PSEUDO-DIFFERENTIAL OPERATORS ON V-MANIFOLDS AND FOLIATIONS

(First Part)

by

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Owing to its length, this paper is published in two parts: Part I consisting of Chapters 1 and 2 and Part II consisting of Chapters 3, 4 and References. Part II is to be found at the next issue of this journal.

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Introduction

The notion of a V-manifold was introduced by Satake [10] in 1956. Reinhart [9] related this concept to foliations by proving that the quotient space $M/\mathcal{F}$ of a manifold M by a closed metric foliation $\mathcal{F}$ of codimension n is a V-manifold of dimension n. We actually prove a stronger result (theorem 4.2 of Chap. 4). The main purpose of this paper is to give a more or less self-contained development of the theory of pseudo-differential operators on V-manifolds in order to prove the existence of parametrices for differential operators on such manifolds. This theory is new, and we would like to point out that one of the difficulties we have had to overcome has been to find a good definition of pseudo-differential operators on a V-manifold.

Baily [2] proved the existence of parametrices for strongly elliptic differential operators (of order 2) on compact V-manifolds. We should also mention the recent paper by Kitahara [7]. We actually prove the existence of parametrices for elliptic differential operators (of any order). (Theorem 3.5). By applying this result we get the decomposition theorem 3.6. As an application of this result to complex analytic foliations with bundle-like Hermitian metric, we relate the cohomology spaces of base-like forms to the spaces of base-like harmonic forms. We obtain in this way Theorem 4.4 and its corollary. This theorem is analogous to Reinhart's one [8] for the complex case and vector bundle valued forms. The first author of the present paper will employ this result in a subsequent paper [5] in order to prove some rigidity theorems for complex analytic foliations.

Concerning Chapter 1, (an introduction to V-manifolds and V-vector bundles) we would like to point out that our definition of V-manifold requires fewer conditions than the usual ones (Satake [11], Baily [2]). We do not require (as Baily does) that if λ is an injection of local uniformizing systems then, for any $\sigma \in G$ there is $\sigma' \in G'$ satisfying $\lambda \circ \sigma = \sigma' \circ \lambda$. We prove this fact as a proposition (corollary of prop. 1.2) in the same way that Satake does [11], but without his supplementary assumption about the dimension of the set of fixed points of G and G'. That is why our definition is more useful to verify that a concrete example constitutes a V-manifold. For instance, in Th. 4.2, to prove that the quotient space $B = M/\mathcal{F}$ of M by a compact Hausdorff foliation is a V-manifold, (according to Baily's definition) we should show that the holonomy group of any leaf in a small neighborhood of a leaf L is a subgroup of the holonomy group of L. We do not need to verify this fact using our definition.

CHAPTER 1

INTRODUCTION TO V-MANIFOLDS AND V-VECTOR BUNDLES

V-MANIFOLDS

Definition 1.1. Let B be a connected Hausdorff paracompact space. Let U be an open set in B. A local uniformizing system (abbreviated in the following as *l. u. s*) of dimension n corresponding to U is a collection $\{\widetilde{U}, G, \varphi\}$ of the following objets:

(a) $\widetilde{U}$ is a connected open set in $\mathbf{R}^n$.
(b) G is a finite group of C^∞ automorphisms of $\widetilde{U}$.
(c) φ is a continuous map from $\widetilde{U}$ onto U such that $\varphi \circ \sigma = \varphi$ for any $\sigma \in G$ and that φ induces a homeomorphism from $\widetilde{U}/G$ onto U.

Example 1.1. Let $D = \{x \in \mathbf{R}^2, |x| < 1\}$. Let G be the group of rotations of D of angles $0, \pi/2, \pi, 3\pi/2$. Let $B = D/G$ with the quotient topology. B can be thought as the set of points of D in the first quadrant with the identification of each point $(0, a)$, $0 \leqslant a < 1$, to $(a, 0)$. B is then a cone. Let us denote by φ the canonical projection $D \to B$. Consider, for instance, the following two open sets in D:

1) An open disk $\widetilde{U}$ of radius < 1 centered at the origin.
2) A small open disk $\widetilde{V}$ contained in the interior of the first quadrant.

Let $U = \varphi(\widetilde{U})$, $V = \varphi(\widetilde{V})$. $\{\widetilde{U}, G, \varphi\}$ is a *l. u. s.* corresponding to U and $\{\widetilde{V}, \{I\}, \varphi\}$ is a *l. u. s.* corresponding to V (we denote by I the identity).

Definition 1.2. Let U, U' be two open sets in B. Let us suppose that $\{\widetilde{U}, G, \varphi\}$ and $\{\widetilde{U}', G', \varphi'\}$ are *l. u. s.* corresponding to U and U'. An injection λ: $\{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ is a diffeomorphism from $\widetilde{U}$ onto an open set in $\widetilde{U}'$ such that $\varphi = \varphi' \circ \lambda$.

Any $\sigma \in G$ can be considered as an injection of $\{\widetilde{U}, G, \varphi\}$ into itself.

Definition 1.3. A C^∞ V-manifold of dimension n is a connected paracompact Hausdorff topological space B with a family $\mathcal{A}$ of *l. u. s's* of dimension n corresponding to open sets in B satisfying the following conditions:

(a) If $\{\widetilde{U}, G, \varphi\}$ and $\{\widetilde{U}', G', \varphi'\} \in \mathcal{A}$ and $\varphi(\widetilde{U}) \subset \varphi'(\widetilde{U}')$ there exists an injection $\lambda \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$.
(b) Let $\mathcal{H}$ be the family of open sets U in B for which there exists a *l. u. s.* $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$. $\mathcal{H}$ satisfies the two following conditions:

(i) For any $p \in B$ there exists $U \in \mathcal{H}$ with $p \in U$.
(ii) For any $p \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{H}$, there exists $U_3 \in \mathcal{H}$ such that $p \in U_3 \subset U_1 \cap U_2$.

A family $\mathcal{A}$ satisfying (a) and (b) is called a defining family.

Remark. We do not require that $\mathcal{H}$ is a basis of open sets in B as Baily does. For example, if $\pi : E \to M$ is a vector bundle on a differentiable manifold M and $\mathcal{A}$ is a basis of open sets in M, the family $\mathcal{B} = \{\pi^{-1}(U), U \in \mathcal{A}\}$ satisfies conditions (i) and (ii) but it is not a basis of open sets in E. If we required that $\mathcal{H}$ were a basis of open sets in B we could not define the notion of V-vector bundle as a V-manifold, as we shall do in the next section.

Example 1.2. The cone B of example 1.1 is a V-manifold in a natural way.

It can be proven that a quotient space M/G of a differentiable manifold M by a properly discontinuous group G of C^∞ automorphisms of M (not necessarily finite) can be endowed with a V-manifold structure.

Example 1.3. We shall see in Chapter 4 that the quotient space $M/\mathcal{F}$ of a manifold M by a compact Hausdorff foliation is a V-manifold in a natural way.

Example 1.4. Let M be a differentiable manifold and let $\mathcal{B}$ be the maximal atlas. For any local chart $(U, \Psi) \in \mathcal{B}$ with connected U, set $\widetilde{U} = \Psi(U), \varphi = \Psi^{-1}$ and $G = \{I\}$. $\{\widetilde{U}, G, \varphi\}$ is a *l. u. s.* corresponding to U. Let $\mathcal{A}$ be the family of these *l. u. s' s.* corresponding to connected $U \in \mathcal{B}$. $\mathcal{A}$ defines a structure of V-manifold on M. We can then see that the notion of V-manifold generalizes the concept of differentiable manifold.

Definition 1.4. We shall say that two defining families $\mathcal{A}$ and $\mathcal{A}'$ are directly equivalent if both of them are contained in another defining family. We shall say that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if there exists a chain of defining families $\{\mathcal{A}_i\}$, $i = 1 \dots r$, such that $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_r = \mathcal{A}'$ and that each $\mathcal{A}_i$ is directly equivalent to $\mathcal{A}_{i+1}$. We shall say that equivalent families define the same V-manifold structure.

Proposition 1.1. Let $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$. For each $\widetilde{p} \in \widetilde{U}_{\widetilde{p}}$ let $G_{\widetilde{p}}$ be the isotropy group of $\widetilde{p}$. Let $\widetilde{S} = \{\widetilde{p} \in \widetilde{U}$ such that $G_{\widetilde{p}} \neq \{I\}\}$. Each $\widetilde{p} \in \widetilde{S}$ has an open neighborhood $\widetilde{V}$, $\widetilde{V} \subset \widetilde{U}$, such that $\widetilde{V} \cap \widetilde{S}$ is a finite union of submanifolds of dimension $< n$. Moreover, for each $\widetilde{q} \in \widetilde{V}$ one has $G_{\widetilde{q}} = \{\alpha \in G_{\widetilde{p}}$ such that $\alpha(\widetilde{q}) = \widetilde{q}\}$. In particular, $G_{\widetilde{q}}$ is a subgroup of $G_{\widetilde{p}}$.

Proof. Let $\widetilde{p} \in \widetilde{S}$. Let $(u^1 \dots u^n)$ be the coordinates in $\widetilde{U}$. For any $\sigma \in G_{\widetilde{p}}$ set

$$a_{ij}(\sigma) = \left(\frac{\partial (u^i \circ \sigma)}{\partial u^j}\right)_{\widetilde{p}}$$

Let $n(\widetilde{p})$ be the order of $G_{\widetilde{p}}$. Define the functions

$$v^i = \frac{1}{n(\widetilde{p})} \sum_{\sigma \in G_{\widetilde{p}}} a_{ij}(\sigma^{-1})\, u^j \circ \sigma$$

We shall have

$$\left(\frac{\partial v^i}{\partial u^j}\right)_{\widetilde{p}} = \frac{1}{n(\widetilde{p})} \sum_{\sigma \in G_{\widetilde{p}}} \sum_k a_{ik}(\sigma^{-1})\, a_{kj}(\sigma)$$

The product of the two matrices $(a_{ij}(\sigma^{-1}))$ and $(a_{ij}(\sigma))$ is the identity since the first one is the Jacobian matrix of the map $u \to \sigma^{-1}(u)$ at $\sigma^{-1}(\widetilde{p}) = \widetilde{p}$ and the second one is the Jacobian matrix of the map $u \to \sigma(u)$ at $\widetilde{p}$. Therefore, there exists an open neighborhood $\widetilde{V}_1$ of $\widetilde{p}$, $\widetilde{V}_1 \subset \widetilde{U}$, such that the functions $(v^1 \dots v^n)$ constitutes a coordinate system on $\widetilde{V}_1$. Let $\widetilde{V}$ be an open neighborhood of $\widetilde{p}$, $\widetilde{V} \subset \widetilde{V}_1$ such that $\sigma(\widetilde{V}) \subset \widetilde{V}_1$ for any $\sigma \in G_{\widetilde{p}}$ and that $\sigma(\widetilde{V}) \cap \widetilde{V} = \varnothing$ for any $\sigma \in G$ such that $\sigma \notin G_{\widetilde{p}}$. We want to know the expression of any $\tau \in G_{\widetilde{p}}$ on $\widetilde{V}$ in the coordinates $(v^1 \dots v^n)$. We shall have

$$v^i \circ \tau = \frac{1}{n(\widetilde{p})} \sum_{\sigma \in G_{\widetilde{p}}} a_{ij}(\sigma^{-1})\, u^j \circ \sigma \circ \tau$$

Let us call $\sigma' = \sigma \circ \tau$. We shall have

$$v^i \circ \tau = \sum_k a_{ik}(\tau) \left(\frac{1}{n(\widetilde{p})} \sum_{\sigma' \in G_{\widetilde{p}}} a_{kj}(\sigma'^{-1})\, u^j \circ \sigma' \right) = \sum_k a_{ik}(\tau) v^k .$$

We then see that τ acts on $\widetilde{V}$ as a linear map whose matrix with respect to the coordinates v^i is $a_{ij}(\tau)$. Hence, if $\tau \neq I$ then the fixed points of τ on V will constitute a linear submanifold of dimension $< n$. Let $\widetilde{X}$ be the union of such submanifolds corresponding to any $\tau \in G_{\widetilde{p}}$, $\tau \neq I$. (Since $\widetilde{p} \in \widetilde{S}$, there exists $\tau \in G_{\widetilde{p}}$, $\tau \neq I$). Let $\widetilde{q}$ be any point of $\widetilde{V}$. Let us prove that $G_{\widetilde{q}} = \{\alpha \in G_{\widetilde{p}}$ such that $\alpha(\widetilde{q}) = \widetilde{q}\}$. The inclusion $\supset$ is obvious. Let us prove the inclusion $\subset$. Let $\alpha \in G_{\widetilde{p}}$. We shall have $\alpha(\widetilde{V}) \cap \widetilde{V} \neq \varnothing$. But from the way that we have chosen $\widetilde{V}$ it follows that $\alpha \in G_{\widetilde{p}}$.

Let us prove $\widetilde{X} = \widetilde{S} \cap \widetilde{V}$. Let $\widetilde{q} \in \widetilde{S} \cap \widetilde{V}$. Since $G_{\widetilde{q}} \neq \{I\}$ there exists $\tau \neq I$ such that $\tau(\widetilde{q}) = \widetilde{q}$. Hence $\widetilde{q} \in \widetilde{X}$. Let $\widetilde{q} \in \widetilde{V}$, $\widetilde{q} \notin \widetilde{S}$. Since $G_{\widetilde{q}} = \{I\}$ there is not $\tau \in G_{\widetilde{p}}$, $\tau \neq I$, such that $\tau(\widetilde{q}) = \widetilde{q}$. Then $\widetilde{q} \notin \widetilde{X}$. Hence $\widetilde{X} = \widetilde{S} \cap \widetilde{V}$.

Proposition 1.2. Let $\{\widetilde{U}, G, \varphi\}$, $\{\widetilde{U}', G', \varphi'\} \in \mathcal{A}$ *with* $\varphi(\widetilde{U}) \subset \varphi'(\widetilde{U}')$. *Let* λ *and* μ *be two injections* $\{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$. *There exists a unique* $\sigma_1' \in G'$ *such that* $\mu = \sigma_1' \circ \lambda$.

In order to prove this proposition we need the following

Lemma. Let G be a finite group of linear automorphisms of $\mathbf{R}^n$, $G \neq \{I\}$. *We suppose that there exists a* $(n-1)$*-dimensional vector subspace V of* $\mathbf{R}^n$ *such that each point of V is fixed by G. Then G consists of two elements, the identity and a symmetry with respect to V.*

Proof of lemma. Take a basis $e_1 \dots e_n$ of $\mathbf{R}^n$ such that $e_1 \dots e_{n-1}$ constitutes a basis of V. The matrices of the elements of G in this basis have the following expression:

$$\begin{pmatrix} 1 \; 0 \dots\dots 0 \; a^1 \\ 0 \; 1 \dots\dots 0 \; a^2 \\ \dots\dots\dots\dots\dots\dots \\ 0 \; 0 \dots\dots 1 \; a^{n-1} \\ 0 \; 0 \dots\dots 0 \; a^n \end{pmatrix}$$

We shall abbreviate this matrix as $g(a^1 \dots a^n)$. We shall have the following composition law:

$$g(a^1 \dots a^n) \circ g(b^1 \dots b^n) = g(b^1 + a^1 b^n, \dots, b^{n-1} + a^{n-1} b^n, a^n b^n)$$

Observe that a^n must be ± 1. Otherwise, since for any positive integer r we have $g(a^1 \dots a^n)^r = g(\dots, (a^n)^r)$, G would not be finite. Observe that $g(a^1 \dots a^n)$ with $a^n = 1$ must be the identity. Otherwise, suppose that $a^i \neq 0$ for some i, $1 \leqslant i \leqslant n-1$. Since $g(a^1 \dots a^{n-1}, 1)^r = g(ra^1, \dots, ra^{n-1}, 1)$, G is not finite. Observe that G only have a unique element $g(a^1 \dots a^{n-1}, -1)$. Otherwise, let $g(b^1 \dots b^{n-1}, -1)$ be another element of this form. We have $g(a^1 \dots a^{n-1}, -1) \circ g(b^1 \dots b^{n-1}, -1) = g(b^1 - a^1, \dots, b^{n-1} - a^{n-1}, 1)$. But we have seen that an element of this form had to be the identity. Hence $a^i = b^i$. This concludes the proof of lemma.

Proof of proposition. Since $\lambda(\widetilde{U})$ is an open set, by virtue of proposition 1.1 we can pick $\widetilde{p}_1 \in \widetilde{U}$ such that $G'_{\lambda(\widetilde{p}_1)} = \{I\}$. We shall have $\varphi'(\mu(\widetilde{p}_1)) = \varphi(\widetilde{p}_1) = \varphi'(\lambda(\widetilde{p}_1))$. Hence there exists $\sigma'_1 \in G'$ such that $\mu(\widetilde{p}_1) = \sigma'_1(\lambda(\widetilde{p}_1))$. Let $S = \{\widetilde{p} \in \widetilde{U}$ such that $G'_{\lambda(\widetilde{p})} \neq \{I\}\}$. Let $C = \{\widetilde{p} \in \widetilde{U} - S$ such that $\mu(\widetilde{p}) = \sigma'_1(\lambda(\widetilde{p}))\}$.

It is clear that $\widetilde{p}_1 \in C$. Hence $C \neq \varnothing$.

Step 1. C is open and closed in $\widetilde{U} - S$.

Given $\widetilde{p} \in C$ we wish to show that $\widetilde{p}$ is interior. For each $\sigma' \in G'$ pick an open neighborhood $V'_{\sigma'}$ of $\sigma'(\lambda(\widetilde{p}))$ in $\widetilde{U}'$ such that

(a) $V'_{\sigma'} \cap V'_{\tau'} = \varnothing$ if $\sigma' \neq \tau'$ (since $G_{\lambda(\widetilde{p})} = \{I\}$, $\sigma'\lambda(\widetilde{p}) \neq \tau'\lambda(\widetilde{p})$ if $\sigma' \neq \tau'$).

(b) $\sigma'(V'_I) \subset V'_{\sigma'}$ for any $\sigma' \in G'$.

Let $V_{\widetilde{p}}$ be a neighborhood of $\widetilde{p}$ in $\widetilde{U} - S$ such that $\lambda(V_{\widetilde{p}}) \subset V'_I$ and that $\mu(V_{\widetilde{p}}) \subset V'_{\sigma'_1}$. We wish to show that $V_{\widetilde{p}} \subset C$. Let $\widetilde{q} \in V_{\widetilde{p}}$. Since $\varphi'(\mu(\widetilde{q})) = \varphi(\widetilde{q}) = \varphi'(\lambda(\widetilde{q}))$ there exists $\sigma'_2 \in G'$ such that $\mu(\widetilde{q}) = \sigma'_2 \lambda(\widetilde{q})$. If $\sigma'_1 = \sigma'_2$ then $\widetilde{q} \in C$. Let us suppose that $\sigma'_1 \neq \sigma'_2$. We shall have $\mu(\widetilde{q}) = \sigma'_2\lambda(\widetilde{q}) \in \sigma'_2(\lambda(V_{\widetilde{p}})) \subset \sigma'_2(V'_I) \subset V'_{\sigma_2}$. But on the other hand $\mu(\widetilde{q}) \in \mu(V_{\widetilde{p}}) \subset V'_{\sigma'_1}$. Hence $\mu(\widetilde{q}) \in V'_{\sigma'_1} \cap V'_{\sigma'_2}$. But if $\sigma'_1 \neq \sigma'_2$ then $V'_{\sigma'_1} \cap V'_{\sigma'_2} = \varnothing$ (condition (a)). This is a contradiction. Hence $V_{\widetilde{p}} \subset C$. Hence C is an open subset in $\widetilde{U} - S$. On the other hand, from the identity $\mu(\widetilde{q}) = \sigma'_1\lambda(\widetilde{q})$ one deduces by continuity that C is closed.

STEP 2. $C = \widetilde{U} - S$.

If $\widetilde{U} - S$ is a connected set this equality is then obvious. Let us suppose that $\widetilde{U} - S$ is not connected. Since C is open and closed in $\widetilde{U} - S$, C will be a union of connected components of $\widetilde{U} - S$. Denote by A_1 the connected component of $\widetilde{U} - S$ that contains $\widetilde{p}_1$. $A_1 \subset C$, hence $\mu(\widetilde{q}) = \sigma'_1\lambda(\widetilde{q})$ for any $\widetilde{q} \in A_1$. Since $\widetilde{U}$ is connected, the connected components of $\widetilde{U} - S$ are separated by S. Since we have supposed that $\widetilde{U} - S$ is not connected, $A_1 \neq \widetilde{U} - S$.

Let A_0 be another connected component of $\widetilde{U} - S$ such that the intersection of closures (in $\widetilde{U}$) $\bar{A}_1 \cap \bar{A}_0$ is not empty. Let $\widetilde{p}_0 \in \bar{A}_1 \cap \bar{A}_0$. We know that S is a union of submanifolds of dimension $< n$ in a small neighborhood of $\widetilde{p}_0$. Observe that $\widetilde{p}$ has to belong to one of these submaniflods of dimension $n - 1$, say S_{n-1}, and that S_{n-1} can be chosen in such a way that $S_{n-1} \cap \bar{A}_1$ contains, itself, a submanifold of dimension $n - 1$. (In fact, the submanifolds of dimension $\leqslant n - 2$ do not locally separate two distinct connected components). In this situation we can choose a point $\widetilde{p} \in S_{n-1} \cap \bar{A}_1$ such that there exists a neighborhood $W_{\widetilde{p}}$ of $\widetilde{p}$ with $W_{\widetilde{p}} \cap S = W_{\widetilde{p}} \cap S_{n-1}$. (Notice that $\widetilde{p}$ might be $\neq \widetilde{p}_0$). Then $W_{\widetilde{p}} - \bar{A}_1$ is clearly contained in a connected component of $\widetilde{U} - S$ (possibly $\neq A_0$), namely A_2. Let $\widetilde{p}_2 \in A_2$ with $G'_{\lambda(\widetilde{p}_2)} = \{I\}$. Since $\varphi'(\mu(\widetilde{p}_2)) = \varphi(\widetilde{p}_2) = \varphi'(\lambda(\widetilde{p}_2))$, there exists $\sigma'_2 \in G'$ such that $\mu(\widetilde{p}_2) = \sigma'_2\ (\lambda(\widetilde{p}_2))$. By applying the preceding reasoning to $\widetilde{p}_2$ and σ'_2 instead of $\widetilde{p}_1$ and σ'_1 we would arrive to the equality $\mu(\widetilde{q}) = \sigma'_2(\lambda(\widetilde{q}))$ for any $\widetilde{q} \in A_2$. Let us show that $\sigma'_1 = \sigma'_2$ necessarily. By continuity, $\mu(\widetilde{p}) = \sigma'_1\ (\lambda(\widetilde{p}))$ and $\mu(\widetilde{p}) = \sigma'_2\ (\lambda(\widetilde{p}))$.

Hence $\sigma_1'^{-1} \circ \sigma_2' \in G'_{\lambda(\tilde{p})}$. We can choose a neighborhood $V_{\tilde{p}} \subset W_{\tilde{p}}$ of $\tilde{p}$ in $\tilde{U}$ such that, on $V'_{\lambda(\tilde{p})} = \lambda(V_{\tilde{p}})$, there exists a coordinate system centered at $\lambda(\tilde{p})$ such that $S'_{n-1} = \lambda(S_{n-1} \cap V_{\tilde{p}})$ is a linear submanifold (in these coordinates). Set $V''_{\mu(\tilde{p})} = \mu(V_{\tilde{p}})$. By the choice of $\tilde{p}$ and $V_{\tilde{p}}$ we can assert that the subset of $V_{\tilde{p}}$ consisting of the fixed points by $G'_{\lambda(\tilde{p})}$ is, exactly, S'_{n-1}. Hence, since $\sigma_1'^{-1} \circ \sigma_2' \in G'_{\lambda(\tilde{p})}$, either $\sigma_1'^{-1} \circ \sigma_2' = I$ or (by lemma) there exists a symmetry s with respect to S'_{n-1} (with respect to the coordinates taken on $V'_{\lambda(\tilde{p})}$) such that $\sigma_2' = \sigma_1' \circ s$.

We know that $\mu(\tilde{q}) = \sigma_1'\lambda(\tilde{q})$ for any $\tilde{q} \in A_1$ and that $\mu(\tilde{q}) = \sigma_2'\lambda(\tilde{q})$ for any $\tilde{q} \in A_2$. This means that σ_1' is equal to $\mu \circ \lambda^{-1}$ on $\lambda(A_1)$ and that σ_2' is equal to $\mu \circ \lambda^{-1}$ on $\lambda(A_2)$. Hence σ_2' maps $\lambda(A_2)$ into $\mu(A_2)$ and $\lambda(A_2) \cap V'_{\lambda(\tilde{p})}$ into $\mu(A_2) \cap V''_{\mu(\tilde{p})}$. But, on the other hand, the symmetry s maps $\lambda(A_2) \cap V'_{\lambda(\tilde{p})}$ into $\lambda(A_1) \cap V'_{\lambda(\tilde{p})}$ and σ_1' maps $\lambda(A_1) \cap V'_{\lambda(\tilde{p})}$ into $\mu(A_1) \cap V''_{\mu(\tilde{p})}$. From this we deduce that $\sigma_2' = \sigma_1' \circ s$ maps $\lambda(A_2) \cap V'_{\lambda(\tilde{p})}$ into $\mu(A_1) \cap V''_{\mu(\tilde{p})}$. This is a contradiction. Hence $\sigma_2' = \sigma_1'$. Hence $A_2 \subset C$. If $A_1 \cup A_2 = \tilde{U} - S$ we shall have $C = \tilde{U} - S$ as we want. Otherwise, there will exist another connected component A_3 such that $\bar{A}_3 \cap S$ contains a submanifold of dimension $n - 1$ in common with $\bar{A}_1 \cap S$ (or $\bar{A}_2 \cap S$). By applying the preceding reasoning to A_3 instead of A_2 and, perhaps, to A_2 instead of A_1, we would prove $A_3 \subset C$. Hence we would arrive, by recurrence, to $C = \tilde{U} - S$.

Step 3. End of the proof.

We shall have $\mu(\tilde{q}) = \sigma_1'\lambda(\tilde{q})$ for any $\tilde{q} \in \tilde{U} - S$. We obtain, by continuity, $\mu = \sigma_1' \circ \lambda$ on $\tilde{U}$. This proves the existence of σ_1'. The uniqueness is obvious since it suffices to choose $\tilde{q}$ such that $G'_{\lambda(\tilde{q})} = \{I\}$ and observe that for this $\tilde{q}$ there exists a unique σ_1' such that $\mu(\tilde{q}) = \sigma_1'\lambda(\tilde{q})$.

Corollary. Let λ be an injection $\{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$. There exist a unique monomorphism $\eta : G \to G'$ such that $\lambda \circ \sigma = \eta(\sigma) \circ \lambda$ for any $\sigma \in G$.

Proof. Given $\sigma \in G$, $\lambda \circ \sigma$ is an injection. Hence, there exists a unique $\sigma' \in G'$ such that $\lambda \circ \sigma = \sigma' \circ \lambda$. The correspondence $\sigma \to \sigma'$ is our homomorphism.

Proposition 1.3. Let $\{\widetilde{U}, G, \varphi\}$ *and* $\{\widetilde{U}', G', \varphi'\} \in \mathcal{A}$ *with* $U = \varphi(\widetilde{U}) \subset \varphi'(\widetilde{U}') = U'$. *Let* λ *be an injection* $\{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$. *If* $\sigma'(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U}) \neq \varnothing$ *for some* $\sigma' \in G'$ *then* $\sigma'(\lambda(\widetilde{U})) = \lambda(\widetilde{U})$. *Moreover* $\sigma' \in \eta(G)$.

PROOF. Assume that $\sigma'(\lambda(U)) \cap \lambda(U) \neq \varnothing$. Then there exist $\widetilde{p}, \widetilde{q} \in \widetilde{U}$ such that $\sigma'\lambda(\widetilde{p}) = \lambda(\widetilde{q})$. Hence $\varphi(\widetilde{p}) = \varphi'(\lambda(\widetilde{p})) = \varphi'(\sigma'\lambda(\widetilde{p})) = \varphi'(\lambda(\widetilde{q})) = \varphi(\widetilde{q})$. Then, there exists $\tau \in G$ such that $\tau(\widetilde{p}) = \widetilde{q}$. Let $\tau' = \eta(\tau) \in G'$. By virtue of proposition 1.1, since $\sigma'(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U})$ is open, we can choose $\widetilde{p}$ such that $\lambda(\widetilde{p}) \in \sigma'(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U})$ and that $G'_{\lambda(\widetilde{p})} = \{I\}$. Hence we shall necessarily have $\sigma' = \tau' = \eta(\tau)$. Since $\tau(\widetilde{U}) = (\widetilde{U})$, we shall have $\sigma'(\lambda(\widetilde{U})) = \tau'(\lambda(\widetilde{U})) = \lambda(\tau(\widetilde{U})) = \lambda(\widetilde{U})$.

Remark. From proposition 1.3 it follows that any injection $\lambda: \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ with $\varphi(\widetilde{U}) = \varphi'(\widetilde{U}')$ gives rise to a diffeomorphism between $\widetilde{U}$ and $\widetilde{U}'$ and that the homomorphism $\eta : G \to G'$ induced by λ is an isomorphism. Then λ^{-1} is also an injection. In fact, if $\varphi(\widetilde{U}) = \varphi'(\widetilde{U}')$ one has $\widetilde{U}' = \bigcup_{\sigma' \in G'} \sigma'(\lambda(\widetilde{U}))$. But, by virtue of proposition 1.3, $\sigma'(\lambda(\widetilde{U})) = \lambda(\widetilde{U})$ for any σ', since $\widetilde{U}'$ would not be connected if this equality did not hold. Hence, $\widetilde{U}' = \bigcup_{\sigma' \in G'} \sigma'(\lambda(\widetilde{U})) = \lambda(\widetilde{U})$. This proves that λ is onto. Proposition 1.3 also implies that η is an epimorphism.

For the general case $\varphi(\widetilde{U}) \subset \varphi'(\widetilde{U}')$ we have the following

Proposition 1.4. Let λ *be an injection* $\{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$. *Given* $\sigma', \tau' \in G'$, *one of the two following assertions holds:*

(*a*) $\sigma'(\lambda(\widetilde{U})) \cap \tau'(\lambda(\widetilde{U})) = \varnothing$. *Then there is no* $\sigma'_1 \in \eta(G)$ *such that* $\sigma' = \tau' \circ \sigma'_1$.

(*b*) $\sigma'(\lambda(\widetilde{U})) = \tau'(\lambda(\widetilde{U}))$. *Then* $\tau'^{-1} \circ \sigma' \in \eta(G)$.

The proof is immediate from proposition 1.3.

Remark. From proposition 1.4 one follows that there exists a one-to-one correspondence between $G'/\eta(G) = \{\sigma'\eta(G)$ for any $\sigma' \in G'\}$ and the connected components of $V_\lambda = \bigcup_{\sigma' \in G'} \sigma'(\lambda(\widetilde{U}))$. If $\mu : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ is another injection and $\sigma' \in G'$ is the unique element such that $\mu = \sigma' \circ \lambda$ (prop. 1.2) then, either $\sigma' \in \eta(G)$ and $\mu(\widetilde{U}) = \lambda(\widetilde{U})$ or $\sigma' \notin \eta(G)$ and $\mu(\widetilde{U})$ is a connected component of V_λ distinct from $\lambda(\widetilde{U})$. In particular V_λ is independent of the choice of the injection λ and it depends only on $\{\widetilde{U}, G, \varphi\}$ and $\{\widetilde{U}', G', \varphi'\}$. *We define* $V(\widetilde{U}, \widetilde{U}') = V_\lambda$.

Proposition 1.5. Let $(B, \mathcal{A})$ *be a* V*-manifold and let* $p \in B$. *Let* $\{\widetilde{U}, G, \varphi\}$, $\{\widetilde{U}', G', \varphi'\} \in \mathcal{A}$ *such that there exist* $\widetilde{p} \in \widetilde{U}$ *and* $\widetilde{p}' \in \widetilde{U}'$ *with* $\varphi(\widetilde{p}) = \varphi'(\widetilde{p}') = p$. *Then* $G_{\widetilde{p}} = G_{\widetilde{p}'}$.

PROOF. Since we can always find $\{\widetilde{U}'', G'', \varphi''\} \in \mathcal{A}$ with $p \in \varphi''(\widetilde{U}'') \subset \varphi(\widetilde{U}) \cap \varphi'(\widetilde{U}')$ it suffices to prove the proposition in the case $\varphi(\widetilde{U}) \subset \varphi'(\widetilde{U}')$. We know that there exists an injection $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ such that $\lambda(\widetilde{p}) = \widetilde{p}'$. Let η be the homomorphism $G \to G'$ associate to λ. Let us prove that the restriction of η to $G_{\widetilde{p}}$ induces an isomorphism between $G_{\widetilde{p}}$ and $G'_{\widetilde{p}'}$. Let $\sigma \in G_{\widetilde{p}}$. We have $\eta(\sigma)\,(\widetilde{p}') = \eta(\sigma)\,(\lambda(\widetilde{p})) = \lambda\sigma(\widetilde{p}) = \lambda(\widetilde{p}) = \widetilde{p}'$. Hence $\eta(G_{\widetilde{p}}) \subset G'_{\widetilde{p}'}$. Reciprocally, if $\sigma' \in G'_{\widetilde{p}'}$ then $\widetilde{p}' = \sigma'(\widetilde{p}') = \sigma'(\lambda(\widetilde{p}))$, hence $\sigma'(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U}) \neq \varnothing$.

Therefore there exists $\sigma \in G$ such that $\eta(\sigma) = \sigma'$. Let us finally prove that $\sigma \in G_{\widetilde{p}}$. By construction $\lambda\sigma(\widetilde{p}) = \eta(\sigma)\,(\lambda(\widetilde{p})) = \widetilde{p}'$.

Hence $\lambda(\sigma(\widetilde{p})) = \lambda(\widetilde{p})$. Since λ is injective we shall necessarily have $\sigma(\widetilde{p}) = \widetilde{p}$.

Definition 1.5. By virtue of proposition 1.5 we can define the isotropy group of each $p \in B$ to be $G_{\widetilde{p}}$, where $\widetilde{p}$ is a point belonging to some $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ such that $\varphi(\widetilde{p}) = p$. A point $p \in B$ is called singular if its isotropy goup is non-trivial.

C^∞ MAPS BETWEEN V-MANIFOLDS. V-VECTOR BUNDLES.

Definition 1.6. Let $(B_1, \mathcal{A}_1)$ and $(B_2, \mathcal{A}_2)$ be two V-manifolds. We mean by a C^∞ map from B_1 into B_2 a system of mappings $\{h_{\widetilde{U}_1}\}$ for each $\{\widetilde{U}_1, G_1, \varphi\} \in \mathcal{A}_1$ satisfying the following conditions:

(i) There exists a correspondence $\{\widetilde{U}_1, G_1, \varphi_1\} \to \{\widetilde{U}_2, G_2, \varphi_2\}$ from $\mathcal{A}_1$ into $\mathcal{A}_2$ such that for any $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}_1$, $h_{\widetilde{U}_1}$ is a C^∞ map from $\widetilde{U}_1$ into $\widetilde{U}_2$.

(ii) Let $\{\widetilde{U}_1, G_1, \varphi\}$, $\{\widetilde{U}'_1, G'_1, \varphi'_1\} \in \mathcal{A}_1$ and $\{\widetilde{U}_2, G_2, \varphi_2\}$, $\{\widetilde{U}'_2, G'_2, \varphi'_2\} \in \mathcal{A}_2$ be corresponding *l.u.s's.* (in the sense of (i)). Then, given an injection $\lambda_1: \{\widetilde{U}_1, G_1, \varphi_1\} \to \{\widetilde{U}'_1, G'_1, \varphi'_1\}$ there exists an injection $\lambda_2: \{\widetilde{U}_2, G_2, \varphi_2\} \to \{\widetilde{U}'_2, G'_2 \varphi'_2\}$ such that $\lambda_2 \circ h_{\widetilde{U}_1} = h_{\widetilde{U}'_1} \circ \lambda_1$.

One can easily prove the following

Proposition 1.6. Given a C^∞ *map,* $\{h_{\widetilde{U}_1}\}$, *from* $(B_1, \mathcal{A}_1)$ *into* $(B_2, \mathcal{A}_2)$ *there exists a unique continuous map* $h: B_1 \to B_2$ *such that for any* $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}_1$ *and its corresponding* $\{\widetilde{U}_2, G_2, \varphi_2\} \in \mathcal{A}_2$ *one has* $\varphi_2 \circ h_{\widetilde{U}_1} = h \circ \varphi_1$.

Remark. We shall identify in the following a C^∞ map $\{h_{\widetilde{U}_1}\}$ to its associate $h: B_1 \to B_2$.

Definition 1.7. Consider $\mathbf{R}$ (resp. $\mathbf{C}$) endowed with the V-manifold structure defined by the single *l.u.s.* $\{\mathbf{R}, \{I_{\mathbf{R}}\}, I_{\mathbf{R}}\}$ (resp. $\{\mathbf{C}, \{I_{\mathbf{C}}\}, I_{\mathbf{C}}\}$). A real (resp. complex) C^∞ map on a V-manifold $(B, \mathcal{A})$ is a C^∞ map from B into $\mathbf{R}$ (resp. $\mathbf{C}$).

Definition 1.8. Let B and E be two V-manifolds and $\pi: E \to B$ a C^∞ map. We say that $\pi: E \to B$ is a C^∞ V-vector bundle with fibre $\mathbf{R}^m$ (resp. $\mathbf{C}^m$) if we can find defining families $\mathcal{A}$ and $\mathcal{A}^*$ of B and E respectively, satisfying the following conditions:

(i) There exists a one-to-one correspondence $\{\widetilde{U}, G, \varphi\} \longleftrightarrow \{\widetilde{U}^*, G^*, \varphi^*\}$ between $\mathcal{A}$ and $\mathcal{A}^*$ such that $\widetilde{U}^* = \widetilde{U} \times \mathbf{R}^m$ (resp. $\widetilde{U} \times \mathbf{C}^m$) and that $\pi \circ \varphi^* = \varphi \circ \pi_{\widetilde{U}^*}$ where $\pi_{\widetilde{U}^*}$ denotes the canonical projection $\pi_{\widetilde{U}^*}: \widetilde{U}^* \to \widetilde{U}$.

(ii) To each injection $\lambda: \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ it corresponds an injection $\lambda^*: \{\widetilde{U}^*, G^*, \varphi^*\} \to \{\widetilde{U}'^*, G'^*, \varphi'^*\}$ such that for any $(\widetilde{p}, q) \in \widetilde{U}^* = \widetilde{U} \times \mathbf{R}^m$ (resp. $\widetilde{U} \times C^m$) one has $\lambda^*(\widetilde{p}, q) = (\lambda(\widetilde{p}), g_\lambda(\widetilde{p})q)$ where $g_\lambda(\widetilde{p}) \in GL(m, \mathbf{R})$ (resp. $GL(m, \mathbf{C})$) and that the map $g_\lambda: \widetilde{U} \to GL(m, \mathbf{R}$ or $\mathbf{C})$ is C^∞.

If $\lambda: \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ and $\mu: \{\widetilde{U}', G', \varphi'\} \to \{\widetilde{U}'', G'', \varphi''\}$ are two injections, one has $g_{\mu \circ \lambda}(\widetilde{p}) = g_\mu(\lambda(\widetilde{p})) \circ g_\lambda(\widetilde{p})$ for any $\widetilde{p} \in \widetilde{U}$. In fact, $\mu^* \lambda^* (\widetilde{p}, q) = \mu^*(\lambda(\widetilde{p}), g_\lambda(\widetilde{p})\, q) = (\mu\lambda(\widetilde{p}), g_\mu(\lambda(\widetilde{p})) \circ g_\lambda(\widetilde{p})q)$.

Remark. The terminology used in Definition 1.8 is perhaps abusive. As we are going to see, if $\pi: E \to B$ is a V-vector bundle with fibre $\mathbf{R}^m$, it is not true that for each $p \in B$ the fibre $\pi^{-1}(p)$ is isomorphic to $\mathbf{R}^m$.

Let $p \in B$ and $\{\tilde{U}, G, \varphi\} \in \mathcal{A}$ such that there exists $\tilde{p} \in \tilde{U}$ satisfying $\varphi(\tilde{p}) = p$. Consider $\tilde{p} \times \mathbf{R}^m \subset \tilde{U}^*$. If $(\tilde{p}, q)$ belongs to $\tilde{p} \times \mathbf{R}^m$ then $\varphi^*(\tilde{p}, q) \in \pi^{-1}(p)$. If $(\tilde{p}, q_1)$ and $(\tilde{p}, q_2)$ satisfy $\varphi^*(\tilde{p}, q_1) = \varphi^*(\tilde{p}, q_2)$ there exists $\sigma^* \in G^*$ such that $\sigma^*(\tilde{p}, q_1) = (\tilde{p}, q_2)$. Using the correspondence $\sigma \longleftrightarrow \sigma^*$ established above we have $(\tilde{p}, q_2) = \sigma^*(\tilde{p}, q_1) = (\sigma(\tilde{p}), g_\sigma(\tilde{p})q_1)$. Hence $\sigma(\tilde{p}) = \tilde{p}$ (in other words, $\sigma \in G_{\tilde{p}}$) and $q_2 = g_\sigma(\tilde{p})\, q_1$. Then, the correspondence

$$\begin{array}{ll} \mathbf{R}^m/\{g_\sigma(\tilde{p}) \text{ with } \sigma \in G_{\tilde{p}}\} & \to \pi^{-1}(p) \\ \text{class of } q & \to \varphi^*(\tilde{p}, q) \end{array}$$

is a well defined one-to-one correspondence. Hence $\pi^{-1}(p) \simeq \mathbf{R}^m$ only if p is non-singular.

Definition 1.9. Let $\pi : E \to B$ be a V-vector bundle. We shall say that a C^∞ V-map $s : B \to E$ is a cross section of E if, with respect to the defining families $\mathcal{A}$ and $\mathcal{A}^*$ of Definition 1.8, s is given by a system $\{s_{\tilde{U}}\}$ satisfying the conditions of Definition 1.6 in such a way that the correspondence $\mathcal{A} \to \mathcal{A}^*$ of (i) Definition 1.6 is given by the correspondence $\mathcal{A} \longleftrightarrow \mathcal{A}^*$ of (i) of Definition 1.8 and that for any $\{\tilde{U}, G, \varphi\} \in \mathcal{A}$ one has $\pi_{\tilde{U}^*} \circ s_U = I$ (Clearly, this implies that $\pi \circ s = I$).

Each $s_{\tilde{U}}$ will be a cross section of $\tilde{U}^* \to U$ in the usual sense in such a way that for each injection $\lambda : \{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$ one has $s_{\tilde{U}'} \circ \lambda = \lambda^* \circ s_{\tilde{U}}$. In particular, each $s_{\tilde{U}}$ is G-invariant. That is, $s_{\tilde{U}} \circ \sigma = \sigma^* \circ s_{\tilde{U}}$ for any $\sigma \in G$.

Definition 1.10. Let $\pi : E \to B$ be a V-vector bundle with fibre $\mathbf{C}^m$. A Hermitian metric h in this V-vector bundle is an assignment of a C^∞ Hermitian metric $h_{\tilde{U}}$ in the vector bundle $\pi_{\tilde{U}^*} : \tilde{U}^* \to \tilde{U}$ to each $\{\tilde{U}, G, \varphi\} \in \mathcal{A}$ in such a way that for any injection $\lambda : \{\tilde{U}', G', \varphi'\} \to \{\tilde{U}, G, \varphi\}$ ($\{\tilde{U}', G', \varphi'\} \in \mathcal{A}$ and $\varphi'(\tilde{U}') \subset \varphi(\tilde{U})$) one has

$$h_{\tilde{U}'}(\tilde{x})\,(s_1(\tilde{x}), s_2(\tilde{x})) = h_{\tilde{U}}(\lambda(\tilde{x}))\,(\lambda^*(s_1(\tilde{x})), \lambda^*(s_2(\tilde{x})))$$

for any cross sections s_1, s_2 of $\tilde{U}'^* \to \tilde{U}'$ and $\tilde{x} \in \tilde{U}'$.

In an analogous way we can define the concept of a Riemannian metric on a V-manifold by the following

Definition 1.11. A Riemannian metric g on B is an assignment of a C^∞ Riemannian metric $g_{\widetilde{U}}$ on $\widetilde{U}$ to each $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$ in such a way that for any injection λ: $\{\widetilde{U}', G', \varphi'\} \to \{\widetilde{U}, G, \varphi\}$ ($\{\widetilde{U}', G', \varphi'\{ \in \mathcal{A}$ and $\varphi'(\widetilde{U}') \subset \varphi(\widetilde{U})$) one has $g_{\widetilde{U}'}(\widetilde{X}, \widetilde{Y}) = g_{\widetilde{U}}(\lambda(\widetilde{X}), \lambda(\widetilde{Y}))$ for any vector fields $\widetilde{X}, \widetilde{Y}$ on $\widetilde{U}'$, where $\lambda(\widetilde{X})$ and $\lambda(\widetilde{Y})$ mean the corresponding vector fields on $\lambda(\widetilde{U}')$ by means of the diffeomorphism $\widetilde{U}' \to \lambda(\widetilde{U}') \subset \widetilde{U}$.

Chapter 2

C^∞ PARTITIONS OF UNITY ON V-MANIFOLDS

Existence of C^∞ partitions of unity on V-manifolds

Definition 2.1. Let $(B, \mathcal{A})$ be a V-manifold. Let $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$. A function $h_{\widetilde{U}} : \widetilde{U} \to \mathbf{R}$ (or $\mathbf{C}$) is called allowable if it is a C^∞ function and it satisfies $h_{\widetilde{U}} = h_{\widetilde{U}} \circ \sigma$ for any $\sigma \in G$.

Proposition 2.1. Let $(B, \mathcal{A})$ be a V-manifold. Let $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ be an injection ($\{\widetilde{U}, G, \varphi\}$, $\{\widetilde{U}', G', \varphi'\} \in \mathcal{A}$). Let $h_{\widetilde{U}'}$ be an allowable function on $\widetilde{U}'$. Define the function $h_{\widetilde{U}}$ on $\widetilde{U}$ by $h_{\widetilde{U}} = h_{\widetilde{U}'} \circ \lambda$. Then, $h_{\widetilde{U}}$ does not depend on the injection λ and it is an allowable function.

Proof. Let us prove that it is allowable. We have $h_{\widetilde{U}} \circ \sigma = h_{\widetilde{U}'} \circ \lambda \circ \sigma = h_{\widetilde{U}'} \circ \eta(\sigma) \circ \lambda = h_{\widetilde{U}'} \circ \lambda = h_{\widetilde{U}}$. Let us prove that it does not depend on λ. Let μ be another injection μ: $\{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$. By virtue of proposition 1.2, there exists $\sigma' \in G'$ such that $\mu = \sigma' \circ \lambda$. Hence, $h_{\widetilde{U}} \circ \mu = h_{\widetilde{U}'} \circ \sigma' \circ \lambda = h_{\widetilde{U}'} \circ \lambda$.

Definition 2.2. In the situation of proposition above we shall say that $h_{\widetilde{U}}$ is the restriction of $h_{\widetilde{U}'}$ to $\{\widetilde{U}, G, \varphi\}$.

Proposition 2.2. Let $(B, \mathcal{A})$ be a V-manifold. Let $\{\widetilde{U}, G, \varphi\}$ and $\{\widetilde{U}', G', \varphi'\} \in \mathcal{A}$ such that $\varphi(\widetilde{U}) \subset \varphi'(\widetilde{U}')$. Let $h_{\widetilde{U}}$ be an allowable function on $\widetilde{U}$ with compact support $\widetilde{K} \subset \widetilde{U}$. Given an injection λ: $\{\widetilde{U}, G, \varphi\} \to$ $\to \{\widetilde{U}', G', \varphi'\}$ (call $\eta : G \to G'$ the associate homomorphism) and given representatives $\sigma'_1, \ldots, \sigma'_k$ of the classes of $G'/\eta(G)$ with $\sigma'_1 = I$, define the function $h_{\widetilde{U}'}$ on $\widetilde{U}'$ by

$$h_{\widetilde{U}'}(\widetilde{p}') \left\{ \begin{array}{l} = 0 \text{ if } \widetilde{p}' \notin V(\widetilde{U}, \widetilde{U}') \text{ (Recall that } V(\widetilde{U}, \widetilde{U}') \text{ has been defined} \\ \qquad \text{in the Remark following Prop. 1.4)} \\ = h_{\widetilde{U}}(\lambda^{-1}(\sigma_i'^{-1}(\widetilde{p}'))) \text{ if } \widetilde{p}' \in \sigma'_i(\lambda(\widetilde{U})) \end{array} \right.$$

Then, one has:

(a) $h_{\widetilde{U}'}$ is independent of the choice of the representatives $\sigma'_1, \ldots, \sigma'_k$ of the classes of $G'/\eta(G)$.

(b) $h_{\widetilde{U}'}$ is independent of the injection λ.

(c) $h_{\widetilde{U}'}$ is allowable and its support is contained in $\widetilde{K}' = \bigcup_{\sigma' \in G'} \sigma'(\lambda(\widetilde{K}))$.

PROOF. (a). Let τ'_i be another representative of the class of σ'_i. There exists $\sigma' \in \eta(G)$ such that $\tau'_i = \sigma'_i \circ \sigma'$. Let $\sigma \in G$ such that $\eta(\sigma) = \sigma'$. We have $\tau'_i(\lambda(\widetilde{U})) = \sigma'_i(\sigma'(\lambda(\widetilde{U}))) = \sigma'_i(\lambda(\widetilde{U}))$. Fix $\widetilde{p}'$ in $\tau'_i(\lambda(\widetilde{U}))$. Let $\widetilde{p}'_1 = \sigma_i'^{-1}(\widetilde{p}') \in \lambda(\widetilde{U})$ and $\widetilde{p}'_2 = \tau_i'^{-1}(\widetilde{p}') \in \lambda(\widetilde{U})$. Let $\widetilde{p}_1 = \lambda^{-1}(\widetilde{p}'_1)$ and $\widetilde{p}_2 = \lambda^{-1}(\widetilde{p}'_2)$. Since $\sigma'(\widetilde{p}'_2) = \sigma_i'^{-1}(\tau'_i(\widetilde{p}'_2)) =$ $= \sigma_i'^{-1}(\widetilde{p}') = \widetilde{p}'_1$, we have $\sigma(\widetilde{p}_2) = \widetilde{p}_1$. From this fact we deduce $h_{\widetilde{U}}(\lambda^{-1}(\tau_i'^{-1}(\widetilde{p}'))) = h_{\widetilde{U}}(\lambda^{-1}(\widetilde{p}'_2)) = h_{\widetilde{U}}(\widetilde{p}_2) = h_{\widetilde{U}}(\sigma^{-1}(\widetilde{p}_1)) = h_{\widetilde{U}}(\widetilde{p}_1) =$ $= h_{\widetilde{U}}(\lambda^{-1}(\widetilde{p}'_1)) = h_{\widetilde{U}}(\lambda^{-1}(\sigma_i'^{-1}(\widetilde{p}')))$.

(b) and (c) are proven in an analogous way with the help of Proposition 1.2.

Definition 2.3. In the situation of proposition above we shall say that $h_{\widetilde{U}'}$ is the extension of $h_{\widetilde{U}}$ to $\{\widetilde{U}', G', \varphi'\}$.

Proposition 2.3. Let $(B, \mathcal{A})$ be a V-manifold. Let $\{\widetilde{U}', G', \varphi'\}$ and $\{\widetilde{U}, G, \varphi\} \in \mathcal{A}$. Set $U = \varphi(\widetilde{U})$ and $U' = \varphi'(\widetilde{U}')$. Suppose that $U \cap U' \neq$ $\neq \varnothing$, $U \subsetneq U'$, $U' \subsetneq U$. Let $h_{\widetilde{U}'}$ be an allowable function on $\widetilde{U}'$ with compact support $\widetilde{K}' \subset \widetilde{U}'$. Then, there exists a unique allowable function $h_{\widetilde{U}}$ on $\widetilde{U}$ such that:

(*a*) $h_{\widetilde{U}} = 0$ *on* $\varphi^{-1}(U - U')$.

(*b*) *Given a l.u.s.* $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}$ *corresponding to an open set* $U_1 \subset U' \cap U$ *and an injection* λ: $\{\widetilde{U}_1, G_1, \varphi_1\} \to \{\widetilde{U}, G, \varphi\}$, *then, if we denote by* $h_{\widetilde{U}_1}$ *the restriction of* $h_{\widetilde{U}'}$ *to* $\{\widetilde{U}_1, G_1, \varphi_1\}$, *we have* $h_{\widetilde{U}_1} = h_{\widetilde{U}} \circ \lambda$.

PROOF. Fix a l.u.s. $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}$ corresponding to an open set $U_1 \subset U' \cap U$. Fix an injection λ: $\{\widetilde{U}_1, G_1, \varphi_1\} \to \{\widetilde{U}, G, \varphi\}$. Denote by $\eta : G_1 \to G$ the morphism associate to λ. Fix $\sigma_1 \dots \sigma_k \in G$ representatives of the elements of $G/\eta(G_1)$. Define a function $h(\widetilde{U}_1, \widetilde{U})$ on $V(\widetilde{U}_1, \widetilde{U}) = \bigcup_{i=1}^{k} \sigma_i(\lambda(\widetilde{U}_1))$ by $h(\widetilde{U}_1, \widetilde{U})(\widetilde{p}) = h_{\widetilde{U}_1}\ (\lambda^{-1}(\sigma_i^{-1}(\widetilde{p})))$ if $\widetilde{p} \in \sigma_i(\lambda(\widetilde{U}_1))$, where $h_{\widetilde{U}_1}$ denotes the restriction of $h_{\widetilde{U}'}$ to $\{\widetilde{U}_1, G_1, \varphi_1\}$. By the same reasoning that in proposition above this definition depends neither on the choice of the representatives $\sigma_1 \dots \sigma_k$ nor on the injection λ. Let $\{\widetilde{U}_2, G_2, \varphi_2\} \in \mathcal{A}$ such that $\varphi_2(\widetilde{U}_2) = U_2 \subset U_1 \subset \subset U' \cap U$. We shall have, in the same way, the function $h(\widetilde{U}_2, \widetilde{U})$ on $V(\widetilde{U}_2, \widetilde{U}) \subset V(\widetilde{U}_1, \widetilde{U})$. We want to show that $h(\widetilde{U}_2, \widetilde{U}) = h(\widetilde{U}_1, \widetilde{U})$ on $V(\widetilde{U}_2, \widetilde{U})$. Let μ be an injection $\{\widetilde{U}_2, G_2, \varphi_2\} \to \{\widetilde{U}_1, G_1, \varphi_1\}$. Let $\eta_2 : G_2 \to G_1$ be its associate homomorphism. Let $\tau_1 \dots \tau_r \in G_1$ representatives of the elements of $G_1/\eta_2(G_2)$. Then, the products $\sigma_i \circ \eta(\tau_j)$, $i = 1 \dots k, j = 1 \dots r$ are representatives of the elements of $G/\eta\eta_2(G_2)$. Since the definition of $h(\widetilde{U}_2, \widetilde{U})$ does not depend on the choice of the injection we can take the injection $\lambda \circ \mu$. We shall have

$$h(\widetilde{U}_2, \widetilde{U})\ (\widetilde{p}) = h_{\widetilde{U}_2}(\mu^{-1}(\lambda^{-1}(\eta(\tau_j)^{-1}(\sigma_i^{-1}(\widetilde{p}))))) \text{ if } \widetilde{p} \in \sigma_i(\eta(\tau_j)\ (\lambda(\mu(\widetilde{U}_2)))).$$

Observe that the restriction of $h_{\widetilde{U}'}$ to $\{\widetilde{U}_2, G_2, \varphi_2\}$ is precisely the restriction of $h_{\widetilde{U}_1}$ to $\{\widetilde{U}_2, G_2, \varphi_2\}$. In other words, $h_{\widetilde{U}_2} = h_{\widetilde{U}_1} \circ \mu$. Hence, $h(\widetilde{U}_2, \widetilde{U})\ (\widetilde{p}) = h_{\widetilde{U}_1}(\lambda^{-1}(\eta(\tau_j)^{-1}(\sigma_i^{-1}(\widetilde{p})))) = h_{\widetilde{U}_1}((\lambda \circ \tau_j)^{-1}\ (\sigma_i^{-1}(\widetilde{p})))$ if $\widetilde{p} \in \sigma_i(\eta(\tau_j)\ (\lambda(\mu(\widetilde{U}_2))) \subset \sigma_i(\lambda(\mu(\widetilde{U}_2)))$ since $\eta(\tau_j) \in \eta(G_1)$. Since the definition of $h(\widetilde{U}_1, \widetilde{U})$ is independent of the injection, we can take the injection $\lambda \circ \tau_j$ and we shall have:

$$h(\widetilde{U}_1, \widetilde{U})\ (\widetilde{p}) = h_{\widetilde{U}_1}((\lambda \circ \tau_j)^{-1}(\sigma_i^{-1}(\widetilde{p}))) \text{ if } \widetilde{p} \in \sigma_i(\lambda(\widetilde{U}_2)).$$

Then, we can see that $h(\widetilde{U}_2, \widetilde{U}) = h(\widetilde{U}_1, \widetilde{U})$ on $\sigma_i(\eta(\tau_j)\ (\lambda(\mu(\widetilde{U}_2))))$ $\subset \sigma_i(\lambda(\mu(\widetilde{U}_2))) \subset \sigma_i(\lambda(\widetilde{U}_1))$.

Let us define the function $h_{\widetilde{U}}$ whose existence the proposition asserts. Define $h_{\widetilde{U}}$ on $\varphi^{-1}(U \cap U')$ as follows. If $\widetilde{p} \in \varphi^{-1}(U \cap U')$, choose a l.u.s. $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}$ corresponding to $U_1 \subset U \cap U'$ such that $\varphi(\widetilde{p}) \in U_1$. Define $h_{\widetilde{U}}(\widetilde{p}) = h(\widetilde{U}_1, \widetilde{U})(\widetilde{p})$. This definition does not depend on the choice of $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}$. In fact, if $\{\widetilde{U}_2, G_2, \varphi_2\} \in \mathcal{A}$ is a l.u.s. corresponding to $U_2 \subset U \cap U'$ such that $\varphi(\widetilde{p}) \in U_2$ we take $\{\widetilde{U}_3, G_3, \varphi_3\} \in \mathcal{A}$ a l.u.s. corresponding to $U_3 \subset U_1 \cap U_2$. We shall have $h(\widetilde{U}_1, \widetilde{U})(\widetilde{p}) = h(\widetilde{U}_3, \widetilde{U})(\widetilde{p}) = h(\widetilde{U}_2, \widetilde{U})(\widetilde{p})$. Define $h_{\widetilde{U}}$ to be zero on $\varphi^{-1}(U - U')$.

Let us show that $h_{\widetilde{U}}$ is allowable. Let $\sigma \in G$ and $\widetilde{p} \in \widetilde{U}$. If $\widetilde{p} \notin \varphi^{-1}(U \cap U')$ then $\sigma(\widetilde{p}) \notin \varphi^{-1}(U - U')$ for any $\sigma \in G$ and $h_{\widetilde{U}}(\sigma(\widetilde{p})) = 0 = h_{\widetilde{U}}(\widetilde{p})$. Suppose that $\widetilde{p} \in \varphi^{-1}(U \cap U')$. Take a l.u.s. $\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{A}$ corresponding to an open set U_1 such that $\varphi(\widetilde{p}) \in U_1$. Let λ: $\{\widetilde{U}_1, G_1, \varphi_1\} \to \{\widetilde{U}, G, \varphi\}$ be an injection. Since $\widetilde{p} \in \lambda(\widetilde{U}_1)$ then $\sigma(\widetilde{p}) \in \sigma\lambda(\widetilde{U}_1)$, hence $h_{\widetilde{U}}(\sigma(\widetilde{p})) = h_{\widetilde{U}_1}(\lambda^{-1}(\sigma^{-1}(\widetilde{p})))$. On the other hand $h_{\widetilde{U}}(\widetilde{p}) = h_{\widetilde{U}_1}((\sigma \circ \lambda)^{-1}(\widetilde{p}))$ since $\widetilde{p} \in \lambda(\widetilde{U})$. (We have taken here the injection $\sigma \circ \lambda$ instead of λ). We have then $h_{\widetilde{U}} \circ \sigma = h_{\widetilde{U}}$. In order to see that $h_{\widetilde{U}}$ is C^∞ it suffices to observe:

(a) On $\varphi^{-1}(U \cap U')$ $h_{\widetilde{U}}$ is C^∞ by construction.

(b) $\widetilde{K} = \varphi^{-1}(\varphi'(\widetilde{K}'))$ is contained in $\varphi^{-1}(U \cap U')$ and it is closed in $\widetilde{U}$.

(c) On $\widetilde{U} - \widetilde{K}$ one has $h_{\widetilde{U}} = 0$.

(d) The open sets $\widetilde{U} - \widetilde{K}$ and $\varphi^{-1}(U \cap U')$ cover U.

The uniqueness is immediate.

Definition 2.4. In the situation of proposition above we shall say that the function $h_{\widetilde{U}}$ is the prolongation of $h_{\widetilde{U}'}$ to $\{\widetilde{U}, G, \varphi\}$.

Proposition 2.4. Let $(B, \mathcal{A})$ be a V-manifold. Let $\{\widetilde{U}_0, G_0, \varphi_0\} \in \mathcal{A}$. Given an allowable function $f_{\widetilde{U}_0} : \widetilde{U}_0 \to \mathbf{R}$ (or $\mathbf{C}$) with compact support, there exists a unique C^∞ function f on B, $f = \{f_{\widetilde{U}}\}$, with compact support contained in $U_0 = \varphi_0(\widetilde{U}_0)$ such that $f \circ \varphi_0 = f_{\widetilde{U}_0}$.

PROOF. If $\varphi(\widetilde{U}) \subset U_0$ we define $f_{\widetilde{U}}$ to be the restriction of $f_{\widetilde{U}_0}$. If $U_0 \subset \varphi(\widetilde{U})$ we define $f_{\widetilde{U}}$ by extension of $f_{\widetilde{U}_0}$. If $U_0 \subsetneq \varphi(U)$, $\varphi(\widetilde{U}) \subsetneq U_0$ and $\varphi(\widetilde{U}) \cap U_0 \neq \varnothing$, we define $f_{\widetilde{U}}$ to be the prolongation of $f_{\widetilde{U}_0}$. If

$\varphi(\widetilde{U}) \cap U_0 = \varnothing$ we define $f_{\widetilde{U}} = 0$. The function $f = \{f_{\widetilde{U}}\}$ satisfies the required conditions.

Proposition 2.5. Let $(B, \mathcal{A})$ *a V-manifold. Given* $p \in B$ *and an open neighborhood* U' *of* p, *there exists a* C^∞ *real function* $f = \{f_{\widetilde{U}}\}$ *with* $0 \leqslant f_{\widetilde{U}} \leqslant 1$ *such that the support of* f *is a compact contained in* U' *and that there exists a compact neighborhood* K *of* p *such that* $f|K = 1$.

Proof. Let $\{\widetilde{U}_0, G_0, \varphi_0\}$ be a l.u.s. of $\mathcal{A}$ corresponding to U_0 such that $p \in U_0$. Let $\widetilde{p} \in U_0$ such that $\varphi_0(\widetilde{p}) = p$. Let $\{\widetilde{p}_1 = \widetilde{p}, \widetilde{p}_2, ..., \widetilde{p}_r\} = \{\sigma(\widetilde{p})$ with $\sigma \in G_0\}$. Let $\sigma_i \in G_0$, $i = 1... r$, such that $\widetilde{p}_i = \sigma_i(\widetilde{p})$. Let $V'_{\widetilde{p}}$ be an open neighborhood of $\widetilde{p}$ such that $\varphi_0(V'_{\widetilde{p}}) \subset U'$ and that the sets $V'_{\widetilde{p}_i} = \sigma_i(V'_{\widetilde{p}})$ are mutually disjoint. Consider $V_{\widetilde{p}} = \bigcap_{\sigma \in (G_0)_{\widetilde{p}}} \sigma(V'_{\widetilde{p}})$. Let K_0 be a compact neighborhood of $\widetilde{p}$ contained in $V_{\widetilde{p}}$. Let $f : \widetilde{U}_0 \to \mathbf{R}$ be a C^∞ function with $f(\widetilde{U}_0) \subset [0, 1]$ such that $f|K_0 = 1$ and that sup $f \subset V_{\widetilde{p}}$. Define

$$f_{\widetilde{U}_0} = \frac{1}{n(\widetilde{p})} \sum_{\sigma \in G_0} f \circ \sigma,$$

where $n(\widetilde{p})$ is the order of $(G_0)_{\widetilde{p}}$. It is clear that $f_{\widetilde{U}_0}$ is an allowable function with $f_{\widetilde{U}_0}(\widetilde{U}_0) \subset [0, 1]$. Let $K' = \bigcap_{\sigma \in (G_0)_{\widetilde{p}}} \sigma(K_0)$. K' is a compact neighborhood of $\widetilde{p}$. Moreover $f_{\widetilde{U}_0}|V'_{\widetilde{p}}$ has compact support contained in $V'_{\widetilde{p}}$ and one has $f_{\widetilde{U}_0}|K' = 1$. Take $K = \varphi_0(K')$ as the compact whose existence the proposition assetrs. Since φ_0 is continuous and open, K is a compact neighborhood of p. Take as $f = \{f_{\widetilde{U}}\}$ whose existence the proposition asserts the function f given by proposition 2.4. It is clear that all the required conditions are fulfilled.

From proposition 2.5 a standard well known reasoning gives the following

Theorem 2.1. Let B *be a V-manifold. Let* $\{U_\alpha\}$ *be an open cover of* B. *There exits a countable partition of unity* $\{h_i, i \in \mathbf{N}\}$ *subordinate to* $\{U_\alpha\}$ *such that each* h_i *is a* C^∞ *function with compact support.*

Corollary. Let K *be a compact in* B. *Let* U *be an open set such that* $K \subset U$. *There exists a* C^∞ *function* $f\colon B \to \mathbf{R}$ *such that* $f|K = 1$ *and* $f(x) = 0$ *for any* $x \notin U$.

PROOF. For each $p \in K$ let U_p be an open neighborhood of p such that $U_p \subset U$. If $p \notin K$ let U_p be an open neighborhood of p such that $U_p \cap K = \varnothing$. Let $\{U_\alpha\}_{\alpha \in A}$ be a locally finite open refinement of the cover $\{U_p\}_{p \in B}$. Let $\{h_i\}_{i \in \mathbf{N}}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. Let $C = \{i \in \mathbf{N}$ such that sup $h_i \cap K \neq \varnothing$. C is a finite set. The function $f = \sum_{i \in C} h_i$ satisfies all the required conditions.

INTEGRATION OVER V-MANIFOLDS

We shall say that a V-manifold B is oriented if we have chosen a defining family $\mathcal{A}$ such that if $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\} \in \mathcal{A}$ with $\varphi(\tilde{U}) \subset \varphi'(\tilde{U}')$, any injection λ: $\{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$ preserves orientations (we consider in $\tilde{U}$ and $\tilde{U}'$ the orientations induced by the canonical orientation in $\mathbf{R}^n$).

An r-differential form on B is an assignment of a C^∞ r-form $\omega_{\tilde{U}}$ on $\tilde{U}$ to each $\{\tilde{U}, G, \varphi\} \in \mathcal{A}$ in such a way that, if $\lambda : \{\tilde{U}, G, \varphi\} \to \{U', G', \varphi'\}$ is an injection, one has $\omega_{\tilde{U}} = \lambda^* (\omega_{\tilde{U}'})$.

Let $(B, \mathcal{A})$ be an oriented V-manifold of dimension n. Let ω be an n-form on B with compact support. Let us go to define the integral $\int_B \omega$. If the support of ω is contained in $U = \varphi(\tilde{U})$ with $\{\tilde{U}, G, \varphi\} \in \mathcal{A}$ we define

$$\int_B \omega = \frac{1}{n(G)} \int_{\tilde{U}} \omega_{\tilde{U}} ,$$

where $n(G)$ is the order of G. In the general case, let $\{f_i\}$ be a C^∞ partition of unity subordinate to a locally finite covering $\{U_i\}$ of B. We define

$$\int_B \omega = \sum_i \int_B f_i \, \omega .$$

This sum has only a finite number of non-vanishing terms. It can be proven that this definition does not depend on the particular partition of unity.