = 2l$.

THEOREM. *If the upper density of small coefficients in (1) is greater than* $1 - 1/l$, *then the singularities* (1) *are located at* k *of the vertices of a regular polygon. This polygon has*

$$k, \tag{2}$$

$$\text{(a)} \quad k + w, \qquad \text{(b)} \quad k + 2w' \tag{3}$$

or

$$j\,l \tag{4}$$

sides, where w *divides* k, $3w' = k$ *and* j *is an integer greater than 3.*

In (3 a) the k singularities are located at the vertices of k/w regular w-agons (2), in (3 b) at the vertices of three regular w'-agons, and in (4) at those of two regular l-agons.

This Theorem is a generalization of a result of MACINTYRE and WILSON (1940, p. 79, Theorem 6) and depends on the argument from Function theory there developed in § 8 and § 6.

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(1) The qualifying phrase «on the circle of convergence» is to be understood here and throughout this paper.

(2) We take a single point to be a regular 1-agon, and a pair of diametrically opposite points to be a regular 2-agon.

The cases of $k = 4$ and $k = 6$ were dealt with in an earlier paper [1] (PONTING, 1953), where it was shown that theorem 1 is a best possible result for any even k. Examples were also given in that paper of functions whose singularities were distributed as in (2), (3 a), 3 b) or (4). In fact, the treatment forms a model for the present paper.

Small latin letters (except a, c, e, h, i and z) will denote non-negative integers.

For convenience we take the radius of the circle of convergence of (1) to be unity and the singularities thereon to be located at the points $\exp(i\alpha_t)$ which are the zeros of

$$F(z) = a_0 - a_1 z + a_2 z^2 - + \dots + a_k z^k, \ (a_0 = 1). \tag{5}$$

By a simple rotation of the z-plane we can take $a_k = 1$ and these simplifications will involve no loss of generality. We also have

$$a_t = \bar{a}_{k-t} = \Sigma \exp \{ - i(\alpha_1 + \alpha_2 + \dots + \alpha_t) \}, \tag{6}$$

where the bar denotes the complex conjugate. We make frequent use of (6) in the subsequent calculations.

We use the results and definitions of [**P**] § 2. Let c_{n+1} be the first coeficient of a favourable situation which occurs with positive upper density, and let [2]

$$c_{n+p}, \quad c_{n+q}, \quad c_{n+r}, \quad c_{n+s}$$

be the first four non-small coefficients. Then from the hypotheses of theorem 1

$$l < p \leqslant 2l < q, \quad 3l < r, \quad 4l < s \tag{7}$$

and

$$q - p \leqslant k, \quad r - q \leqslant k, \quad s - r \leqslant k. \tag{8}$$

We consider the following set of cases which is exhaustive.

Case 1: $\qquad p = k;$

Case 2: $\qquad p \leqslant k - 1, \quad q = k + 1 = r - p;$

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(1) Frequent reference will be made to this paper which will be denoted by [**P**].

(2) The letters k, l, p, q, r and s will have the same significance throughout.

Case 3 : $p \leqslant k-1$, $r-p \geqslant k+1$, $r-p$ and q not both equal to $k+1$; subdivided into : (3a) $s-q > k$, (3b) $s-q \leqslant k$;

Case 4 : $p \leqslant k-1$, $r-p \leqslant k$; subdivided into :
(4a) $k-r+p+1 \leqslant r-q-1$, (4b) $k-r+p+1 > r-q-1$.

The method of proof is to deal with particularly simple subcases and then with the others. We frequently use the fact that $a_k = 1 \neq 0$ and show that some positions lead to the contradiction $a_k = 0$.

These cases will be considered in §§ 4, 5, 6 and 7 respectively, using certain relations between the a_j which will be obtained in § 2. In this way the following results will arise amongst others :

$$a_{tw}\, a_j = a_{tw+j} \,(t = 1, 2, \ldots, d-1\,;\; j = 1, 2, \ldots, w-1),$$
$$(t = d\,;\; j = 1, 2, \ldots, k-dw), \tag{9}$$

and when $dw + w > k+1$,

$$a_{k-dw+j} = 0 \quad (j = 1, 2, \ldots, dw + w - k - 1), \tag{10}$$

where (¹)

$$dw + w > k \geqslant dw. \tag{11}$$

From (9), (10) and (11) equation (5) becomes

$$F(z) = (1 - a_1 z + a_2 z^2 - + \ldots + a_{k-dw}\,(-z)^{k-dw}) \times$$
$$(1 + a_w\,(-z)^w + \ldots + a_{dw}\,(-z)^{dw}). \tag{12}$$

In addition we shall have one of the following :

$$a_w \neq 0,\;\; a_{tw} = a_w^t\,,\;\; (t = 1, 2, \ldots, d)\,; \tag{13}$$

$$d = 2,\;\; -a_w = \delta_1 + \delta_2,\;\; a_{2w} = \delta_1 \delta_2,\;\; \delta_1^t = \delta_2^t\,,\;\; (t > 3)\,; \tag{14}$$

$$d = 3,\;\; a_w = R\varepsilon,\;\; a_{2w} = R\varepsilon^2,\;\; a_{3w} = \varepsilon^3,\;\; R^2 = R + 1,\;\; 1\,\varepsilon\,1 = 1. \tag{15}$$

Equations (13) and (15) will arise directly from the consideration od §§ 4, 5, 6, 7 ; it will also be shown in these sections that if (13) does not hold, then either $d = 2$ and $q - p = w$ or (15) holds with $q - p = w$ or $q - p = 2w$. When $d = 2$ we deduce in § 3 that (14) holds.

(¹) The letters d and w will have the same significance throughout.

We note that $t \neq 1$ in (14), for then $\delta_1 = \delta_2$ contradicting the hypothesis that the singularities are distinct. If $t = 2$, then

$$\delta_1 + \delta_2 = 0, \qquad a_w = 0,$$

and we have (13) with $d = 1$ and w replaced by $2w$. If $t = 3$, then

$$\delta_1^2 + \delta_1 \delta_2 + \delta_2^2 = 0, \qquad a_w^2 - a_{2w} = 0,$$

and thus (13) arises with $d = 2$.

We show in § 3 that k must be equal to dw. From this and (12)-(15) it follows that $F(z)$ takes one of the forms: (1)

$$(1 - a_w (-z)^w)^{-1} (1 - a_w^{d+1} (-z)^{dw+w}); \tag{16}$$

$$1 - (\delta_1 + \delta_2)(-z)^w + \delta_1 \delta_2 z^{2w}, \; (k = 2w, \; \delta_1^j = \delta_2^j, \; j > 3); \tag{17}$$

$$(1 - R\varepsilon z^w + \varepsilon^2 z^{2w})^{-1} (1 + \varepsilon^2 z^{5w}), \; (k = 3w, \; \varepsilon^3 = 1). \tag{18}$$

Equation (16) gives (2) when $d = 1$, (3 a) when $d > 1$; (17) gives (4); (18) gives (3 b).

2. We construct from the k by Nl matrix (2)

$$\left(\exp\{-i\tau\alpha_\sigma\}\right), \quad (\sigma = 1, 2, \dots, k; \; \tau = 1, 2, \dots, Nl), \tag{19}$$

a k-rowed matrix

$$A, \tag{20}$$

by omitting columns p, q, r, s, and all other columns of (19) which correspond to non-small coefficients. As in § 8 of the paper of Macintyre and Wilson (1940), it follows that all k-th order minors of (20) vanish, and this forms the logical basis of the suceeding arguments. The details of the discussions from Function theory are given for three singularities in § 6 of the above paper.

In the following, the a_j are the symetric functions defined by (5) with the convention that

$$a_j = 0 \quad \text{if} \quad j > k \quad \text{or} \quad j < 0. \tag{21}$$

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(1) See [**P**], §§ 3.21, 4.52 for examples of (16); for (17) see §§ 3.1, 4.2; for (18), see §§ 4.31, 4.9.

(2) See [**P**], § 2, for the definition of N.

We deduce the following relations, in which we write [1]

$$q - p = u, \qquad r - q = v$$

and make use of (6).

(A) If $q > k + 1$, then

$$a_j = 0 = a_{k-j}, \quad (j = k - p + 1,\ k - p + 2, \ldots,\ u - 1).$$

(B) If $r - p > k + 1$, then

$$a_j = 0 = a_{k-j}, \quad (j = k - v + 1,\ k - v + 2, \ldots,\ u - 1).$$

(C) If $u < k$ and $v > 1$, then

$$a_{u+j} = a_u\, a_j, \quad (j = 0,\ 1,\ 2, \ldots, v - 1).$$

(D) If $r - p > k$, $v < k$ and $s - r > 1$, then

$$a_{v+j} = a_v\, a_j, \quad (j = 0,\ 1,\ 2,\ \ldots,\ s - r - 1).$$

(E) If $u < k$ and $s - r > 1$, then

$$a_j\,(a_{u+v} - a_u\, a_v) = a_{u+v+j} - a_u a_{v+j}, \quad (j = 0,\ 1,\ 2, \ldots, s - r - 1).$$

(F) If $r - p > k$, $v < k$ and $s_1 - s > 1$, where c_{n+s}, is the next non-small coefficient in (1) after c_{n+s}, then

$$a_j\,(a_{s-q} - a_v\, a_{s-r}) = a_{s-q+j} - a_v\, a_{s-r+j}, \quad (j = 0,\ 1,\ 2, \ldots, s, - s - 1).$$

Case 1 is best treated differently from the other cases, particularly when $q = k + 1$ and relations (A) do not hold. Neither (A) nor (B) may be used for case 2 but (C) and (D) are sufficient. In case 3 at least one of (A) or (B) will hold as well as both (C) and (D); when $s - q \leqslant k$ we need (F) as well. Relations (E) are of no use in case 3, for then $a_{u+v} = 0$ since $u + v = r - p > k$ and then these relations are a consequence of (D). We are unable to use (B) and (D) in case 4. When $q = k + 1$, (C) and (E) are sufficient for case $4a$, where we make use of § 2.41, but when $q > k + 1$ we need (A) as well. It is deduced in § 2.43 that $q > k + 1$ in case $4b$, so that relations (A) hold; these are needed. The other results of § 2.43 are also used.

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[1] The letters u and v will have the same significance throughout.

2.1 Relations (A) and (B) are consequences of well known results on determinants and may be obtained by the methods of MACINTYRE and WILSON (1940, p. 78). For the others we need to use a few properties of S-functions [1], and we shall use the notation and definitions of [**P**]. In the proof of (E) and (F) we shall need the more general version of lemma 2 of [**P**] in which $\lambda_1 < k$; in this case (μ) has λ_1 parts and the determinant $| a\mu_\sigma - \sigma + \tau |$ is of order λ_1.

As an example, we deduce relations (A). Columns

$$j,\ j+1,\ j+2, \ldots, p-1,\ p+1, \ldots, k+j$$

of (19) are retained in (20) when

$$1 \leqslant j \leqslant q-(k+1).$$

The vanishing of the determinant formed by these columns implies [2] that

$$0 = \{ 1^{k+j-p} \} = a_{k+j-p}.$$

The other relations in (A) follow on using (6).

2.2 [3] Proof of (C). When $1 < u < k$, i.e. $q-k < p < q-1$, columns

$$q-k, q-k+1, \ldots, p-1,\ p+1, \ldots, q-1,\ q+t,\ (t = 1, 2, \ldots, v-1)$$

of (19) are retained in (20). Then [4]

$$\lambda_k = \lambda_{k-1} = \ldots = \lambda_{u+1} = 0,\ \lambda_u = \lambda_{u-1} = \ldots = \lambda_2 = 1,\ \lambda_1 = t+1$$

and hence $\{ t+1, 1^{u-1} \} = 0$. When $u = 1$, then $p = k, q = k+1$ from (7), and we have $\{ t+1 \} = 0$ from columns $1, 2, \ldots, k-1, k+t+1$.

The partition conjugate to $(t+1, 1^{u-1})$ is $(u, 1^t)$. Hence

$$\begin{vmatrix} a_u\ \mathbf{a}^{(t)} \\ \mathbf{e}_t\ H_t \end{vmatrix} = 0, \tag{22}$$

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(1) Cf. LITTLEWOOD (1940).

(2) We have $\lambda_1 = \lambda_2 = \ldots = \lambda_{k+j-p} = 1,\ \lambda_{k+j-p+1} = \ldots = \lambda_k = 0$. As usual, zero parts are omitted.

(3) From (8), $u \leqslant k$ and $v \leqslant k$; the cases of $u = k$, $v = k$ are considered in §§ 6.11, 6.13 respectively.

(4) See [P], footnote to § 3.1.

where $\mathbf{e}_t$ is the column vector of length t with unity in the first row and zero below, and

$$\mathbf{a}^{(t)} = (a_{u+1}, a_{u+2}, \ldots, a_{u+t}),$$
$$H_t = (a_{\tau-\sigma+1}), \quad (\sigma, \tau = 1, 2, \ldots, t).$$

When $t = 1$, (22) becomes

$$a_u \, a_1 = a_{u+1} \, a_0 = a_{u+1}.$$

When $t > 1$, we assume that (C) is true for $j = 1, 2, \ldots, t-1$. If $a_u = 0$, then from this assumption,

$$a_{u+1} = a_{u+2} = \ldots = a_{u+t-1} = 0.$$

Hence, from (22),

$$a_{u+t} = 0,$$

since the cofactor of this term has unity on the principal diagonal and zero below, and thus $a_{u+t} = 0 = a_u \, a_t$.

If $a_u \neq 0$, subtract a_u times row 2 of (22) from row 1; the new first row is

$$0, \, 0, \ldots, a_{u+t} - a_u \, a_t.$$

Hence, as above,

$$a_{u+t} - a_u \, a_t = 0,$$

relations (C) follow, but this process stops when $t = v - 1$.

2.21. Relations (D) are similarly established by using columns

$$r-k, \; r-k+1, \ldots, q-1, \; q+1, \ldots, r-1, \; r+j,$$
$$(j = 1, 2, \ldots, s-r-1).$$

2.3. Proof of (E).

Columns

$$q-k, \; q-k+1, \ldots, p-1, \; p+1, \ldots, q-1, \; r+j,$$
$$(j = 1, 2, \ldots, s-r-1),$$

of (19) are retained in (20), and as in § 2.2,

$$\begin{vmatrix} a_u & \mathbf{a}^{(v+j)} \\ \mathbf{e}_{v+j} & H_{v+j} \end{vmatrix} = 0. \tag{23}$$

If $a_u \neq 0$, subtract a_u times row 2 of (23) from row 1 and use ([1]) (C). Then the new first row is $(\mathbf{0},\ a^*_{v+u}\ \mathbf{a}^{*(j)})$, where

$$\mathbf{0} = (0, 0, ..., 0), \qquad a^*_t = a_t - a_u\, a_{t-u},$$
$$\mathbf{a}^{*(j)} = (a^*_{v+u+1},\ a^*_{v+u+2}, ..., a^*_{v+u+j}).$$

Hence (23) becomes

$$\begin{vmatrix} a^*_{v+u} & \mathbf{a}^{*(j)} \\ \mathbf{e}_j & H_j \end{vmatrix} = 0. \tag{24}$$

When $j = 1$ (24) becomes $a^*_{v+u}\, a_1 = a^*_{v+u+1}$, i.e.

$$a_1\,(a_{v+u} - a_v\, a_u) = a_{v+u+1} - a_u\, a_{v+1},$$

and a similar procedure to that of § 2.2 gives (E).

If $a_u = 0$, then relations (C) still give (24) from (23).

2.31. Relations (F) are similarly established.

2.4. We now enumerate some consequences of these relations in forms chosen to simplify their application.

2.41. If, as in case $4a$, and case 1 when $r = 2k$,

$$1 < k - r + p + 1 < v - 1,$$

we have from (C),

$$a_{k-r+p+j}\, a_u = a_{k-v+j}, \qquad (j = 1,\ 2, ..., v - (k - r + p + 1)).$$

However, $a_{r-p-j} = a_{v-j}\, a_u$ from (C), and (6) gives

$$a_{k-r+p+j} = a_{k-v+j}\, a_{k-u}.$$

Hence

$$a_{k-r+p+j} = a_u\, a_{k-u}\, a_{k-r+p+j}.$$

Thus we have either $a_u\, a_{k-u} \neq 1$ and

$$a_j = 0 = a_{k-j}, \quad (j = k - r + p + 1,\ k - r + p + 2, ..., v - 1), \tag{25}$$

or

$$a_u\, a_{k-u} = 1. \tag{26}$$

([1]) If $v = 1$, the new first row is $(0,\ a^*_{v+1}.\ a^*_{v+2}, ..., a^*_{v+1+j})$; the remainder of the argument is unaltered.

When (26) holds, relations (C) become

$$a_{k-u}\, a_{u+j} = a_j, \qquad (j = 1,\ 2, \ldots, v-1),$$

and then (6) gives

$$a_u\, a_{k-u-j} = a_{k-j}, \qquad (j = 1,\ 2, \ldots, v-1). \tag{27}$$

Now $k - v + 1 \leqslant r - p - 1$, and thus ($C$), (26) and (27) give

$$a_u\, a_j = a_{u+j}, \qquad (j = 1,\ 2, \ldots, k-u). \tag{28}$$

2.42. Similarly, when $r-p \geqslant k+1$ and $1 \leqslant k-s+q+1 \leqslant s-r-1$ as [1] in part of case $3b$, or case 2 when $s = 2k+1$, we have either $a_v\, a_{k-v} \neq 1$ and

$$a_j = 0 = a_{k-j}, \qquad (j = k-s+q+1,\ k-s+q+2, \ldots, s-r-1), \tag{29}$$

or $a_v\, a_{k-v} = 1$ and

$$a_{v+j} = a_v\, a_j, \qquad (j = 1,\ 2, \ldots, k-v). \tag{30}$$

2.43. When $k-r+p+1 \geqslant v$, then $k+p+q+1 \geqslant 2r \geqslant 3k+2$ from (7), and thus $p+q > 2k$ in case $4b$. We consider this together with other relations from case $4b$ and assume

$$a_{u+v} \neq a_u\, a_v, \quad r-p \leqslant k, \quad p \leqslant k-1, \quad p+q > 2k. \tag{31}$$

Since $p \leqslant k-1$ it follows that $q > k+1$ and so we may use relations (A). From the last two inequalities of (31),

$$k - u + 1 \leqslant 2k - q \leqslant p - 1,$$

and hence relations (A) give

$$a_j = 0, \qquad (j = k-u+1,\ k-u+2, \ldots, 2k-q).$$

Now, by definition

$$a_j = 0, \qquad (j = k+1,\ k+2, \ldots, 2k-p).$$

Hence relations (E) contain

$$0 = a_{k+j} - a_u\, a_{k-u+j} = a_{k-r+p+j}\,(a_{u+v} - a_u\, a_v), \qquad (j = 1,\ 2, \ldots, k-p),$$

(1) See §§ 6.421, 5.21.

since $2k - z \leqslant s - r - 1$ from (7). Then since $a_{u+v} \neq a_u a_v$ from (31),

$$a_j = 0 = a_{k-j} \quad (j = k - r + p + 1,\ k - r + p + 2, \ldots, 2k - r). \tag{32}$$

2.44 We consider as in part (1) of case $3b$,

$$a_{s-q} \neq a_v a_{s-r}, \qquad s - q \leqslant k, \qquad r - p > k + 1, \qquad s_1 - r > u. \tag{33}$$

Since, from (33), $u = r - p + q - r > k - r + q + 1 > s - r$ and $s_1 - r > u$, we find from (B) that relations (F) contain

$$a_j (a_{s-q} - a_v a_{s-r}) = a_{s-q+j} - a_v a_{s-r+j} = 0,$$

for $j = k - s + q + 1,\ k - s + q + 2, \ldots, u - 1 - s + r$. Hence

$$a_j = 0 = a_{k-j}, \quad (j = k - s + q + 1,\ k - s + q + 2, \ldots, u - 1 - s + r), \tag{34}$$

since $a_{s-q} \neq a_v a_{s-r}$ from (33).

3. We assume (2) in this section that:

(i), (9), (10) and (11) have been established,

(ii), $u = fw$,

(iii), if (13) does not hold, then either $d = 2$ and $u = w$, or (15) holds with $u = w$ or $u = 2w$.

We define the constants

$$\eta_j$$

by

$$(1 + a_w \zeta + a_{2w} \zeta^2 + \ldots + a_{dw} \zeta^d)^{-1} = \sum_{j=0}^{\infty} \eta_j \zeta^j. \tag{35}$$

Lemma 1. *If $u = fw$, if (9), (10) and (11) are true, and if column*

$$p + (t - 1) w, \qquad (t > f + 1)\ (3)$$

of (19) is retained in (20), then

$$\eta_{t-1} + \eta_{t-2} a_u + \ldots + \eta_{t-f} a_{fw-w} = 0.$$

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(1) In the discussion of case $3b$, the elimination of certain simple sub-cases leads first to the situation of § 2.44 and then the discussion of a sub-case involves § 2.42. See §§ 6.42, 6.421.

(2) As explained in § 1, the results (i), and (iii) will be established in §§ 4, 5, 6, and 7 ; (ii) will also be established in these sections.

(3) Since $u = g - p$, this ensures that this colum comes after column q.

Proof.

Columns

$$q-k, \quad q-k+1, \dots, p-1, \quad p+1, \dots, q-1, \quad p+(t-1)\,w$$

of (19) are retained in (20), hence, as in § 2.2,

$$\{(t-f-1)\,w+1,\ 1^{fw-1}\}=0. \tag{36}$$

The partition conjugate to $((t-f-1)\,w+1,\ 1^{fw-1})$ is

$$(fw,\ 1^{(t-f-1)\,w}). \tag{37}$$

We define the w by w matrices B_g by

$$B_g=(a_{gw-w+\tau-\sigma}),$$

where (21) holds. The first row and the first column of B_g will be denoted by

$$\beta_g,\ \gamma_g,$$

respectively.

Then (36) and (37) imply that the determinant of the $(t-f-1)\,w+1$ by $(t-f-1)\,w+1$ matrix

$$\begin{bmatrix} \beta_{f+1} & \beta_{f+2} & \cdots & \beta_{t-1} & a_{tw-w} \\ B_1 & B_2 & \cdots & B_{t-f-1} & \gamma_{t-f} \\ 0 & B_1 & \cdots & B_{t-f-2} & \gamma_{t-f-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_1 & \gamma_2 \end{bmatrix}$$

is zero. We denote this determinant by Δ.

From (35), we have

$$\eta_j+a_w\,\eta_{j-1}+\dots+a_{jw}=0, \qquad (j>0), \tag{38}$$

so that (9) applied to (38) gives

$$\beta_{j+1}+\eta_1\,\beta_j+\dots+\eta_j\,\beta_1=0,$$
$$\gamma_{j+2}+\eta_1\,\gamma_{j+1}+\dots+\eta_{j+1}\,\gamma_1=0, \qquad (j=1,\ 2,\ \dots).$$

Hence in Δ, we add

$$\sum_{j=0}^{t-f-2} \{\eta_{t-f-1-j} \times \text{column } (jw+1)\}$$

to the last column.

The first element of this new column will be

$$\begin{aligned} a_{tu-w} &+ \eta_1 a_{tw-2w} + \dots + \eta_{t-f-1} a_{fw} \\ &= -(\eta_{t-f} a_{fw-w} + \dots + \eta_{t-1}), \end{aligned} \tag{39}$$

from (38). All the other elements, except the last $w-1$, will be zero. Now add

$$\sum_{j=1}^{t-f-2} \{\eta_j \times \text{row } (jw+2)\}$$

to row 2, which becomes,

$$1, \ a_1, \ a_2, \cdot a_{w-1}, \ 0, \ 0, \dots, 0.$$

Then subtract a_{fw} times this new row from the first row, and thus remove the first w terms. Similar operations will remove every term in the first row except the last, which remains unaltered. The cofactor, of this last term has unity on the principal diagonal and zeros below. Since Δ is zero, the lemma now follows from (39).

Lemma 2. *If the upper density of small coefficients of* (1) *is greater than* $1 - 1/l$, *if* (9), (10) *and* (11) *are true with*

$$w = u, \qquad d = 2, \qquad u_w^2 \neq a_{2w}, \ a_w \neq 0,$$

and if δ_1, δ_2 *are defined by*

$$1 + a_w \zeta + a_{2w} \zeta^2 = (1 - \delta_1 \zeta)(1 - \delta_2 \zeta), \tag{40}$$

then for some $t > 3$.

$$\delta_1^t = \delta_2^t.$$

Proof. From [**P**], theorem 2, there will be a block of l consecutive columns of (19) which come at some point after column q and which are retained in (20). Since $d = 2$, we have from (11) that $k \geqslant 2w$ and so $w \leqslant l$ and for some t, column

$$p + tw - w$$

will be one of these columns. Since $u = w$ then $f = 1$ and so by lemma 1,

$$\eta_{t-1} = 0.$$

From (35) and (40),

$$\eta_{t-1} = (\delta_1 - \delta_2)^{-1} (\delta_1^t - \delta_2^t)$$

and hence

$$\delta_1^t = \delta_2^t.$$

As mentioned in the paragraph after (15), $t + 1$, and t cannot have the values 2 or 3 when $a_w \neq 0$ and $a_w^2 \neq a_{2w}$.

This lemma establishes (14), the stipulations $a_u \neq 0$ and $a_w^2 \neq a_{2w}$ ensuring that (13) and (14) are distinct.

The favourable situation [**P**] from which all our results are deduced begins with C_{n+1} for a sequence of n of positive upper density. Let the last coefficient of the second **0** block in situation (ii), or the second **1** block in situation (iii), or the third **0** block in situation (iv) be

$$C_{n+n_0}$$

and let

$$N_0$$

be any suitably chosen finite positive integer. Then there are only a finite number of possible distributions of non-small coefficients in the N_0 coefficients following c_{n+n_0}. Hence there will be a subsequence of n of positive upper density for which the $n_0 + N_0$ coefficients:

$$c_{n+1},\ c_{n+n},\ \ldots,\ c_{n+n_0+N_0}$$

have the same distribution of non-small coefficients. In the subsequent argument we consider only those c_{jw} for which

$$n + 1 < jw \leqslant n + n_0 + N_0.$$

Lemma 3. *If the conditions of lemma 2 hold and if t_0 is the least t such that*

$$\delta_1^t = \delta_2^t$$

then $\eta_j = 0$ if, and only if, $j + 1$ is a multiple of t_0.

Proof. The condition is clearly sufficient. If $\eta_j = 0$, we have

$$\delta_1^{j+1} = \delta_2^{j+1}$$

from (35) and (40). If

$$j + 1 = gt_0 + t_1, \qquad (t_0 > t_1 \geqslant 0),$$

then $\delta_1^{t_1} = \delta_2^{t_1}$ which is only possible when $t_1 = 0$.

LEMMA 4. *If* (9), (10), (11) *and* (15) *are satisfied, then*

$$\eta_{5j} = (-\varepsilon)^{5j}, \quad \eta_{5j+1} = R(-\varepsilon)^{5j+1}, \quad \eta_{5j+2} = (-\varepsilon)^{5j+2}, \quad \eta_{5j+3} = \eta_{5j+4} = 0.$$

Proof. From (15) we have $d = 3$ and

$$(1 + R\varepsilon\zeta + R\varepsilon^2\zeta^2 + \varepsilon^3\zeta^3)(1 - R\varepsilon\zeta + \varepsilon^2\zeta^2) = 1 + \varepsilon^5\zeta^5.$$

Hence in (35)

$$\Sigma\eta_j\zeta^j = (1 - R\varepsilon\zeta + \varepsilon^2\zeta^2)(1 + \varepsilon^5\zeta^5)^{-1}$$

and the lemma follows.

LEMMA 5. *If* (9), (10), (11) *and* (13) *hold, then*

$$\eta_{jd+j} \neq 0, \quad \eta_{jd+j+1} \neq 0, \quad \eta_t = 0 \; (t \neq 0, 1 \; (\mathrm{mod}\; d+1)). \tag{41}$$

Proof. If $d = 1$, then (41) follows from (35) since a_u cannot now be zero. If $d > 1$, (35) becomes

$$\Sigma\eta_j\zeta^j = (1 - a_w\zeta)(1 - a_w^{d+1}\zeta^{d+1})^{-1}$$

from (13), and the lemma follows.

LEMMA 6. *When* (9), (10) *and* (11) *are true, one of the following sets of columns of* (19) *is not retained in* (20):

(i) $p + jt_0 w, \quad p + (jt_0 + 1)w, \ldots, p + (jt_0 + t_0 - 2)w$;

(ii) $p + 5jw, \quad p + (5j + 1)w, \quad p + (5j + 2)w$;

(iii) $p + 5jw, \quad p + (5j + 2)w, \quad p + (5j + 3)w$;

(iv) $p + j(d + 1)w, \quad p + j(d + 1)w + u$;

for $j = 1, 2, \ldots$. *These results hold when*: (i), $d = 2$, $u = w$, $t_0 > 3$; (ii), $d = 3$, $u = w$ *and* (15) *holds*; (iii), $d = 3$, $u = 2u$ *and* (15) *holds*; (iv), relations (13) *hold.*

Proof. If (13) is not true and $d = 2$ then $u = w$, $f = 1$ and the lemma follows from lemmas 1, 2 and 3.

When (15) holds and the set (ii) of columns of (19) is retained in (20) then $u = w$, $f = 1$, and from lemmas 1 and 4,

$$0 = \eta_{5j} = (-\varepsilon)^{5j}, \quad 0 = \eta_{5j+1} = R(-\varepsilon)^{5j+1}, \quad 0 = \eta_{5j+2} = (-\varepsilon)^{5j+2}.$$

However $|\varepsilon| = 1$, $R^2 = R + 1$ from (15), and thus each of these three results is in contradiction to (15). When (15) holds and the set (iii) is retained, then $u = 2w$, $f = 2$, and Lemmas 1 and 4 give

$$0 = \eta_{5j} + a_w \eta_{5j+1} = (-\varepsilon)^{5j}, \quad 0 = \eta_{5j+2} + a_w \eta_{5j+1} = (1 - R^2)(-\varepsilon)^{5j+2},$$
$$0 = \eta_{5j+3} + a_w \eta_{5j+2} = -R(-\varepsilon)^{5j+3},$$

since $a_w = -\eta_1 = R\varepsilon$ from (35). Each of these three results is in contradiction to (15) as above.

If (13) is true and the set (iv) of columns of (19) is retained in (20) then since $fw = u \leqslant k < (d + 1)w$ from (11), we have $f < d + 1$ and so from lemma 5, $f - 1$ consecutive η_j in lemma 1 are zero.

The relation of lemma 1 thus reduces to

$$\eta_{jd+j} = 0, \qquad \eta_{jd+j+1} a_{fw-w} = 0,$$

each of which is in contradiction with the results of lemma 5.

We define

$$\begin{aligned} \varphi_m(z) &= \frac{1}{w'} \sum_{j=0}^{w'-1} \omega^{-mj} \varphi(\omega^j z), \quad (m = 0, 1, \ldots, w' - 1) \\ &= \sum_{j=0}^{\infty} c_{m+jw'} z^{m+jw'}, \end{aligned} \tag{42}$$

where w' is a prime factor of w and ω is a primitive w'-th root of unity.

From (12), $\varphi(z)$ of (1) has dw singularities on dw/w' regular w'-agons, and $k - dw$ other singularities. Hence each $\varphi_m(z)$ has at most

$$dw + (k - dw)w' \tag{43}$$

singularities.

Our favourable situation begins with c_{n+1} for a consequence of n of positive upper density. There will be subsequences of n where the n are all congruent modulo w' and at least one of these must occur with a positive upper density. We may thus consider the behaviour of the coefficients of any particular $\varphi_m(z)$ in some favourable situation and know that this pattern will be repeated in a sequence of positive upper density. We will deal only with such favourable situations.

3.1 (1) When (13) does not hold and $d = 2$ and $t_0 > 3$, lemma 6 (i) shows that at least a fraction (2)

$$(t_0 - 1)/(t_0 u)$$

of the coefficients in (1) are non-small and occur in one $\varphi_m(z)$, say $\varphi_{m_0}(z)$, since $u = w$. However, from the hypotheses of the theorem less than a fraction

$$1/l$$

of the coefficients can be non-small. Thus, at most, the remaining non-small coefficients are a fraction

$$1/l - (t_0 - 1)/(t_0 u) \tag{44}$$

of the coefficients and these have to be shared among the remaining $w' - 1$ functions $\varphi_m(z)$. We assume that (44) is not negative for then we have a contradiction and the situation is impossible.

If less than

$$1/(2u + (k - 2u)w')$$

of the coefficients of a particular $\varphi_m(z)$ are non-small, then from (43) the only possibility is that this $\varphi_m(z)$ has no singularities.

Now, fron (44) and (11), since $d = 2$,

$$ut_0/(t_0 - 1) \geqslant l \geqslant u. \tag{45}$$

Since $t_0 \geqslant 4$, we have from (45),

$$\frac{t_0 u - l(t_0 - 1)}{lt_0 u} \leqslant \frac{1}{lt_0} < \frac{1}{4l} < \frac{1}{4(d' + l - u)} = \frac{\frac{1}{2}w}{2u + (k - 2u)w'}, \tag{46}$$

(1) The arguments of §§ 3.1, 3.2 and 3.3 are extensions of those of [**P**] §§ 3.32, 4.11, 4.33. The argument of the present paper shows that the distribution of § 4.33 of [**P**] es impossible ; this is immaterial for it is sufficient to show that the theorem follows from any possible distribution of the coefficients.

(2) In this case, $u = w$.

where $u = d'\,w'$. Hence at most [1]

$$\frac{1}{2}(w' - 1)$$

of the remaining $\varphi_m(z)$ can have singularities, so that at least

$$\frac{1}{2}(w' - 1)$$

have none. This can only happen when the singularities of $\varphi(z)$ cancel in these $\varphi_m(z)$.

Let the singularities of $\varphi(z)$ other than those at the vertices of the $(2\,u/w')\,w'$-agons be at

$$l_1,\ l_2,\ \ldots,\ l_{k-2u}. \tag{47}$$

These singularities of $\varphi(z)$ can only cancel in $\varphi_m(z)$ if they are in sets of two, or more, at some of the vertices of regular w'-agons. Suppose that exactly t of (47) are at the vertices of some regular w'-agon. Let these vertices be

$$\beta,\ \beta\,\omega_1,\ \beta\,\omega_2,\ \ldots,\ \beta\,\omega_{t-1}$$

where

$$\omega_j = \omega^{-e_j}, \qquad (w' - 1 \geqslant e_1 > e_2 > \ldots > e_{t-1} \geqslant 1).$$

Then of the w' functions

$$\varphi(\omega^j z),$$

only

$$\varphi(z),\ \varphi(\omega_1 z),\ \ldots,\ \varphi(\omega_{t-1} z)$$

have a singularity at $z = \beta$.

Suppose that

$$t < \frac{1}{2}(w' - 1) \tag{48}$$

and let

$$\varphi_{n_j}(z), \qquad (w' - 1 \geqslant n_1 > n_2 > \ldots > n_t \geqslant 0),$$

be t of those $\varphi_m(z)$ which have no singularities.

---

(1) If $w' = 2$, we find that only $\varphi_{m_0}(z)$ can have singularities, the subsequent argument still holds. In future we will not deal with $w' = 2$ separately.

Now $\varphi(\omega^j z)$ is regular at $z = \beta$ unless $j = 0, -e_1, -e_2, \ldots, -e_{t-1}$, and thus $\varphi_{n_m}(z)$ can only be regular at $z = \beta$ if

$$\Phi_m(z) = \varphi(z) + \omega_1^{-n_m} \varphi(\omega_1 z) + \ldots + \omega_{t-1}^{-n_m} \varphi(\omega_{t-1} z)$$

is also regular at $z = \beta$.

We need the following:

LEMMA 7. *If g is a prime, if b is any positive integer, and if $\omega \neq 1$ is a g^b-th root of unity, then the t-th order determinant.*

$$\Delta_0 = \left|\omega_{\tau-1}^{n_\sigma}\right|, \ (\sigma, \tau = 1, 2, \ldots, t; \ \omega_0 = 1, \ \omega_j = \omega^{-e_j}, \ j > 0), \quad (49)$$

is non-zero when n_j and e_j are integers which satisfy

$$g - 1 \geqslant e_1 > e_2 > \ldots > e_{t-1} \geqslant 1, \quad g - 1 \geqslant n_1 > n_2 \ldots > n_t \geqslant 0. \quad (50)$$

Proof. The determinant (49) is the product of

$$\prod_{0 \leqslant x < y \leqslant t-1} (\omega_x - \omega_y) \quad (51)$$

and the S-function

$$\{n_1 - t + 1, \ n_2 - t + 2, \ldots, n_{t-1} - 1, \ n_t\} \quad (52)$$

of

$$1, \ \omega_1, \ \omega_2, \ldots, \omega_{t-1}.$$

It follows from the conditions of this Lemma that the product (51) cannot be zero. MITCHELL (1882) has shown ([1]) that (52) may be expanded as a polynomial in ω with positive integral coefficients whose sum is (54). Since $\omega^{g^b} = 1$ we find that (52) will have the form

$$f_0 + f_1 \omega + f_2 \omega^2 + \ldots + f_\gamma \omega^\gamma, \qquad (\gamma = g^b - 1) \quad (53)$$

where the f_j are non-negative and

$$\sum_{j=1}^{\gamma} f_j = \prod_{x<y} \left(\frac{n_x - n_y}{y - x}\right). \quad (54)$$

([1]) Mitchell proved this result when $t = 4$ and stated that his method would establish the general result. Schur (1901, § 23, particularly (44) (45) and (46″)), proved this result by the use of characteristics of irreducible invariant transformations.

Now (53) can only vanish if it has one, or more, factors [1] of the form

$$1 + \omega^{j} + \omega^{2j} + \ldots + \omega^{gj-j}, \qquad (j = 1,\ g,\ g^2, \ldots, g^{b-1}). \qquad (55)$$

From (50), the prime g cannot divide (54); thus none of the factors (55) can divide (53). The lemma now follows since (52) cannot vanish.

Let

$$\Omega_{\sigma\tau}$$

be the cofactor of the element in row σ column τ of (49). Since the $\Phi_j(z)$ are regular at $z = \beta$, it follows that

$$\bar{\Omega}_{1},\ \Phi_1(z) + \bar{\Omega}_{2},\ \Phi_2(z) + \ldots + \bar{\Omega}_{t},\ \Phi_t(z) = \bar{\Delta}_0\, \varphi(z)$$

is also regular at $z = \beta$. From lemma 7, $\Delta_0 \neq 0$, and thus we have the contradiction that $\varphi(z)$ is regular at $z = \beta$, and so (48) is not true. Hence there are at least $\frac{1}{2}(w' + 1)$ of the singularities at (47) which lie on each w'-agon.

In this case there will be less than $(k - 2u)/\left(\frac{1}{2} w'\right)$ of these polygons and hence the number of singularities of each $\varphi_m(z)$ will be less than

$$2u + (k - 2u)\,2.$$

But $2k - 2u < 2k = 4l$ and thus from (46),

$$\frac{1}{l} - \frac{t_0 - 1}{t_0 u} \leqslant \frac{1}{4l} < \frac{1}{2u + (k - 2u)\,2}.$$

Hence, from the argument preceding (45), none of the other $\varphi_m(z)$ has any singularities. Thus only $\varphi_{m_0}(z)$ has any singularities. The remaining $w' - 1$ of the $\varphi_m(z)$ have none. From Lemma 7, this implies that there is a singularity at each vertex of every w'-agon, and that w' divides $k - 2u$.

3.11. If w'^2 is a factor of u, then we define

$$\Psi_m(z) = \frac{1}{w'^2} \sum_{j=0}^{w'^2-1} \xi^{m-j}\, \varphi(\xi^j z), \quad (m = 0, 1, \ldots, w'^2 - 1),$$

$$= \frac{1}{w'} \sum_{j=0}^{w'-1} \xi^{-mj}\, \varphi_{m'}(\xi^j z), \quad (m \equiv m' \ (\mathrm{mod}\ w')),$$

---

[2] There are no other polynomials in w with rational coefficients which divide these factors.

from (42), where ξ is a primitive w'-th root of ω and hence a primitive w'^2-th root of unity.

The functions $\Psi_m(z)$ can have

$$2u + (k - 2u)\,w'$$

singularities at most, and as in § 3.1 we see that at least

$$w'^2 - \frac{1}{2}(w' + 1)$$

have none.

We now argue as in § 3.1 with w' replaced by w'^2 and $\varphi(z)$ replaced by $\varphi(z)$, for only $\varphi_{m_0}(z)$ contributes singularities to $\Psi_m(z)$. Then $\varphi_{m_0}(z)$ will have dw singularities on dw/w'^2 regular w'^2-agons and $k-dw$ singularities on $(k-dw)/w'$ regular w'-agons. The vertices of two or more of these w'-agons must be at the vertices of a regular w'^2-agon. Suppose that there are exactly t of these w'-agons imbedded in a w'^2-agon and let their vertices be

$$\beta\xi^{-j},\ \beta\xi^{-j}\omega,\ \beta\xi^{-j}\omega^2, \ldots, \beta\xi^{-j}\omega^{w'-1}, \quad (j = 0,\ f_1,\ f_2, \ldots f_{t-1})$$

where β is a singularity of $\varphi_{m_0}(z)$ and

$$w' - 1 \geqslant f_1 > f_2 > \ldots > f_{t-1} > 0.$$

If

$$t \leqslant \frac{1}{2}(w' - 1), \tag{56}$$

we can find t values of m which are congruent to m_0 modulo w', where the functions $\Psi_m(z)$ are regular at $z = \beta$.

A similar argument to that of § 3.1 will show that $\varphi_{m_0}(z)$ is also regular at $z = \beta$. This is a contradiction and so (56) is false. As in § 3.1 we deduce that w'^2 divides $k - 2u$.

3.12. We continue this argument with all the powers of w' which divide u, and then repeat the whole process with any other prime factor of u. This shows that u will divide $k - 2u$, which contradicts (11) if $k > 2u$. Thus we are left with $k = 2u$.

3.2. When (15) holds, similar arguments to those of § 3.1, etc., will show that $k = 3w$. The expression (44) is replaced by

$$1/l - 3/(5w)$$

and (45) by

$$5\,w/3 \geqslant l \geqslant 3\,w/2.$$

Then, similarly to (46), when $w = d'\,w'$,

$$\frac{1}{l} - \frac{3}{5\,w} = \frac{5\,w - 3\,l}{5\,wl} < \frac{1}{4\,l} < \frac{1}{6\,d' + 4\,l - 6\,w} = \frac{\frac{1}{2}\,w'}{3\,w + (k - 3\,w)\,w'},$$

and as before we deduce that w divides $k - 3\,w$. Hence $k = 3\,w$ from (11).

3.3. When (13) holds, slightly modified arguments will show that

$$k = dw.$$

Relation (44) becomes

$$1/l - 2/(dw + w).$$

Now, when d is a positive integer,

$$(4\,l - d\,w)\,(dw + w - 2\,l) - l\,(d+1)\,w = -\,8\,l^2 - w^2\,d\,(d+1) + \\ + lw\,(5\,d + 3) \leqslant 0, \qquad (57)$$

since

$$(5\,d + 3)^2 - 32\,d\,(d+1) = -\,7\,d^2 - 2\,d + 9 \leqslant 0.$$

When $d'\,w' = w$,

$$d' \leqslant \frac{1}{2}\,w,$$

so that

$$\frac{\frac{1}{2}\,w'}{dw + (k - dw)\,w'} = \frac{1}{4\,l - 2\,d\,(w - d')} \geqslant \frac{1}{4\,l - dw} \geqslant \frac{1}{l} - \frac{2}{dw + w},$$

from (57).

The remainder of the argument follows as before.

4. [1] Case 1, $p = k$.

Columns

$$1,\ 2,\ \ldots,\ k-1,\ k+j$$

---

[1] All the distributions of § 4 are included in the favourable situations (ii) (iii) or (iv) of [P].

of (19) are retained in (20), and as in [**P**] §§ 3.21, 4.8,

$$h_j = \{j\} = 0,$$

$$(j = 1, 2, \ldots, q-k-1,\ q-k+1, \ldots, r-k-1,\ r-k+1, \ldots, s-k-1), \quad (58)$$

where h_j is the j-th complete homogeneous symmetric function of

$$\exp(-i\alpha_1),\ \exp(-i\alpha_2),\ \ldots,\ \exp(-i\alpha_k).$$

We note that $s - k - 1 \geqslant k$ from (7).

We use the well-known relations

$$a_g - a_{g-1}h_1 + a_{g-2}h_2 - + \ldots + (-1)^g h_g = 0, \qquad (g = 1, 2, \ldots). \quad (59)$$

When $q > k + 1$, we have immediately from (58) and (59),

$$a_j = 0,\ (j = 1, 2, \ldots, u - 1) \quad (60)$$

since $q - k - 1 = q - p - 1 = u - 1$.

We consider the cases of $q = k + 1$, $q > 3l$ and $3l \geqslant q > k + 1$ in §§ 4.1-4.122, § 4.2 and §§ 4.3-4.332 respectively; the further subdivision depends on the position of r and certain other factors. 4.1, $q = k + 1$ (1). There is no general result which follows immediately from this situation and we proceed to the discussion of the subcases $r > 2k$ and $r \leqslant 2k$ in § 4.11 and §§ 4.12-4.122. The results of § 4.12 apply to §§ 4.121, 4.122 but those of § 4.121 do not necessarily hold in § 4.122. The decimal numbering of the sections and sub-sections will be used in this manner.

4.11. If $r > 2k$ (58) and (59) now give

$$a_2 = a_1^2,\ a_3 = a_1^3,\ \ldots,\ a_k = a_1^k,$$

and thus (16) occurs with $w = 1$ and $d = k$.

When $r > 2k + 1$ we shall have the contradiction that $a_1^{k+1} + 0$, but this will not affect the general result. Unless we show directly that a certain distribution of non-small coefficients is impossible, it is sufficient to show that the theorem is true even tough consideration of the positions of further non-small coefficients may show that the original distribution was impossible.

(1) Cf. [**P**], §§ 3.2 - 3.22, 4.8 - 4.82.

4.12. If $r \leqslant 2k$, equations (58) and (59) give

$$a_t = a_1^t, \qquad (t = 1,\, 2, \ldots, r - k - 1), \tag{61}$$

$$a_{r-k} - a_{r-k-1}\, h_1 + (-1)^{r-k}\, h_{r-k} = 0. \tag{62}$$

From (7), $r > 3l$, and hence $r - k - 1 \geqslant l$.

4.121. If $r - k - 1 > l$, (6) and (61) give

$$a_{l-1} = \bar{a}_{l+1}, \qquad a_1^{l-1} = \bar{a}_1^{l+1}, \qquad a_1^k = 1 \quad \text{or} \quad 0,$$

$$a_{r-k} = \bar{a}_{2k-r} = \bar{a}_1^{2k-r} = a_1^{r-k}.$$

Then (62) gives $h_{r-k} = 0$ and hence (16) arises as in § 4.11 when $a_1 \neq 0$; we have the contradiction $a_k = 0$ when $a_1 = 0$.

4.122. [1] If $r - k - 1 = l$, then equations (59), (58) give

$$a_k - a_{k-1}\, h_1 + (-1)^{r-k}\, a_{2k-r}\, h_{r-k} = 0. \tag{63}$$

Set

$$h = (-1)^l\, h_{l+1},$$

then (62) and (63) [2] become, on using (61) and (6),

$$\bar{a}_1^{l-1} = a_1^{l+1} + h, \qquad 1 = a_1\, \bar{a}_1 + a_1^{l-1}\, h.$$

Now $a_1^l = a_l = \bar{a}_l = \bar{a}_1^l$, from (61) and (6), and hence

$$1 - a_1\, \bar{a}_1 = (a_1\, \bar{a}_1)^{l-1}\, (1 - a_1\, a_1).$$

Thus

$$a_1\, \bar{a}_1 = 1, \qquad h = 0,$$

from (63) and so (16) again follows.

4.2. When $q > 3l$, then $u = q - p > l$ and (60) gives

$$a_1 = a_2 = \ldots = a_l = 0$$

and (16) follows from (6).

4.3, $3l \geqslant q > k + 1$.

---

(1) This case has to be treated separately from § 4.121, for (61) does not now contain $a_{l+1} = a_1^{l+1}$ since $r - k - 1 = l$; and if $r = 2k$ then (63) is the same as (62). Cf. [P] § 3.22.

(2) Since these two equations are the same when $k = 2$, the argument must be modified here and elsewhere. The case of $k = 2$ was considered by MACINTYRE and WILSON (1940), and hence we assume $k > 2$.

4.31. If $r > 2k$, then $r - p - 1 \geqslant k$ so that the relations (C) contain $a_k = a_u a_{u-k}$. Now (60) gives

$$a_1 = a_2 = \dots = a_{u-1} = 0$$

and using (C) we must have u dividing k in order to avoid the contradiction $a_k = 0$. Relations (C) also give

$$a_u^{k/u} = a_k = 1, \qquad a_{tu} = a_u^t, \qquad (t = 1, 2, \dots, k/u),$$

and hence (16) follows since (12) now becomes

$$F(z) = 1 + a_u(-z)^u + a_u^2 z^{2u} + \dots + a_u^{k/u} z^k.$$

4.32. When $r = 2k$, then $k - r + p + 1 = 1 < v - 1$, since $3l \geqslant q$.

Then the conditions of § 2.41 are satisfied and we have either (25) or (28).

4.321. If (28) holds then (16) follows as in § 4.31.

4.322. If (25) holds then

$$a_1 = a_2 = \dots = a_{v-1} = 0.$$

If (1) $v > l$, then (16) follows as in § 4.2.

If (2) $v = l$, then (64) and (6) give

$$F(z) = 1 + a_l(-z)^l + z^k.$$

If $a_l^2 = 1$ or $a_l = 0$, then (16) follows; if $a_l^2 \neq 1, 0$ then (17) follows from Lemma 2.

4.33. When $r < 2k$, then from (7), $l < r - k < k$. From (59), (5) and (58),

$$(F(z))^{-1} = \Sigma h_j z^j = 1 + h_u z^u + h_{r-k} z^{r-k} + 0(z^{k+1}).$$

Set

$$z = -\zeta, \qquad h_u = (-1)^{u+1}\nu_1, \qquad h_{r-k} = (-1)^{r-k-1}\nu_2,$$

then

$$1 + a_1\zeta + \dots + a_k\zeta^k = [1 - \nu_1\zeta^u - \nu_2\zeta^{r-k} + 0(\zeta^{k+1})]^{-1}$$
$$= 1 + \nu_1\zeta^u + \nu_1^2\zeta^{2u} + \dots + \nu_2\zeta^{r-k}(1 + 2\nu_1\zeta + 3\nu_1^2\zeta^{2u} + \dots). \qquad (65)$$

(1) In this case v cannot be less than l, for $r = 2k$ and $3l \geqslant q$.
(2) We now have favourable situation (ii) of [P].

We need only consider

$$\nu_1 \nu_2 \neq 0, \tag{66}$$

for if $\nu_1 = 0$ then since $r - k < k$, (65) gives the contradiction $a_k = 0$, and if $\nu_2 = 0$ then u must divide k, and (65) gives (16).

4.331. If u does not divide $r - k$, then $a_{r-k} = \nu_2$ from (65).

Since $2k - r < r - k$, from (7), then $\bar{\nu}_2 = \bar{a}_{r-k} = a_{2k-r} = 0$ unless $2k - r$ is a multiple of u, from (65). Let

$$2k - r = gu.$$

Hence u does not divide k. Now for some b,

$$bu > r - k > (b - 1)u.$$

Then from (65),

$$a_{bu} = \nu_1^b.$$

Now $k - bu < 2k - r < r - k$ and $k - bu$ cannot be a multiple of u, and hence from (60) and (65),

$$\bar{\nu}_1^b = \bar{a}_{bu} = a_{k-bu} = 0,$$

contradicting (66). Thus we consider

4.332, $r - k = bu$.

In this case u must also divide k, otherwise (65) gives $a_k = 0$. Let

$$k = (b + g)u, \qquad 2k - r = gu,$$

with $g > 0$ since $r < 2k$.

Equation (65) now gives

$$\bar{\nu}_1^g = \bar{a}_u^g = \bar{a}_{2k-r} = a_{r-k} = \nu_2 + \nu_1^b \tag{67}$$

$$\bar{\nu}_1^{g-1} = \bar{a}_u^{g-1} = \bar{a}_{2k-r-u} = a_{r-k+u} = 2\nu_2\nu_1 + \nu_1^{b+1}. \tag{68}$$

If $g > 1$, then we also have

$$\bar{\nu}_1^{g-2} = \bar{a}_u^{g-2} = \bar{a}_{2k-r-2u} = a_{r-k+u} = 3\nu_2\nu_1^2 + \nu_1^{b+2}. \tag{69}$$

Then (67), (68) and (69) give $\nu_1\bar{\nu}_1 = 1$. Hence (67) and (68) give $\nu_1^g \nu_2 = 0$, contradicting (66). This leaves $g = 1$ so that

$$k = (b + 1)u.$$

Hence $k-2u < bu = r-k$ and if $b+1 \geqslant 4$ then (65) gives

$$\bar{\nu}_1^{t} = \bar{a}_{tu} = \bar{a}_{k-tu} = a_{(b+1-t)u} = \nu_1^{b+1-t}, \qquad (t = 2,\ 3).$$

Thus $\bar{\nu}_1 = \nu_1^{-1}$ since $\nu_1 \neq 0$ from (66), and hence $\nu_1^{b-1} = 1$. Then $2\nu_2\nu_1 = 0$ from (68) since $g = 1$; this is in contradiction to (66). Hence $b+1 < 4$. Moreover, $bu = r-k > l \geqslant u$ since $g \leqslant 3l$. Thus we are left with $b = 2$ and $g = 1$ so that $k = 3u$.

Equations (67) and (68) are now

$$\bar{\nu}_1 = \nu_2 + \nu_1^2, \qquad 1 = 2\nu_2\nu_1 + \nu_1^3,$$

and hence

$$1 = 2\nu_1\bar{\nu}_1 - \nu_1^3.$$

Thus ν_1^3 is real and

$$\nu_1 = R\omega, \qquad (\omega^3 = 1,\ R^3 - 2R^2 + 1 = 0).$$

Then (65) gives

$$a_u = R\omega, \qquad a_{2u} = R\omega^2, \qquad a_{3u} = \omega^3 = 1, \qquad k = 3u.$$

If $R = 1$, then $\nu_1^3 = 1$ and $2\nu_2\nu_1 = 0$ as above, in contradiction to (66). Thus $R^2 = R + 1$ and we have (18).

5. [1] Case 2, $p \leqslant k-1$, $q = k+1 = r-p$. The last conditions, which prevent the use of (A) and (B) in the discussion of this case, are of great help when we use (C) and (D). We note that in this case $r \leqslant 2k$.

From (7), $p > l$ so that $u = q - p < k$ and since $r - p \geqslant k+1$ in cases 2 and 3, we have

$$a_u \neq 0. \tag{70}$$

Otherwise (C) gives $1 = a_k = 0$. Since $p \leqslant k-1$ and $q \geqslant k+1$ in cases 2,3 and 4, we have

$$u \geqslant 2. \tag{71}$$

We consider the cases of $r = 2k$ and $r < 2k$ in §§ 5.1-5.13 and §§ 5.2-5.22 respectively. The sub-division depends on the position of s, and we note that $s > 2k$ from (7).

5.1. If $r = 2k$, then $p = k-1$ and

$$u = 2. \tag{72}$$

---

[1] All these situations arise in favourable situation (ii) of [**P**].

Relations (C) become

$$a_{2+j} = a_2 a_j, \qquad (j = 1, 2, \ldots, k-2). \tag{73}$$

5.11. If $s > 2k+2$, columns $k+2$, $k+3, \ldots, 2k-1$, $2k+1$, $2k+2$ of (19) are retained in (20) and thus $a_2 = 0$ as in § 2.1; this is in contradiction to (70) since $u = 2$ from (72).

5.12. If $s = 2k+1$ and c_{n+s_1} is the next non-small coefficient after c_{n+s}, then $s_1 > 2k+l$ since we are in favourable situation (ii) of [**P**]. Hence, when $k > 4$, columns $k+2$, $k+3, \ldots, 2k-1$, $2k+2$, $2k+2$, $2k+3$ of (19) are retained in (20). In the notation of [**P**] Lemma 2, $\lambda_k = \lambda_{k-1} = \ldots = \lambda_3 = 0$, $\lambda_2 = \lambda_1 = 2$. Hence $\{2^2\} = 0$, i.e. $a_2^2 - a_1 a_3 = 0$. Then $a_2^2 = a_1^2 a_2$ from (72) and (73), and $a_1^2 = a_2$ from (70). Then (73) gives (16) with $w = 1$.

The case of $k = 2$ was considered by Macintyre and Wilson (1940, §§ 1-5) for a variety of types of singularity, and the cases of $k = 4$, $k = 6$ for isolated essential points of finite exponential order by Ponting (1), 1953.

5.13. If $s = 2k+2$. relations (D) contain $a_1 a_{k-1} = a_k$, and hence

$$a_1 a_1 a_2^{l-1} = a_2^{l}, \qquad a_1^2 = a_2$$

from (73), (72) and (70), and (16) follows.

5.2. If $r < 2k$, then $s - r > 1$ since $s > 2k$ from (7).

5.21. If $s = 2k+1$, then the conditions of § 2.42 hold since

$$k - s + q + 1 = 1 \leqslant s - r - 1$$

and $r - p = k + 1$. Thus we have (29) or (30).

If (29) holds, then

$$a_1 = a_2 = \ldots = a_{u-2} = 0. \tag{74}$$

since $1 \leqslant s - r - 1 = 2k - (p + k + 1) = q - p - 2 = u - 2$.

From (C),

$$k = gu \quad \text{or} \quad k = gu - 1$$

for some g, since $a_k \neq 0$.

---

(2) See, in particular §§ 3.41, 4.52.

If $k = gu$, then since $a_1 = 0$.

$$a_{k-1} = 0, \qquad a_{u-1} = 0,$$

and (16) follows on using (74) and (C).

If $k = gu - 1$, then from (74), equation (12) becomes

$$F(z) = (1 + a_{u-1}(-z)^{u-1})(1 + a_u(-z)^u + \ldots + a_u^{g-1}(-z)^{gu-u}).$$

This is (12), (with (13)), and since $u - 1 > 0$ from (71), this situation is impossible as shown in § 3. This leaves (30) which will be considered below.

5.22. $s \geqslant 2k + 2$. In this case relations (D) contain (30), namely

$$a_{v+j} = a_v a_j, \qquad (j = 1, 2, \ldots, k - v).$$

If $v = bu$, then $k = (b + 1)u - 1$ since $u + v = r - p = k + 1$, and using (C),

$$F(z) = (1 - a_1 z + a_2 z^2 - + \ldots + a_{n-1}(-z)^{u-1})$$
$$(1 + a_u(-z)^u + \ldots + a_u^b(-z)^{bu}),$$

as in (12) and (13), and thus an impossible situation arises as in § 5.21.

[1] If $bu > v > (b - 1)u$, we define

$$\begin{aligned} v - (b_1 - 1)u &= w_1, \qquad (b_1 = b)\\ u - (b_2 - 1)w_1 &= w_2,\\ &\ldots\ldots\ldots\ldots\\ w_{m-1} - (b_{m+1} - 1)w_m &= w_{m+1},\\ w_m - (b_{m+2} - 1)w_{m+1} &= 0, \end{aligned}$$

where $u > w_1 > w_2 > \ldots > w_{m+1} \geqslant 1$.

We set

$$w_{m+1} = w.$$

Now from (C) and (30), for $j = 1, 2, \ldots, k - v$,

$$a_u^{b_1-1} a_{w_1+j} = a_{v+j} = a_v a_j = a_u^{b_1-1} a_{w_1} a_j. \tag{75}$$

---

[1] Since $q = k + 1 = r - p$ and $p \geqslant l + 1$ from (7), then $r \geqslant 3l + 2$ and $v > l \geqslant u$, so that $b \geqslant 2$. Cf. [P] § 4.22 where $u = 3$ and $w = w_1 = 1$.

Since $a_u \neq 0$ from (70) and $k - v = u - 1$, then (75) gives

$$a_{w_1+j} = a_{w_1} a_j, \qquad (j = 1, 2, \dots, u-1). \tag{76}$$

If

$$(g+1)u \geqslant t > gu$$

and if $t - gu \geqslant w_1$, then $u + w_1 - 1 \geqslant t - gu$ and (76) gives with (C),

$$a_t = a_u^g a_{t-gu} = a_u^g a_{t-gu-w_1} a_{w_1} = a_{t-w_1} a_{w_1}.$$

If $t - gu < w_1$, the $w_1 < u < t - (g-1)u \leqslant u + w_1 - 1$, and

$$a_t = a_u^{g-1} a_{t-(g-1)u} = a_u^{g-1} a_{t-(g-1)u-w_1} a_{w_1} = a_{t-w_1} a_{w_1}.$$

Hence (76) is extended to

$$a_{w_1+j} = a_{w_1} a_j, \qquad (j = 1, 2, \dots, k - w_1).$$

We repeat this process, ending with

$$a_{w+j} = a_w a_j, \qquad (j = 1, 2, \dots, k - w). \tag{77}$$

We note that w is the greatest common divisor of u and v.
Let

$$k + 1 = u + v = (b_0 + 1)w.$$

The from (77), (12) becomes

$$F(z) = (1 - a_1 z + a_2 z^2 - + \dots + a_{w-1}(-z)^{w-1})$$
$$(1 + a_w(-z)^w + \dots + a_w^{b_0}(-z)^{b_0 w}).$$

From § 3 (in particular § 3.3) this is only possible when

$$w - 1 = 0,$$

that is when u and v are prime to one another. In this case we have (16).

6. [1] Case 3., $p \leqslant k - 1$, $r - p \geqslant k + 1$, not both $r - p$ and q equal to $k + 1$. We deal first with

$$u = k, \quad \text{or} \quad v = 1, \quad \text{or} \quad v = k.$$

[1] All the distributions of § 6 are included in the favourable situations (ii) (iii) or (iv) of [P].

In § 6.2 we consider the sub-cases of $u > l$ and of u dividing v when $u \leqslant l$. When $s-q > k$ we have case $3a$ and deal with this in § 6.3 making use of the argument of § 5.22; to do this we need $a_v a_{k-v} = a_k$ from relations (D) and this is possible when $s - q > k$. Case $3b$, $s - q \leqslant k$ is considered in § 6.4 and its sub-sections. We deal first with the special sub-cases $a_v = 0$ and $a_{s-q} = a_v a_{s-r}$. In § 6.42 we deduce certain inequalities concerning u and v when neither of these relations hold. The inequalities are used in dealing with the remaining portion of case $3b$.

6.11, $u = k$. (¹) Since $p > l$ from (7), then $q > k + 1$, the relations (A) exist and contain

$$a_l = a_{l+1} = \dots = a_{k-1} = 0.$$

Then (16) follows on using (6).

6.12. The case of $v = 1$ is only possible when $u = k$ since $r - p \geqslant k + 1$.

6.13. If $v = k$, then

$$r > 2\,k, \qquad r - p > k + 1.$$

Also

$$a_1 = a_2 = \dots = a_{u-1} = 0$$

from (B), and with (C) gives (16) as in § 4.31.

Thus, from (8) we take

6.2. $$2 \leqslant u < k, \qquad 2 \leqslant v < k.$$

We define b by

$$k \geqslant bu > k - u. \tag{78}$$

If $p \geqslant v = r - q$, then $q \geqslant r - p$ so that $q > k + 1$ from the conditions of case 3, and then relations (A) exist. Similarly, when $v \geqslant p$, then $r - p > k + 1$ and we have (B). Hence from (78), $bu \geqslant \max(p, v)$, otherwise $a_{bu} = 0$ from (A) or (B) so that $a_u = 0$ from (C) in contradiction to (70).

If $b = 1$ in (78), then $u > l$ and $q > k + 1$ since $p > l$ from (7). In this case the relations (A) overlap so that (²) $u \geqslant p$ and relations (A) become

$$a_{k-u+1} = a_{k-u+2} = \dots = a_{k-p+1} = a_{k-p+2} = \dots = a_{p-1} = \dots = a_{u-1} = 0.$$

(¹) Cf. [P] §§ 3,4.

(²) In order to avoid $a_u = 0$ which contradicts (70).

This, with (C), gives (12) and (13) [1] with $w = u$, $d = 1$.

Thus we consider

$$b \geqslant 2, \qquad k \geqslant bu \geqslant \max(p, v) > k - u. \tag{79}$$

Hence using (7),

$$u < l, \qquad q < 3l. \tag{80}$$

6.21. If $bu = v$ then $a_v \neq 0$ from (C) and (70). Hence $v \geqslant p$ and relations (B) exist, otherwise relations (A) exist as in § 6.2 and then $a_v = 0$ since $v \geqslant k - u + 1$. Relations (B) become

$$a_j = 0, \qquad (j = u-1,\ u-2, \ldots, k - bu + 1).$$

Then from (C) we have (12) and (13) with $w = u$, $d = b$. Hence we are left to consider

$$bu > v > k - u \geqslant (b-1)u, \quad b \geqslant 2. \tag{81}$$

6.3. Case $3a$, $s - q > k$.

In this case (D) includes $a_k = a_{k-v}\, a_v$ and thus

$$a_v \neq 0 \tag{82}$$

and as in § 6.21, $v \geqslant p$ and relations (B) exist. Then (B) and (C) give

$$a_j = 0, \qquad (j = bu - 1,\ bu - 2, \ldots, k - v + (b-1)u + 1).$$

Since $bu > v$ from (81) and $a_v \neq 0$ from (82), then from the above relations

$$k - v + (b-1)u \geqslant v, \quad \text{i.e.} \quad k - w_1 \geqslant v,$$

with the notation of § 5.22. Arguing as in (75),

$$a_{w_1 + j} = a_{w_1} a_j, \qquad (j = 1, 2, \ldots, k - v). \tag{83}$$

Also (B) gives, with (C),

$$a_j = 0, \quad (j = k - v + 1,\ k - v + 2, \ldots, u - 1)$$
$$a_j = 0, \quad (j = k - (b-1)u + 1,\ k - (b-1)u + 2, \ldots, v - (b-2)u - 1).$$

---

[1] Hence $k = u$ and (16) follows. This step will be omitted in future. Cf. [**P**] § 4.11, where $u = p = 5$, $v = 2$ and we have an impossible situation.

This last set is obtained by using (C) with the complex conjugates of the preceding set. Hence, since $w_1 = v - (b - 1) u_1$

$$a_{w_1+j} = 0 = a_{w_1} a_j, \qquad (j = k - v + 1,\ k - v + 2, \ldots, u - 1).$$

These relations combine with (83) to give (76).

We argue as in § 5.22 and with the same definitions, to reach (77). This gives (12) and hence (16) follows.

6.4. Case $3b$, $s - q \leqslant k$.

Since $p \leqslant k - 1$ from (7) and $q - p = u \leqslant l$ from (80), then

$$s \leqslant 5l - 1. \tag{84}$$

Since $l < p \leqslant 2l < q < 3l$ from (7) and (80), we are in favourable situation (ii) of [**P**]. Now $s < 5l$ from (84) and thus, if $r \leqslant 2k$, then

$$s_1 > 5l, \tag{85}$$

where c_{n+s_1} is the next non-small coefficient after c_{n+s}. If $r > 2k$ then in the notation of [**P**] § 2 we have

0 1 1 0.

The fifth block contains at least two non-small coefficients since $4l < v < s < 5l$. By arguments similar to those of Lemma 1 of [**P**] we find that there will be a modified favourable situation of the form

0 1 1 0 2 1 1 ... 1 0 or **0 1 1 0 2 0.**

Thus (85) still holds.

6.411. When $a_v = 0$, relations (B) and (D) give

$$a_{k-u+1} = a_{k-u+2} = \ldots = a_{v-1} = a_v = \ldots = a_{s-q-1} = 0. \tag{86}$$

Relations (B) do not hold when $r - p = k + 1$ but in this case $k - u + 1 = k - q + p + 1 = r - q = v$; relations (D) do not hold when $s - r = 1$, but then $v = r - q = s - q - 1$ so that (86) holds in both these cases.

Moreover, when $a_v = 0$, relations (F) become

$$a_j a_{s-q} = a_{s-q+j}, \qquad (j = 1, 2, \ldots, s_1 - s - 1). \tag{87}$$

If $s - q = k$, then (86) and (6) give

$$a_1 = a_2 = \dots = a_{u-1} = 0$$

and hence (16) follow as in § 4.31. When $s - q < k$ we argue in a similar manner to § 6.3 with v (B) and (D) replaced by $s - q$, (86) and (87) respectively. We note that $s_1 - q > 5l - q > k$ from (85) and (80).

6.412. If $a_{s-q} = a_v a_{s-r}$, then (D) and (F) combine to give

$$a_{v+j} = a_v a_j, \qquad (j = 1, 2, \dots, s_1 - r - 1).$$

As in § 6.411, $s_1 - q > k$ so that the above relations contain $a_v a_{k-v} = a_k$. We then argue as in § 6.3.

6.42. Thus we consider

$$a_v \neq 0, \qquad a_{s-q} \neq a_v a_{s-r}. \tag{88}$$

As in § 63, since $a_v \neq 0$, we have

$$v \geqslant p, \qquad r - p > k + 1, \quad v \leqslant k - v + (b - 1) u. \tag{89}$$

But $k \geqslant bu$ from (78), and hence from (89)

$$r - q = v \leqslant 2k - u - v = 2k - q - (v - p) \leqslant 2k - q, \tag{90}$$

$$r \leqslant 2k. \tag{91}$$

Also $s > 2k$ from (7), and hence from (90),

$$s - q > 2k - q \geqslant k - v + (b - 1) u. \tag{92}$$

We now have from (89), (88) and the hypothesis of case $3b$ that,

$$r - p > k + 1, \qquad a_{s-q} \neq a_v a_{s-r}, \qquad s - q \leqslant k,$$

and also

$$s_1 - r > 5l - r \geqslant l \geqslant u,$$

from (85), (91) and (80). The conditions of § 2.44 are satisfied, and hence (34) holds, so that

$$a_{s-q-1} = a_{s-q-2} = \dots = a_{k-u+s-r+1} = 0.$$

Since $a_v \neq 0$, then $a_{k-v+(b-1)u} \neq 0$ from (6), (81), (C) and (70). Hence from (92) and (34) we have

$$k - r + p + s - q + 1 =$$
$$= k - u + s - r + 1 > k - v + (b-1)u = k - r + p + bu,$$

and so

$$s - q \geqslant bu.$$

6.421. When $s - q > bu$, then $k - u + s - r + 1 > bu$, i.e.

$$k + s - r \geqslant (b+1)u. \tag{93}$$

Otherwise $a_{bu} = 0$ from (34), and hence $a_u = 0$ from (C) in contradiction to (70).

Now from (89) and (7),

$$k + u + v - bu \leqslant 2k - v < s - v \leqslant s - p.$$

Hence $u + v - s + q < bu - k - p + q = (b+1)u - k \leqslant s - r$ from (93), and thus from (89),

$$k + 1 - s + q < u + v - s + q < s - r$$

since $u + v = r - p > k + 1$. Thus $s - r - 1 > k + 1 - s + q \geqslant 1$ since $s - q \leqslant k$ in case $3b$, and since $r - p > k + 1$ from (89), the conditions of § 2.42 are satisfied. We therefore have (29) or (30).

If (29) holds, so that

$$a_{s-q-1} = a_{s-q-2} = \dots = a_{k-s+r+1} = 0,$$

then since $s - q > bu$ in this sub-section and to avoid the contradiction $a_{bu} = 0$, we must have

$$k - s + r + 1 > bu, \qquad r > bu + k \tag{94}$$

since $s > 2k$. From (89),

$$q = p + u \leqslant v + u, \qquad v - q \geqslant -u.$$

Hence

$$2v > k + (b-1)u,$$

from (94). This contradicts (89) and thus leaves (30), i.e.

$$a_v a_j = a_{v+j}, \qquad (j = 1, 2, \ldots, k - v).$$

We then argue as for case 3a.

6.422. $s - q = bu$.

If $s_1 - s > u$, relations (F) contain

$$a_u (a_{bu} - a_v a_{s-r}) = 0 - a_v a_{s-r+u},$$

since $(b + 1) u > k$ from (78). Now

$$s - r + u = bu - v + u < v + u - 1,$$

since $v > (b - 1) u$ from (81). Hence, from (C), $a_{s-r} a_u = a_{s-r+u}$ and we have the contradiction $a_u a_{bu} = 0$.

When $s_1 - s \leqslant u$ then $s_1 \leqslant s - q + 2u + p = (b + 2) u + p$ and also

$$s - q = bu, \qquad 2p \leqslant 2v \leqslant k + (b - 1) u \tag{95}$$

from (89). Hence, from (85),

$$5k < 2s_1 \leqslant k + 3(b + 1) u \tag{96}$$

and therefore $3(b + 1) u > 4k \geqslant 4bu$ from (78), so that $b < 3$. Thus from (79) we have only to consider $b = 2$, i.e.

$$s - q = 2u. \tag{97}$$

We now have from (79), (81) and (89),

$$k \geqslant s - q = 2u > v \geqslant p > k - u; \qquad 2v \leqslant k + u. \tag{98}$$

Hence $2v + 1 \leqslant k + u + 1 < v + 2u$ since $u + v = r - p > k + 1$ from (89), and thus $v < 2u - 1$ so that $s - r = 2u - v > 1$. Then since $r - p - 1 > k \geqslant 2u$ and $v > u$ from (98),

$$a_u a_{v-u+j} = a_{v+j} = a_v a_j = a_{v-u} a_v a_j, \qquad (j = 1, 2, \ldots, 2u - v - 1),$$

on using (C) and (D). Then from (70),

$$a_{v-u+j} = a_{v-u} a_j, \qquad (j = 1, 2, \ldots, 2u - v - 1). \tag{99}$$

From (95), $2s < k + 7u$ since $b = 2$; hence from (96)

$$2s_1 - 2s > 5k - k - 7u \geqslant k - u \geqslant 2v - 2u$$

since $k \geqslant 2u$ and $k + u \geqslant 2v$ from (98).

Thus $s_1 - s > v - u$ and so relations (F) contain

$$a_j(a_{2u} - a_v a_{s-r}) = a_{2u+j} - a_v a_{s-r+j}, \qquad (j = 1, 2, \ldots, v - u - 1).$$

Using (C), it follows that

$$a_{2u-v+j} = a_{2u-v} a_j, \qquad (j = 1, 2, \ldots, v - u - 1) \tag{100}$$

since $s - r = 2u - v$ from (97) and $a_v \neq 0$ from (88).

From (89), $u + v = r - p > k$ so that $k - v < u$, and

$$v - u \leqslant k - v, \qquad 2u - v \leqslant k - v$$

from (98), and thus it follows from (99) and (100) that

$$a_{v-u} \neq 0, \qquad a_{2u-v} \neq 0 \tag{101}$$

since $a_{k-v} = \bar{a}_v \neq 0$ from (88).

We define

$$w$$

as the greatest common divisor of u and v, and hence of $v - u$ and $2u - v$. A similar argument to that of § 5.22, using (99), (100) and (101), gives

$$a_{w+j} = a_w a_j, \qquad (j = 1, 2, \ldots, u - w - 1).$$

If $v - u = g_1 w$, $2u - v = g_2 w$, then

$$a_{v-u} = a_w^{g_1}, \qquad a_{2u-v} = a_w^{g_2}.$$

Since $s_1 - q > k$ from (80) and (85), relations (F) contain

$$a_{k-2u}(a_{2u} - a_v a_{2u-v}) = 1 - a_v a_{k-v},$$

and $a_{k-2u} a_{2u} = 1$ from (C), $a_v \neq 0$ from (88), and hence

$$a_{k-2u} a_{2u-v} = a_{k-v}, \qquad a_{k-u} a_{2u-v} = a_{k-v+u},$$

from (C), i.e.

$$\bar{a}_u a_w^{g_2} = \bar{a}_w^{g_1}. \tag{102}$$

If $|a_w| = 1$, then

$$a_u = a_w^{g_1+g_2}$$

and hence

$$a_{s-q} = a_u^2 = a_u a_w^{g_1} a_w^{g_2} = a_u a_{v-u} a_w^{g_2} = a_v a_{s-r},$$

in contradiction to (88). Then since $|a_u| = 1$ from (C), (102) is only possible when $g_1 = g_2 = 1$. Hence $2u - v = v - u$, $s - r = \frac{1}{2} u = w$, $2v = 3u$, $5w = u + v > k \geqslant 2u = 4w$. It follows from (99), (B) and (C) that (9), (10) and (11) are true, and thus Lemma 1 may be used. Now columns

$$p + 3w, \qquad p + 4w$$

of (19) are retained in (20). Hence from Lemma 1,

$$\eta_3 + a_w \eta_2 = 0, \qquad \eta_4 + a_w \eta_3 = 0. \tag{103}$$

From (7), (80), (91), (84) and (85) we have

$$r \leqslant 2l < q < 3l < r \leqslant 4l < s < 5l < s_1$$

and are thus in situation (ii) of [**P**]. Since $w = \frac{1}{2} u \leqslant \frac{1}{2} l$ from (80) and $p \leqslant v \leqslant l + \frac{1}{2} u$ from (98) then $p + a_w \leqslant l + 10u \leqslant 6l$. Thus one of columns

$$p + 8w, \qquad p + 9w$$

of (19) is retained in (20), for $s = s - r + r - p + p = p + 6w$ and if $s_1 = p + 8w$ then column $p + 9w$ is kept.

If the first is retained, then since $u = 2w$, lemma 1, (38) and (C) give

$$\eta_8 + a_w \eta_7 = 0, \qquad a_{2w}(\eta_6 + a_w \eta_5 + a_{2w} \eta_4) = 0.$$

Then $\eta_6 + a_w \eta_5 + a_{2w} \eta_4 = 0$ since $a_{2w} = a_u \neq 0$ from (70), and from (38) and (103),

$$a_{3w} \eta_3 + a_{4w} \eta_2 = 0, \qquad \eta_2 (a_{4w} - a_w a_{3w}) = 0.$$

From (88), $a_{4w} = a_{s-q} \neq a_v a_{s-r} = a_w a_{3w}$ and thus $\eta_2 = 0$.

Relations (103) give $\eta_3 = 0 = \eta_4$, and from (38), the contradiction

$$a_{3w}\eta_1 + a_{4w} = 0, \qquad a_{4w} = a_w a_{3w}$$

arises. Hence colum $p + 8w$ cannot be retained. Similarly (1), column $p + qw$ cannot be retained and thus this situation is impossible.

7. (2) Case 4, $p \leqslant k - 1$, $r - p \leqslant k$.
For this to be possible, we must have

$$u < k,\ r < 2k \quad \text{and} \quad s - r \geqslant 2,$$

since $s > 2k$ from (7).
We first consider the following special cases,

$$a_u = 0, \quad \text{or} \quad a_{r-p} = a_u a_v.$$

We then use the conditions $a_u \neq 0$, $a_{r-p} \neq a_u a_v$ in the discussion of the remainder of case 4. There is a certain similarity between cases $3b$ and $4a$, as in the former we showed that the assumptions $a_v \neq 0$, $a_{s-q} \neq a_v a_{s-r}$ lead in § 6.421 to $1 \leqslant k + 1 - s + q \leqslant s - r - 1$. These compare with the conditions $1 \leqslant k + 1 - r + p \leqslant r - q - 1$. Case $4b$ is discussed in § 7.4 and its sub-sections; we consider the case of u dividing $r - p$ and show that when this does not happen then $a_v = 0 = a_{r-p}$, which gives a contradiction.
7.1. If $a_u = 0$, (A) and (C) give

$$a_j = 0, \quad (j = k - p + 1,\ k - p + 2, \ldots, u - 1,\ u, \ldots, r - p - 1). \tag{104}$$

This is true when $q = k + 1$ and relations (A) do not hold, or when $v = 1$ and relations (C) do not hold; when $v = 1$, $q \geqslant 3l$ since $r > 3l$ from (7) so that relations (A) hold. (3) Relations (E) become

$$a_{r-p}\, a_j = a_{r-p+j}, \qquad (j = 1,\ 2, \ldots, s - r - 1). \tag{105}$$

Now $s - p - 1 > k$ from (7), and thus $a_{r-p} \neq 0$ from (105). Hence from (104) and (6), we must have $r - p \geqslant p$ since $p > l$ and $r - p - 1 \geqslant l$ from (7). Now $k - p + 1 \leqslant l$ and (12), with $w = r - p$, $d = 1$, follows from (104), (6) and (105). As in § 3 this is only possible when $r - p = k$, and we have (16).

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(1) We also use $a^1 \neq 0$, from (101).
(2) Favourable situation (ii) of [P], or (iv) when $q \geqslant 3l + 1$.
(3) Compare [P], §§ 4.51, 4.7, see also § 4.6.

Hence we assume

$$a_u \neq 0.$$

7.2. If $a_{r-p} = a_u a_v$, then (C) and (E) together give

$$a_j a_u = a_{u+j}, \qquad (j = 1, 2, ..., s - q - 1). \tag{106}$$

This is still true when $v = 1$ or $s - r = 1$. We note that $s - p - 1 \geqslant k$ from (7), so that (106) includes $a_u a_{k-u} = a_k$.

Since $a_u \neq 0$ it follows from (106) and (21) that

$$a_j = 0, \qquad (j = s - q - 1, s - q - 2, ..., k - u + 1). \tag{107}$$

Hence

$$s - q \leqslant k$$

otherwise (107) includes $a_k = 0$.

7.21. If $q = k + 1$, then since $s - q \leqslant k$ and $s > 2k$ from (7), we must have $s = 2k + 1$; (107) and (6) give

$$a_j = 0, \qquad (j = 1, 2, ..., u - 1).$$

Hence (16) follows from (106) as in § 4.31. [1]

7.22. When $q > k + 1$ we have (A), and as in (78) we define b by

$$k \geqslant bu > k - u.$$

Since $a_u \neq 0$, and in order to avoid $a_{bu} = 0$ and $a_u = 0 = a_k$ from (106), we have

$$bu \geqslant p, \qquad bu \geqslant s - q$$

from (A) and (107) respectively. (A) and (106) give

$$a_j = 0, \quad (j = bu - 1, bu - 2, ..., k - p + (b - 1)u + 1). \tag{108}$$

Now $bu \leqslant k$ from (78) and $s > 2k$, and thus

$$k - p + (b - 1)u \leqslant 2k - q < s - q,$$

so that (108) and (107) overlap, (or continue when $k = bu$, $s = 2k + 1$) and hence

$$a_j = 0, \qquad (j = k - u + 1, k - u + 2, ..., bu - 1).$$

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[1] Cf. [P], § 4.52.

This, with (106) gives (12), (13) and hence (16) follows.
We thus assume

$$a_u \neq 0, \qquad a_{r-p} \neq a_u a_v. \tag{109}$$

7.3. Case $4a$,

$$k - r + p + 1 \leqslant v - 1. \tag{110}$$

Since $r - p \leqslant k$, then $1 \leqslant k - r + p + 1$ and the conditions of § 2.41 are satisfied. If (26) holds then (28) follows but this contains

$$a_{r-p} = a_u a_v,$$

contradicting (109). Hence

$$a_u a_{k-u} \neq 1 \tag{111}$$

and (25) holds.

If $v > l$ then $k - v + 1 \leqslant l$. Now, if $(g+1)u \geqslant r - p > gu$ for some g, then $gu \geqslant v > k - v + 1$ and $a_{gu} = 0$ from (25). Then relations (C) give $a_u = 0$, contradicting (109). Hence we take

$$v \leqslant l. \tag{112}$$

Taking conjugates of (E), we obtain

$$a_{k-j}(a_{k-r+p} - a_{k-v} a_{k-u}) = a_{k-r+p-j} - a_{k-u} a_{k-v-j}, \\ (j = 0, 1, \ldots, s - r - 1). \tag{113}$$

Since $k - r + p < v - 1$, relations (C) give.[1]

$$a_u a_{k-r+p-j} = a_{k-v-j}, \qquad (j = 0, 1, \ldots, k - r + p).$$

Hence, multiplying (113) by a_u, we obtain.

$$a_{k-j}(a_{k-v} - a_u a_{k-u} a_{k-v}) = a_{k-v-j} - a_u a_{k-u} a_{k-v-j}, \\ (j = 0, 1, \ldots, k - r + p),$$

since $k - r + p \leqslant s - r - 1$ from (7). Then (111) gives, with (6),

$$a_{k-j} a_{k-v} = a_{k-v-j}, \qquad a_v a_j = a_{v+j}, \qquad (j = 0, 1, \ldots, k - r + p). \tag{114}$$

[1] The subsequent argument holds when $r - p = k$.

Relations (E) then give

$$a_{r-p}\, a_j = a_{r-p+j}, \qquad (j = 0,\ 1,\ \dots,\ k - r + p). \tag{115}$$

Since $a_{r-p} \neq a_u a_v$ from (109), then from (114),

$$k - r + p < u, \qquad r - p > k - u. \tag{116}$$

Thus

$$r \geqslant 2\,p, \tag{117}$$

for when $q = k + 1$, (116) becomes (117); and when $q > k + 1$, (117) follows from (A) and (116) since $a_{r-p} \neq 0$ from (115).

If $a_v = 0$, then (114) gives $0 = a_v\, a_{k-r+p} = a_{k-u} = \bar{a}_u$, contradicting (109). Hence $a_{k-v} = \bar{a}_v \neq 0$ and either

$$k - v \geqslant p \quad \text{or} \quad k - v \leqslant k - u.$$

These inequalities follow from (A) when $q > k + 1$; and when $q = k + 1$, then $k - u = p - 1$. We consider the above alternatives in §§ 31-7-312 and § 7.32.

7.31. When $k - r + q = k - v \geqslant p$, then $r - k \leqslant q - p = u$, and hence from (7) and (112),

$$u > l \geqslant v. \tag{118}$$

Then $q > l + p > k + 1$ from (7), and the relations (A) exist. Since $u > l$ and $p - 1 > l$, these relations overlap and contain

$$a_j = 0, \qquad (j = k - u + 1,\ k - u + 2,\ \dots,\ u - 1). \tag{119}$$

Since $k - v \geqslant p$ and $r - p \leqslant k$ from § 7.31 and § 7,

$$r + v \leqslant k - r + p \leqslant 2k < s$$

from (7). Hence

$$v \leqslant s - r - 1 \tag{120}$$

and (E), (25) and (110) give

$$0 = a_j\,(a_{r-p} - a_u\, a_v) = 0 - a_u\, a_{v+j},$$
$$(j = k - r + p + 1,\ k - r + p + 2,\ \dots,\ v - 1).$$

Since $a_u \neq 0$ from (109), we must have $u \geqslant 2\,v$, otherwise the above relations include $a_u^2 = 0$. We consider $u > 2\,v$ and $u = 2\,v$ in § 7.311 and § 7.312.

7.311. If $u > 2v$ then $a_{2v} = 0$ from (119) since $2v > k - u + 1$ from (110), and also $r - p + v > k$ so that $a_{r-p+v} = 0$. Hence from (E) and (120),

$$a_v(a_{r-p} - a_u a_v) = 0 - a_u a_{2v} = 0.$$

This contradicts (109), since $a_v = 0$ implies $a_u = 0$ as in the last paragraph of § 7.3.

7.312. Hence (1) $u = 2v$, $r - p = 3v$ and $4v = 2u > k$ from (118). Now $|a_{r-p}| = 1$, from (115). We therefore set

$$a_{3v} = e^{-i\alpha}, \qquad a_v = Re^{i\beta}, \qquad (R, \alpha, \beta \text{ real}).$$

Now (114) contains $a_{k-3v}\, a_v = a_{k-2v}$, and hence

$$a_{2v} = Re^{-i(\alpha+\beta)}.$$

Then from (E) and (120),

$$a_v(a_{3v} - a_v a_{2v}) = 0 - a_{2v}^2,$$
$$(R - R^3)\, e^{-i(\alpha+3\beta)} = -R^2.$$

Hence, changing the sign of R if necessary,

$$\alpha + 3\beta = 0, \qquad R^2 = R + 1,$$

since $a_v \neq 0$ from § 7.3. Set $\omega = e^{i\beta}$ and then

$$a_v = R\omega, \quad a_{2v} = R\omega^2, \quad a_{3v} = \omega^3, \quad (R^2 = R + 1, \ |\omega| = 1),$$

so that (15) arises. Relations (9), (10) and (11) with $w = v$, $d = 3$ follow from (114), (C), (115), (25) and (119). Hence (18) holds and this distribution is only possible when $k = 3v = r - p$.

7.32. If $k - v \leqslant k - u$ then $v \geqslant u > k - r + p$ from (116). Hence from (25), $v = u$, otherwise $a_u = 0$ in contradiction to (109). Hence $r - p = 2u$ and from (116),

$$3u > k \geqslant r - p = 2u.$$

This is (11) with $d = 2$; (9) and (10) follow from (C), (115) and (25); (14) now follows from lemma 2 since $a_u \neq 0$ and $a_{2u} = a_{r-p} + a_u a_v = a_u^2$

(1) Compare [P] § 4.31.

from (109). As in § 3, this distribution is only possible when $k = 2u = r - p$ and then we have (17). [1]

7.4. Case 4 b,

$$k - r + p + 1 > v - 1, \qquad q > k + 1. \tag{121}$$

The second inequality was deduced in § 2.43. The conditions of § 2.43 are satisfied and hence (32) follows. If $r - p > (g - 1)u \geqslant r - k$ for some $g \geqslant 2$, then from (32) and (C), $a_u^{g-1} = 0$, contradicting (109). Hence we take

$$gu \geqslant r - p > r - k > (g - 1)u,$$

and consider $gu = r - p$, $gu > r - p$ in §§ 7.41-7.412 and § 7.42.

7.41, $gu = r - p$. If $k - r + p = v - 1$ then $a_{k-r+p} = a_{v-1} = a_{gu-u-1} = 0$, from ($A$) and ($C$), and thus $a_{k-v} = a_{k-r+p}\, a_u = 0$ from (C). Hence $a_{r-p} = 0 = a_v$ from (6), in contradiction to (109). Thus we take $k - r + p > v - 1$ from (121), and since $r - p > l$ from (7),

$$2gu > k > r - p + v - 1 = (2g - 1)u - 1. \tag{122}$$

Hence $k - gu < gu = r - p$, so that (C) and (6) give

$$a_{k-gu-tu}\, a_u^t = a_{k-gu}, \qquad a_{gu+tu}\, a_{k-u}^t = a_{gu}, \qquad (t = 0, 1, ..., g - 1). \tag{123}$$

7.411. If $g > 2$, then since $p \leqslant k - 1$ from § 7, we have

$$s - p - 1 > k + 1 > (2g - 1u \geqslant (g + 2)u$$

from (7) and (122). Hence relations (E) contain

$$a_u^j (a_{gu} - a_u a_{gu-u}) = a_{gu+ju} - a_u a_{gu+ju-u}, \qquad (j = 1, 2), \tag{124}$$

and we have

$$a_{gu+2u} - 2a_u a_{gu+u} + a_u^2 a_{gu} = 0.$$

Then from (123), since $a_{k-u} = \bar{a}_u \neq 0$ from (109),

$$a_{gu+2u}(1 - 2a_u a_{k-u} + a_u^2 a_{k-u}^2) = 0.$$

This contradicts (109). For if $a_u a_{k-u} = 1$, then (124) and (123) give

$$a_u(a_{r-p} - a_u a_v) = a_u(a_{gu} - a_u a_{gu-u}) = a_{gu+u} - a_u a_{gu} = 0. \tag{125}$$

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[1] Compare [P] § 3.42.

If $a_{gu+2u} = 0$, then

$$a_{gu} = a_{gu-u} = 0$$

from (123), since $a_{k-u} = \bar{a}_u \neq 0$ from (109). Hence (125) again holds.

7.412. When $g = 2$, i.e. $v - p = 2u$, then from (122),

$$4u > k > 3u - 1, \qquad v = u < l. \tag{126}$$

Thus $3u \leqslant k \leqslant s - p - 1 = s - r + 1 + 2u$ from (7), and relations (E) contain

$$a_j (a_{2u} - a_u^2) = a_{2u+j} - a_u a_{u+j}, \qquad (j = 0, 1, ..., u - 1).$$

Hence from (C)

$$a_{2u} a_j = a_{2u+j}, \qquad (j = 0, 1, 2, ..., u - 1). \tag{127}$$

From (E), $a_u (a_{2u} - a_u^2) = a_{3u} - a_u a_{2u}$ since $s - r > u$ as above. Then from (C) and (E),

$$\begin{aligned} a_j (a_{3u} - a_u a_{2u}) &= a_j a_u (a_{2u} - a_u^2) = a_{u+j} (a_{2u} - a_u^2) \\ &= a_{3u+j} - a_u a_{2u+j}, \qquad (j = 0, 1, 2, ..., k - 3u), \end{aligned}$$

since $k - 3u < u = v$ from (126) and $k \leqslant s - p - 1$ from (7). Hence from (127) and (126),

$$a_{3u} a_j = a_{3u+j}, \qquad (j = 0, 1, 2, ..., k - 3u). \tag{128}$$

From (126), $r - p + u \leqslant k$ and $3u \geqslant k - u + 1$. Hence

$$r - k \leqslant p - u, \qquad 3u \geqslant p. \tag{129}$$

The latter inequality follows from (A) since $a_{3u} \neq 0$ from (128). From (A), (C) and (6),

$$a_j = 0, \qquad (j = p - u - 1, p - u - 2, ..., k - 2u + 1). \tag{130}$$

Then from (126) and (129), (32) overlaps [1] with (130) to give

$$a_j = 0, \quad (j = 2u - 1, 2u - 2, ..., p - u - 1, ..., r - k, ..., k - 2u + 1). \tag{131}$$

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[1] Or continues, when $r - k = p - u$.

As in § 7.312 we set

$$a_u = Re^{i\beta}, \qquad a_{3u} = e^{-i\alpha}, \qquad (R, \alpha, \beta \text{ real}).$$

Then $a_{k-3u}\, a_u = a_{k-2u}$ from (C) since $k - 2u \leqslant 2u - 1 = r - p - 1$ from (126), so that

$$a_{2u} = Re^{-i(\alpha+\beta)}.$$

Now relations (E) contain

$$a_{k-2u}\,(a_{2u} - a_u^2) = 1 - a_u\, a_{k-u}.$$

Thus

$$R^3 e^{i(\alpha+3\beta)} - 2R^2 + 1 = 0, \quad \alpha + 3\beta = 0, \quad (R-1)(R^2 - R - 1) = 0.$$

If $R = 1$ then $a_{r-p} = a_{2u} = e^{-i(\alpha+\beta)} = e^{2i\beta} = a_u^2 = a_u\, a_v$ in contradiction to (109), and hence (15) holds. Relations (9) with $u = w$ and $d = 3$ follow from (C), (127) and (128); (10) follows from (131) and (C); and (126) is (11) with $d = 3$. Hence (18) holds.

7.42. When $gu > r - p$ we show that $a_v = 0$ and then deduce certain inequalities involving $r - p$ and p which must be satisfied in order that $a_{r-p} \neq 0$. Next we show that $a_{k+u-2(r-p)} = 0$ and then use (E) to obtain a contradiction.

When $gu > r - p$, then from § 7.4,

$$gu > r - p > r - k > (g-1)\,u, \tag{132}$$

and hence

$$k - p + (g-1)\,u + 1 \leqslant r - p, \qquad k - p + (g-2)\,u + 1 \leqslant v. \tag{133}$$

From (A) and (C),

$$a_j = 0, \quad (j = (g-1)\,u - 1,\ (g-1)\,u - 2, \ldots, k - p + (g-2)\,u + 1), \tag{134}$$

since $(g-1)\,u - 1 < r - p - 1$ from (132). Then from (6),

$$a_j = 0, \quad (j = p - (g-2)\,u - 1,\ p - (g-2)\,u - 2, \ldots, k - (g-1)\,u + 1). \tag{135}$$

From (132) and (133), $(g-1)\,u - 1 \geqslant v \geqslant k - p + (g-2)\,u + 1$, and thus

$$a_v = 0. \tag{136}$$

from (134). Then since $a_{u+v} - a_u a_v \neq 0$ from (109),

$$\bar{a}_{k-r+p} = a_{r-p} \neq 0. \tag{137}$$

If $r - p \geqslant p - (g - 2) u$, then (C) and (135) give

$$a_j = 0, \quad (j = p - (g - 1) u - 1,\ p - (g - 1) u - 2, \ldots, k - gu + 1);$$

but $p - (g - 1) u - 1 \geqslant k - r + p > k - gu$ from (133) and (132), and hence $a_{k-r+p} = 0$ in contradiction to (137). Thus

$$r - p < p - (g - 2) u. \tag{138}$$

Since $a_{r-p} \neq 0$ from (137), it follows from (135) and (138) that

$$r - p \leqslant k - (g - 1) u. \tag{139}$$

We now show that

$$r - p \geqslant p - (g - 1) u. \tag{140}$$

Fom (7), $r - p > l > k - r + p$, and if $r - p = k - p + (g - 1) u + 1$ then (140) follows. Thus we consider $r - p > k - p + (g - 1) u + 1$ from (133). Then (134) and (C) give

$$a_j = 0, \quad (j = r - p - 1,\ r - p - 2, \ldots, k - p + (g - 1) u + 1) \tag{141}$$

since $gu - 1 > r - p - 1$ from (132). Then (140) follows since $r - p > k - r + p$ and $a_{k-r+p} \neq 0$ from (137).

From (132) (140) and (139),

$$(2g - 1) u > r - p + (g - 1) u \geqslant p \geqslant r - k + (g - 1) u > (2g - 2) u. \tag{142}$$

Using (140), we apply (6) and (C) to (141) to obtain

$$\begin{gathered} a_j = 0, \qquad (j = p - (2g - 2) u - 1, \\ p - (2g - 2) u - 2, \ldots, k - r + p - (g - 1) u + 1). \end{gathered} \tag{143}$$

We note that $k - r + p \geqslant (g - 1) u$ from (139).

From (139) and (7),

$$2k + 2p - 2(g - 1) u \geqslant 2r > 3k$$

and thus

$$k - p < p - (2g - 2)u < u,$$

since $p < (2g - 1)u$ from (142). Hence (143) and (A) overlap (or continue when $p - (2g - 2)u = k - p + 1$), to give

$$a_j = 0, \qquad (j = u - 1,\ u - 2, \ldots,$$
$$p - (2g - 2)u, \ldots, k - p + 1, \ldots, k - r + p - (g - 1)u + 1).$$

Then

$$a_{k+u-2(r-p)} = 0 \tag{144}$$

since $u > k - 2(r - p) + u > k - r + p - (g - 1)u$ from (7) and (132).

From (7),

$$s - r - 1 \geqslant 2k - r > k + u - 2(r - p)$$

since $2(r - p) > r - k + q - p$, also from (7). Thus relations (E) contain

$$a_{k+u-2(r-p)}(a_{r-p} - a_u a_v) = a_{k-r+p+u} - a_u a_{k-r+p}.$$

However, $k - r + p + u = k - v$, and from (144), (136) and (6)

$$a_u a_{k-r+p} = 0.$$

This result is in contradiction to (109) and (137). Thus the situation of § 7.42 is impossible.

This completes the proof of the theorem.

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