AREA OF THE ELLIPSE DETERMINED BY FIVE TANGENTS

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The aim of this paper is the obtention of a formula permitting the calculation of the area of an ellipse determined by five of its tangents without previously calculating the elements of the curve. The area of the ellipse is obtained in function of the areas of the triangles determined by the five straight lines; we get, therefore, a formula which is invariable in equivalent affine transformations. Every ellipse may be considered as an affine equivalent to a circle; it suffices, therefore, to find the formula for the circle, as it is still valid for the ellipse. (1)

1. Every ellipse may be transformed, by means of an equivalent affinity, into a circumference whose centre is the origin of coordinates and whose equation is, therefore, of the form

$$x^2 + y^2 = r^2, \tag{1}$$

its tangential equation being then

$$u^2 + v^2 = 1 : r^2. \tag{2}$$

If $P_i(x_i, y_i)$, $i = 1, 2, 3$, are three points of the circumference (1),

$$x_i^2 + y_i^2 = r^2, \qquad i = 1, 2, 3, \tag{3}$$

in the same way, if $p_i(u_i, v_i)$, $i = 1, 2, 3$, are three tangents of the same circumference, (2)

$$u_i^2 + v_i^2 = 1 : r^2, \qquad i = 1, 2, 3. \tag{4}$$

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(1) The analogous problem when the ellipse is determined by five of its points is solved in: Rodeja F., E. G. Aire de l'Ellipse déterminée par cinq points. Comment. Math. Helv. 20, 172-176 (1947). And a study on the formula obtained in: Rodeja F., E. G. Sobre una formula del area de la elipse. Rev. de Geofísica. (I) n.° 22, 246-257 (1947). (II), n.° 26, 166-181 (1948). (III), n.° 32, 459-469 (1949).

(2) With (u_i, v_i) we indicate the Plücker coordinates non-homogeneous of the straight line p_i.

By proceedings of a metrical character, we deduce the well known formula of the radius r of the circumference circumscribed to a triangle $P_1 P_2 P_3$,

$$r = \frac{(12)\,(23)\,(31)}{4\,(123)}, \tag{5}$$

where (ij) shows the distance between points P_i, P_j, and (ijk) the area of the triangle $P_i P_j P_k$. This identity is a consequence of the identities (3)

From identities (3), we get identities (4) substituting x, y, r to u, v, $1:r$. If with $(\overline{ij})$ and $(\overline{ijk})$ we indicate the expressions for the straight lines, analogous to (ij) and (ijk), is

$$(\overline{ij}) = [(u_i - u_j)^2 + (v_i - v_j)^2]^{\frac{1}{2}}, \tag{6}$$

$$(\overline{ijk}) = \frac{1}{2} \begin{vmatrix} u_i & v_i & 1 \\ u_j & v_j & 1 \\ u_k & v_k & 1 \end{vmatrix},$$

the identity analogous to (5) is

$$\frac{1}{r} = \frac{(\overline{12})\,(\overline{23})\,(\overline{31})}{4\,(\overline{123})} \tag{7}$$

2. If we call p_0 the straight line of the infinite of the plane, of Plücker coordinates (0,0) and we indicate with $\{\overline{ijk}\}$ the area of the triangle formed by the straight lines $p_i p_j p_k$, we may write the formula which gives the area of the triangle $p_1 p_2 p_3$ in function of the Plücker coordinates of these straight lines [3], in the form

$$\{\overline{123}\} = \frac{(\overline{123})}{4\,(\overline{012})\,(\overline{023})\,(\overline{031})}. \tag{8}$$

From (7) and (8) introducing the symbols $\{\overline{ij}\}$.

$$\{\overline{ij}\} = \frac{(\overline{ij})^2}{(\overline{0\,ij})}, \tag{9}$$

we get

$$\frac{64}{r^2} = \frac{\{12\}\,\{23\}\,\{31\}}{\{\overline{123}\}}. \tag{10}$$

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(3) A. Agostini ed E. Bortolotti. Exercici di Geometria Analitica. Bologna (1925), p. 141.

3. p_i, $i = 1, 2, 3, 4, 5$ being five tangents of the circumference (1), the problem lies in calculating the area of the circle in function of the areas of the $\binom{5}{3} = 10$ triangles that may be formed with the five straight lines. Each one of these ten triangles gives the same value for the radium r of the circumference by means of the formula (10).

By writing the value $64 : r^2$ obtained by means of the four triangles whose sides are p_i, $i = 1, 2, 3, 4$, we get

$$\text{(11)} \quad \frac{\{\overline{12}\}\{\overline{23}\}\{\overline{31}\}}{\{\overline{123}\}} = \frac{\{\overline{23}\}\{\overline{34}\{\{\overline{42}\}}{\{\overline{234}\}} = \frac{\{\overline{34}\}\{\overline{41}\}\{\overline{13}\}}{\{\overline{341}\}} = \frac{\{\overline{41}\}\{\overline{12}\}\{\overline{24}\}}{\{\overline{412}\}}.$$

By multiplying the two first members of these equalities and the two last ones, we find

$$\text{(12)} \quad \frac{\{\overline{12}\}\{\overline{23}\}\{\overline{31}\}\{\overline{23}\}\{\overline{34}\}\{\overline{42}\}}{\{\overline{123}\}\{\overline{234}\}} = \frac{\{\overline{34}\}\{\overline{41}\}\{\overline{13}\}\{\overline{41}\}\{\overline{12}\}\{\overline{24}\}}{\{\overline{341}\}\{\overline{412}\}},$$

that is reduced to

$$\text{(13)} \quad \frac{\{\overline{23}\}^2}{\{\overline{123}\}\{\overline{234}\}} = \frac{\{\overline{41}\}^2}{\{\overline{341}\}\{\overline{412}\}}.$$

Similar formulas to (13), are

$$\text{(14)} \quad \frac{\{\overline{23}\}^2}{\{\overline{523}\}\{\overline{234}\}} = \frac{\{\overline{45}\}^2}{\{\overline{345}\}\{\overline{452}\}} \qquad \frac{\{\overline{45}\}^2}{\{\overline{245}\}\{\overline{451}\}} = \frac{\{\overline{12}\}^2}{\{\overline{512}\}\{\overline{124}\}},$$

whence

$$\text{(15)} \quad \{\overline{23}\}^2 = \frac{\{\overline{523}\}\{\overline{234}\}\{\overline{245}\}\{\overline{451}\}}{\{\overline{345}\}\{\overline{452}\}\{\overline{512}\}\{\overline{124}\}}\{\overline{12}\}^2 = \frac{\{\overline{523}\}\{\overline{234}\}\{\overline{451}\}}{\{\overline{345}\}\{\overline{512}\}\{\overline{124}\}}\{\overline{12}\}^2.$$

The analogous formula for $\{\overline{31}\}$ in function of $\{\overline{12}\}$, is

$$\text{(16)} \quad \{\overline{31}\}^2 = \frac{\{\overline{513}\}\{\overline{134}\}\{\overline{452}\}}{\{\overline{345}\}\{\overline{521}\}\{\overline{214}\}}\{\overline{12}\}^2.$$

The substitution of $\{\overline{23}\}$ and $\{\overline{31}\}$, by means of (15) and (16) in the formula (10), permits the reckonig of the radium of the circumference in function of the length of one only symbol $\{\overline{ij}\}$ and of areas of triangles,

$$(17)\quad \left[\frac{64}{r^2}\right]^2 = \frac{\{\overline{12}\}^2\{\overline{23}\}^2\{\overline{31}\}^2}{\{\overline{123}\}^2} = \frac{\{\overline{523}\}\{\overline{234}\}\{\overline{451}\}\{\overline{513}\}\{\overline{134}\}\{\overline{452}\}\{\overline{12}\}^6}{\{\overline{345}\}\{\overline{512}\}\{\overline{124}\}\{\overline{345}\}\{\overline{521}\}\{\overline{214}\}\{\overline{123}\}^2},$$

and simplifying and ordering

$$(18)\quad \left[\frac{64}{r^2}\right]^2 = \left[\frac{\{\overline{134}\}\,\{\overline{145}\}\,\{\overline{153}\}\,\{\overline{234}\}\,\{\overline{245}\}\,\{\overline{253}\}}{\{\overline{123}\}^2\{\overline{124}\}^2\{\overline{125}\}^2\{\overline{345}\}^2}\right]\{\overline{12}\}^6.$$

If we indicate the expression enclosed in brackets with the symbol $[\overline{1-2}]$, and the analogous in the same way.

$$(19)\quad \{\overline{12}\} = \left[\frac{64}{r^2}\right]^{\frac{1}{3}} \frac{1}{[\overline{1-2}]^{\frac{1}{6}}}.$$

We may write:

$$(20)\quad \begin{aligned} [\overline{1-2}] &= \frac{\{\overline{134}\}\,\{\overline{145}\}\,\{\overline{153}\}\,\{\overline{234}\}\,\{\overline{245}\}\,\{\overline{253}\}}{\{\overline{123}\}^2\{\overline{124}\}^2\{\overline{125}\}^2\{\overline{345}\}^2} \\ [\overline{2-3}] &= \frac{\{\overline{214}\}\,\{\overline{245}\}\,\{\overline{251}\}\,\{\overline{314}\}\,\{\overline{345}\}\,\{\overline{351}\}}{\{\overline{123}\}^2\{\overline{234}\}^2\{\overline{235}\}^2\{\overline{145}\}^2} \\ [\overline{3-1}] &= \frac{\{\overline{234}\}\,\{\overline{345}\}\,\{\overline{352}\}\,\{\overline{124}\}\,\{\overline{145}\}\,\{\overline{152}\}}{\{\overline{123}\}^2\{\overline{314}\}^2\{\overline{315}\}^2\{\overline{245}\}^2}. \end{aligned}$$

By multiplying and simplifying we find

$$(21)\quad [\overline{1-2}]\,[\overline{2-3}]\,[\overline{3-1}] = \frac{1}{\{\overline{123}\}^6}.$$

4. The symbols $\{\overline{ij}\}$, function of the tangential coordinates of two straight lines, not having a simple geometrical interpretation, in order to find a relation connecting such symbols, they are substituted by $\{ij\}$, thus indicating the same function when we put instead of u, v, Plücker coordinates of straight lines, x, y, coordinates of points.

From (9), we get

$$(22)\quad \{ij\} = \frac{(ij)^2}{(0\,ij)},$$

where the numberer is the square of the length of the segment $P_i P_j$ and the denominator is the area of the triangle formed by $P_i P_j$ and the origin of coordinates P_0.

The straight lines p_i, $i = 1, 2, 3$, being tangents to the circumference (1), their Plücker coordinates verify the identities (4) and the points P_i, $i = 1, 2, 3$, whose coordinates have the same values are in a circumference of centre the origin P_0 and radium $1 : r$.

By indicating with α, β, γ the angles $P_2 P_0 P_3$, $P_3 P_0 P_1$, $P_1 P_0 P_2$ is easily deduced,

$$(23) \quad \{23\} = 4 \operatorname{tang} (\alpha : 2), \quad \{31\} = 4 \operatorname{tang} (\beta : 2), \quad \{21\} = 4 \operatorname{tang} (\gamma : 2),$$

and being $\alpha + \beta + \gamma = 0$ or $\alpha + \beta + \gamma = 2\pi$, from the identity.

$$(24) \quad \operatorname{tang} (\alpha : 2) + \operatorname{tang} (\beta : 2) + \operatorname{tang} (\gamma : 2) = \operatorname{tang} (\alpha : 2) \operatorname{tang} (\beta : 2) \operatorname{tang} (\gamma : 2),$$

$$(25) \quad 16 \left[\{12\} + \{23\} + \{31\} \right] = \{12\} \{23\} \{31\}$$

is deduced.

And if in this identity we consider once more the values of the variables as the Plücker coordinates of the straight lines p_i, $i = 1, 2, 3$, we get

$$(26) \quad 16 \left[\{\overline{12}\} + \{\overline{23}\} + \{\overline{31}\} \right] = \{\overline{12}\} \{\overline{23}\} \{\overline{31}\}.$$

5. If we replace in (26) the symbols $\{\overline{ij}\}$ by their values deduced from (19) and from the analogous expressions, we find

$$(27) \quad 16 \left[\frac{64}{r^2} \right]^{\frac{1}{3}} \left[\frac{1}{[\overline{1-2}]^{\frac{1}{6}}} + \frac{1}{[\overline{2-3}]^{\frac{1}{6}}} + \frac{1}{[\overline{3-1}]^{\frac{1}{6}}} \right] = \frac{64}{r^2} \frac{1}{[\overline{1-2}]^{\frac{1}{6}} [\overline{2-3}]^{\frac{1}{6}} [\overline{3-1}]^{\frac{1}{6}}},$$

and keeping in mind (21), we deduce

$$(28) \quad \frac{1}{r^{\frac{4}{3}}} = \left[\frac{1}{[\overline{1-2}]^{\frac{1}{6}}} + \frac{1}{[\overline{2-3}]^{\frac{1}{6}}} + \frac{1}{[\overline{3-1}]^{\frac{1}{6}}} \right] \frac{1}{\{\overline{123}\}}$$

Executing the reckoning of the expression enclosed in brackets, keeping in mind the expressions (20) of $[\overline{1-2}]$, $[\overline{2-3}]$, $[\overline{3-1}]$, we deduce

$$(29) \quad \frac{\{\overline{123}\}^{\frac{1}{2}}}{P^{\frac{1}{6}}} \left[L_1^{\frac{1}{2}} + M_1^{\frac{1}{2}} + N_1^{\frac{1}{2}} \right]$$

where P_1 indicates the product of the areas of the ten triangles,

$$(30)\quad P_1 = \{\overline{123}\}\{\overline{124}\}\{\overline{125}\}\{\overline{134}\}\{\overline{135}\}\{\overline{145}\}\{\overline{234}\}\{\overline{235}\}\{\overline{245}\}\{\overline{345}\},$$

and

$$(31)\quad L_1 = \{\overline{124}\}\{\overline{125}\}\{\overline{345}\} \quad M_1 = \{\overline{234}\}\{\overline{235}\}\{\overline{145}\} \quad N_1 = \{\overline{314}\}\{\overline{315}\}\{\overline{245}\}.$$

Whence we obtain, for the area πr^2 of the circle,

$$(32)\qquad A = \pi r^2 = \frac{\pi \{\overline{123}\}^{\frac{3}{4}} P_1^{\frac{1}{4}}}{[L_1^{\frac{1}{2}} + M_1^{\frac{1}{2}} + N_1^{\frac{1}{2}}]^{\frac{3}{2}}}.$$

Since, as we have already said, this formula keeps invariable in the equivalent affinities, we finally deduce:

The area of an ellipse determined by five tangents in function of the areas of the ten triangles formed by the same, is given by the preceding formula with the said notes.

In this formula, the triangle $p_1 p_2 p_3$ appears in a special way, because it has been obtained from the relation (26) that refers to that triangle; we shall therefore obtain analogous formulas by using the remaining triangles.

6. If we want to find the area of the ellipse in function of the determinants that have been represented with the symbols $(\overline{ijk})$, from (8) we deduce

$$(33)\qquad P_1 = \frac{P_2^2}{4^{10} Q^3}$$

$$L_1 = \frac{(\overline{023})(\overline{031})}{4^3 (\overline{012})} \frac{L_2^2}{Q} \quad M_1 = \frac{(\overline{031})(\overline{012})}{4^3 (\overline{023})} \frac{M_2^2}{Q} \quad N_1 = \frac{(\overline{012})(\overline{023})}{4^3 (\overline{031})} \frac{N_2^2}{Q},$$

where Q is the product of the ten determinants $(\overline{0\,ij})$,

$$(34)\quad Q = (\overline{012})(\overline{013})(\overline{014})(\overline{015})(\overline{023})(\overline{024})(\overline{025})(\overline{034})(\overline{035})(\overline{045}),$$

and P_2, L_2, M_2, N_2 the analogous expressions to the P_1, L_1, M_1, N_1 formed with the symbols $(\overline{ijk})$,

$$(35) \qquad P_2 = (\overline{123})\,(\overline{124})\,(\overline{125})\,(\overline{134})\,(\overline{135})\,(\overline{145})\,(\overline{234})\,(\overline{235})\,(\overline{245})\,(\overline{345})$$

$$L_2 = (\overline{124})\,(\overline{125})\,(\overline{345}) \quad M_2 = (\overline{234})\,(\overline{235})\,(\overline{145}) \quad N_2 = (\overline{314})\,(\overline{315})\,(\overline{245}),$$

by substituting these values in (32) and simplifying we obtain as a formula for the area of the ellipse

$$(36) \quad A = \frac{\pi\,(\overline{123})^{\frac{3}{2}}\,P_1^{\frac{1}{2}}}{4\,[(\overline{023})\,(\overline{031})\,L_2 + (\overline{031})\,(\overline{012})\,M_2 + (\overline{012})\,(\overline{023})\,N_2]^{\frac{3}{2}}}.$$

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