UPPER AND LOWER BOUNDS FOR QUADRATIC FUNCTIONALS (*)

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SUMMARY. The paper gives a systematic procedure for obtaining monotone sequences of upper and lower bounds for quadratic integrals such as those encountered in torsional rigidity, capacity, and other physical quantities. The monotone sequences are obtained from Bessel's inequality, while maximum and minimum principles for the solutions of the boundary value problems are obtained from Schwarz' inequality.

1. **Introduction.** In many problems of mathematical physics, it is desired to find the numerical value of a quadratic integral of an unknown function, where the unknown function is a solution of a certain boundary value problem consisting of a linear partial differential equation plus linear boundary conditions. The quadratic integral in question is usually the quadratic form associated with a bilinear integral which occurs in a Green's identity for the boundary value problem.

For example, consider the determination of the capacity of a « ring shaped » plane domain D bounded externally by a smooth simple closed curve C_0 and internally by another simple closed curve C_1. Here it is desired to evaluate the integral

$$\int_D (v \;+\; v_y^2)\; dx\, dy,$$

where the unknown function v is the solution of the Dirichlet problem :

$$\Delta v = 0, \quad \text{on} \quad D,$$

$$v = 1, \quad \text{on} \quad C_1 ; \qquad v = 0, \quad \text{on} \quad C_0,$$

and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

(*) This work was sponsored by the Office of Naval Research is the Laplacian. Still another example is the determination of the torsional rigidity, or stiffness, S, of a bounded plane domain D whose boundary is C. Consider the formula (Lamé's constant μ is taken to be unity, and P is the polar moment of inertia of D with respect to its centroid)

$$S = P - \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

which has been used by J. B. Diaz and A. Weinstein [2] (1), and is valid for simply or multiply connected D. In this formula, v is the warping function, and the evaluation of the torsional rigidity S is seen to be equivalent to the evaluation of the integral

$$\int_D (v_x^2 + v_y^2)\, dx\, dy,$$

where the warping function v is *a* single-valued solution of the Neumann problem

$$\Delta v = 0, \quad \text{on} \quad D,$$

$$\frac{\partial v}{\partial n} = \frac{\partial}{\partial s}\left[\frac{1}{2}(x^2 + y^2)\right], \quad \text{on} \quad C,$$

and $\dfrac{\partial}{\partial n}$, $\dfrac{\partial}{\partial s}$ denote differentiation in the direction of the outer normal to C, and along C respectively. In both these examples, the bilinear integral associated with the quadratic integral to be evaluated is the Dirichlet integral

$$\int_D (\varphi_x \psi_x + \varphi_y \psi_y)\, dx\, dy,$$

which occurs in Green's identity

$$\int_D \varphi \Delta \psi\, dx\, dy + \int_D (\varphi_x \psi_x + \varphi_y \psi_y)\, dx\, dy = \int_D \varphi \frac{\partial \psi}{\partial n}\, ds.$$

In many instances, known maximum and minimum principles for a solution of the boundary value problem can be used to obtain upper and lower bounds for the desired quadratic integral. However, for numerical purposes it is important to be able to obtain explicitly,

(1) Numbers in square brackets refer to the bibliography at the end of the paper.

in a systematic manner, a sequence of monotonically non-increasing upper bounds as well as a sequence of monotonically non-decreasing lower bounds for the number sought, i. e. for the value of the quadratic integral of the unknown function. In an earlier note (DIAZ [1]) it was shown that in a great many cases (for example, for certain semi-homogeneous boundary value problems, where either the boundary condition or the differential equation is homogeneous) the maximum and minimum principles follow directly from Schwarz' inequality, whereas the desired monotone sequences of upper and lower bounds can be obtained readily from Bessel's inequality. Starting with Schwarz' and Bessel's inequalities, formulated for a linear vector space with a positive semi-definite scalar product, it was shown, without reference to any specific boundary value problem, how inequalities for the square of the length of an (a priori unknown) vector could be obtained immediately, and how the resulting inequalities could be readily applied in special cases, which include Dirichlet's and Neumann's problems for Laplace's equation. In the present paper the results of the previous note are developed further, and inequalities are given which are also applicable to non-homogeneous problems (both differential equation and boundary condition non-homogeneous and also to problems where the boundary condition is of «mixed type» (e. g., $\alpha v + \beta \dfrac{\partial v}{\partial n}$ prescribed on the boundary, in the case of Laplace's equation). The same unified approach is followed as in the previous note, starting with a linear vector space with a positive semi-definite scalar product. Section 2 contains the main inequalities, which are stated in the form of theorems, for convenience of reference. All of these inequalities follow easily from Schwarz' and Bessel's inequalities. Section 3 deals with a few applications and serves to show the wide variety of special cases covered by the inequalities of section 2. Bibliographical remarks have been relegated to section 4, which contains a discussion of several related papers. The bibliography lists related papers which have come to the writer's attention, and, in some instances, papers whose results are readily obtainable from section 2 as special cases. The idea of a linear vector space with a scalar product has been current in the mathematical literature for around fifty years. The derivation of the basic inequalities in a vector space is a comparatively trivial matter, yet when these simple computations are performed (as seems to be the custom) explicitly in terms of the bilinear integral which is used in defining the scalar product, the sheer weight of the symbolism obscures the simplicity of the whole procedure, and many problems of estimating quadratic integrals, which are, as shown here, identical, are not usually recognized to be one and the same.

It may seem at first glance that the determination of upper and lower bounds for a quadratic integral of an unknown solution of the boundary value problem on the one hand, and the approximation (« in the sense of the same quadratic integral ») of the unknown solution of the boundary value problem by a linear combination of known functions on the other, are questions which are not directly related to each other. However, as can easily be seen, (essentially from « least squares »), these two problems are equivalent. The precise sense in which this equivalence is meant is stated at the end of section 2.

A new result concerning the estimation of Dirichlet's integral in Dirichlet's problem deserves special mention. This result is directly related to the papers of J. B. Diaz and A. Weinstein [1], [2], which were the starting point for the present paper. In [2] an upper bound for the Dirichlet integral in Neumann's problem was given in terms of a single function satisfying a certain boundary condition, thus paralleling the classical upper bound for the Dirichlet integral in Dirichlet's problem, given by Dirichlet's principle in terms of a single function satisfying a certain boundary condition. In [1] a lower bound for the Dirichlet integral in Neumann's problem was given in terms of a single arbitrary non-constant function. Inequality (42) of the present paper, which yields a lower bound for the Dirichlet integral in Dirichlet's problem in terms of a single arbitrary non-constant function, thus puts the estimation of Dirichlet's integral in Dirichlet's and Neumann's problems on a parallel basis.

Finally, it is emphasized that the question of existence of a solution of the boundary value problems considered is purposely set aside, and is not considered at all. It is shown in each boundary value problem that, if a solution exists, then such a solution is also given as a solution of certain maximum and minimum principles, and bounds for a quadratic integral of such a solution are obtained. A proof of the existence of a solution, starting from one of the maximum and minimum principles (such as is carried out, for example, for the classical Dirichlet's principle, in Courant [2] and Courant - Hilbert [I] volume II) is not touched upon.

2. **Upper and lower bounds.** To avoid repetition in the various special cases, it is convenient to operate in a real linear vector space with a positive semi definite scalar product. By a real linear vector space

is meant a set of elements (called « vectors », following custom) which can be added in pairs (« vector addition »), can be multiplied by real numbers (« scalar multiplication »), these two operations obeying the customary rules of vector algebra. Besides, there is a positive semi definite scalar product, i. e., a real number (a, b) is associated with each ordered pair of vectors a and b, which satisfies the rules

$$\begin{aligned} (\alpha a, b) &= \alpha (a, b) \\ (a_1 + a_2, b) &= (a_1, b) + (a_2, b), \\ (a, b) &= (b, a), \\ (a, a) &\geqq 0, \end{aligned}$$

for any vectors a and b, and any real number α. The equality $(a, a) = 0$ may hold even if a is not the zero vector, and this, together with the last inequality, explains the use of the adjective « positive semi definite ». In the applications to boundary value problems, the elements of the vector space are usually functions, and addition of functions (vectors) and multiplication of functions by real numbers are defined in the usual way for functions, while the scalar product of two functions is given by a bilinear integral associated with the boundary value problem. A glance at the two examples mentioned in the introduction taking $\left((a, b) \equiv \int_D (a_x b_x + a_y b_y)\, dx\, dy\right)$ shows that the equality $(a, a) = 0$ may indeed hold for functions other than the zero function. As usual, the number (a, a) will be referred to as the square of the length of the vector a. The length (norm) of the vector a will be denoted, as usual, by $|a|$, and then

$$(a, a) = |a|.^2$$

The original problem of obtaining upper and lower bounds for a quadratic integral, phrased in a linear vector space with a positive semi definite scalar product, is translated into finding upper and lower bounds for the square of the length, (v, v) or the length, $|v|$, of a vector v. The vector (function) v is a priori unknown, but some information is known about it, say the partial differential equation and the boundary conditions satisfied by v.

In the present section, various inequalities giving the desired upper and lower bounds will be given together. For the sake of clarity and for convenience of reference they will be listed as numbered theorems and corollaries. Theorems 1 and 2 and corollaries were given earlier (Diaz [1]) as easy consequences of Bessel's and Schwarz' inequalities :

$$\sum_{i=1}^{n} (f, g_i)^2 \leq (f, f), \tag{B}$$

where

$$(g_j, g_i) = \begin{cases} 1, & if \quad j = i, \\ 0, & if \quad j \neq i; \end{cases}$$

and

$$(f, h)^2 \leq (f, f)\ (h, h). \tag{S}$$

Theorem 1 and Corollary 1 yield lower bounds for (v, v) :

Theorem 1 : If v is a vector, p is a positive integer, and $w_1, \ldots, w_p$, are orthonormal vectors, then

$$\sum_{i=1}^{p} (v, w_i)^2 \leq (v, v),$$

with equality if and only if $\left| v - \sum_{i=1}^{p} (v, w_i)\, w_i \right| = 0.$

Corollary 1 : If v is a vector and w is a vector of positive length, then

$$\frac{(v, w)^2}{(w, w)} \leq (v, v),$$

with equality if and only if $\left| v - \frac{(v, w)}{(w, w)}\, w \right| = 0.$

Theorem 2 and Corollary 2 yield upper bounds for (v, v) :

Theorem 2 : If v is a vector, q is a positive integer, $z_1, \ldots, z_q$ are orthonormal vectors such that $(z_i, v) = 0$ for $i = 1. \ldots, q$, and z is a vector such that $(z - v, v) = 0$, then

$$(v, v) \leq (z, z) - \sum_{i=1}^{q} (z, z_i)^2,$$

with equality if and only if $\left| z - v - \sum_{i=1}^{q} (z, z_i)\, z_i \right| = 0.$

(This follows easily from Bessel's inequality, with $f = z - v$, $n = q$, and $g_i = z_i$, $i = 1, \ldots, q$).

Corollary 2 : If v is a vector and z is a vector such that $(z - v, v) = 0$, then

$$(v, v) \leq (z, z),$$

with equality if and only if $| z - v | = 0.$

(In theorem 2 and corollary 2, the condition $(z - v, v) = 0$ may be replaced by $(z - v, v) \geqq 0$, the argument remaining unchanged, thus obtaining a theorem 2′ and a corollary 2′, which will not be stated explicitly, inasmuch as they are included as special cases, respectively, of theorem 4′ and corollary 4′ below, with $y = 0$, and $y_1, \ldots, y_p$ absent).

Collecting the results of theorems 1 and 2, it follows that for each pair of positive integers p and q

$$\sum_{i=1}^{p} (v, w_i)^2 \leq (v, v) \leq (z, z) - \sum_{i=1}^{q} (z, z_i)^2,$$

or, equivalently,

$$\left| \sum_{i=1}^{p} (v, w_i)\, w_i \right|^2 \leq | v |^2 \leq \left| z - \sum_{i=1}^{q} (z, z_i)\, z_i \right|^2;$$

and thus a monotone non-increasing sequence of upper bounds and a monotone non-decreasing sequence of lower bounds for the number (v, v) have been obtained. However, in order that these upper and lower bounds be of practical use, one must be able to evaluate them explicitly, without knowing the vector v. In other words, using only the information available about v (i. e. the partial differential equation and the boundary conditions satisfied by v) one must be able to choose known vectors w_i such that the scalar products (v, w_i) are known, and also choose known vectors z_i and z which fulfill the conditions $(z_i, v) = 0$ and $(z - v, v) = 0$, respectively. Once p vectors w_i, q vectors z_i, and a vector z are chosen, the last inequalities furnish numerically computable upper and lower bounds for the number (v, v).

In choosing the vectors w_i, z_i, and z, the main device employed is Green's identity, written in a form suitable for the boundary value problem under consideration. From the examples given earlier (Diaz [1]) and the ones in the next section it appears that theorems 1 and 2 are particularly adapted for boundary value problems which are semi-homogeneous (that is, either the differential equation or the boundary condition is homogeneous; for example

$$\begin{aligned} \Delta v &= f, \quad \text{on } D, \\ v &= 0, \quad \text{on } C, \end{aligned}$$

where f is a given function on D). In order to be able to deal with non-homogeneous problems (which may, of course, be reduced to semi-homogeneous problems by a proper choice of either a particular solution of the non-homogeneous differential equation or of a function satisfying the non-homogeneous boundary conditions) and also to deal with «mixed» boundary conditions $\left(\text{e. g., } v + \frac{\partial v}{\partial n} = 0, \text{ on } C\right)$, two more theorems will be proved next.

The proofs of these two theorems depend directly on Schwarz' inequality. In order to see this more clearly, a few preliminary remarks will be found useful. If x and y are vectors, then

$$|(x, x)\, y - (x, y)\, x|^2 = ((x, x)\, y - (x, y)\, x, (x, x)\, y - (x, y)\, x) =$$
$$= (x, x) \cdot [(x, x)\,(y, y) - (x, y)^2],$$

and therefore

$$(x, y)^2 \leqq (x, x)\;(y, y), \tag{1}$$

with equality if and only if, either $(x, x) = 0$, or $(x, x) > 0$ and $\left| y - \frac{(x, y)}{(x, x)}\, x \right| = 0$. (Incidentally, [1] (1) implies that the scalar product of two vectors is always zero if just one of the vectors has zero length, a fact which will be employed frequently). When $(x, x) > 0$, the condition for equality may be restated thus: equality holds in (1) if and only if $y = Cx$ plus a vector of zero length, where C is a real number, namely $\frac{(x, y)}{(x, x)}$. This remark serves to reformulate Schwarz' inequality (1) in the precise form in which it will be needed:

$$-|x|\;|y| \leqq (x, y) \leqq |x|\;|y|, \tag{2}$$

where, if $(x, x) = 0$ then both equality signs hold for any y; and, if $|x|^2 = (x, x) > 0$ then right hand equality sign holds if and only if $y = \frac{|y|}{|x|}\, x$ plus a vector of zero length, while if $(x, x) > 0$ then left hand equality sign holds if and only if $y = -\frac{|y|}{|x|}\, x$ plus a vector of zero length. Consider only the determination of the right hand equality condition in (2) when $|x| > 0$, the left hand equality condition being obtained similarly. Clearly the right hand equality holds in (2)

[1] Actually it has only been shown so far that (1) holds if either $(x, x) > 0$ or $(y, y) > 0$ However, if both $(x, x) = 0$ and $(y, y) = 0$ then $0 \leqq (x \pm y, x \pm y) = (x, x) + (y, y) \pm 2\,(x, y) = \pm 2\,(x, y)$, so that $(x, y) = 0$, and (1) again holds.

if $y = \frac{|y|}{|x|} x$ plus a vector of zero length. Conversely, if the right hand equality holds in (2) for some y, then $(x, y) = |x| \, |y|$ and, by squaring, it follows that equality also holds in (1). Therefore $y = \frac{(x, y)}{(x, x)} x$ plus a vector of zero length, i. e. $y = \frac{|y|}{|x|} x$ plus a vector of zero length. Slightly rephrased, (2) is just the triangle inequality

$$(|x| - |y|)^2 \leqq |x + y|^2 \leqq (|x| + |y|)^2.$$

The theorem of Pythagoras will also be needed. If x and y are vectors and $(x, y) = 0$, i. e. x is perpendicular to y, then

$$|x \pm y|^2 = |x|^2 + |y|^2.$$

In particular, if $x_1, \ldots, x_n$ are orthonormal vectors, then

$$x = \left[x - \sum_{i=1}^{n} (x, x_i)\, x_i\right] + \sum_{i=1}^{n} (x, x_i)\, x_i,$$

where the vector $x - \sum_{i=1}^{n} (x, x_i)\, x_i$ is perpendicular to the projection of x on the linear subspace spanned by $x_1. \ldots, x_n$; that is, to the vector $\sum_{i=1}^{n} (x, x_i)\, x_i$; and Pythagoras theorem yields

$$|x|^2 = \left| x - \sum_{i=1}^{n} (x, x_i)\, x_i \right|^2 + \left| \sum_{i=1}^{n} (x, x_i)\, x_i \right|^2 =$$
$$= \left| x - \sum_{i=1}^{n} (x, x_i)\, x_i \right|^2 + \sum_{i=1}^{n} (x, x_i)^2.$$

Theorem 3 : If c is a vector, R is a non-negative number, n is a positive integer, $a_1, \ldots, a_n$ are real numbers, $w_1, \ldots, w_n$ are orthonormal vectors, and w is any vector such that both $|w - c|^2 = R^2$ and $(w, w_i) = a_i$ for $i = 1, \ldots, n$, then

$$\left.\begin{array}{c} \sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} - \left(R^2 - \sum_{i=1}^{n} \{a_i - (c, w_i)\}^2\right)^{\frac{1}{2}}\right]^2 \\ \leqq |w|^2 \leqq \\ \sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} + \left(R^2 - \sum_{i=1}^{n} \{a_i - (c, w_i)\}^2\right)^{\frac{1}{2}}\right]^2. \end{array}\right\} \quad (3)$$

If $\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| = 0$ then both equality signs hold in (3). If $\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| > 0$ then the left hand equality sign holds in (3) if and only if w equals the vector

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right)^{\frac{1}{2}} \right] \frac{c - \sum_{i=1}^{n} (c, w_i) w_i}{\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|}$$

plus a vector of zero length. If $\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| > 0$ then the right hand equality sign holds in (3) if and only if w equals the vector

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| + \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right)^{\frac{1}{2}} \right] \frac{c - \sum_{i=1}^{n} (c, w_i) w_i}{\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|}$$

plus a vector of zero length.

Proof: In view of the conditions $| w - c |^2 = R^2$, and $(w, w_i) = a_i$ for $i = 1, \ldots, n$, it follows that for any vector w

$$\left.\begin{aligned} \left| w - c - \sum_{i=1}^{n} (w - c, w_i) w_i \right|^2 &= | w - c |^2 - \left| \sum_{i=1}^{n} (w - c, w_i) w_i \right|^2 = \\ &= R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2. \end{aligned}\right\} \tag{4}$$

(In particular, (4) implies that if the set of all such vectors w is not empty, then $R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \geq 0$).

On the other hand

$$\begin{aligned} | w |^2 &= \left| \sum_{i=1}^{n} (w, w_i) w_i \right|^2 + \left| w - \sum_{i=1}^{n} (w, w_i) w_i \right|^2 = \\ &= \left| \sum_{i=1}^{n} a_i w_i \right|^2 + \left| w - \sum_{i=1}^{n} (w, w_i) w_i - \left[c - \sum_{i=1}^{n} (c, w_i) w_i \right] + \left[c - \sum_{i=1}^{n} (c, w_i) w_i \right] \right|^2 = \\ &= \sum_{i=1}^{n} a_i^2 + \left| w - c - \sum_{i=1}^{n} (w - c, w_i) w_i \right|^2 + \left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|^2 + \\ &\quad + 2 \left(\left[c - \sum_{i=1}^{n} (c, w_i) w_i \right], \left[w - c - \sum_{i=1}^{n} (w - c, w_i) w_i \right] \right) = \\ &= \sum_{i=1}^{n} a_i^2 + \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right) + \left(| c |^2 - \sum_{i=1}^{n} (c, w_i)^2 \right) + \\ &\quad + 2 \left(\left[c - \sum_{i=1}^{n} (c, w_i) w_i \right], \left[w - c - \sum_{i=1}^{n} (w - c, w_i) w_i \right] \right). \end{aligned}$$

The desired conclusion now follows by applying the result of (2) to the last scalar product appearing in the last equation, choosing $x = c - \sum_{i=1}^{n} (c, w_i)\, w_i$ and $y = w - c - \sum_{i=1}^{n} (w - c, w_i)\, w_i$ and taking (4) into account.

Theorem 3 may be easily interpreted geometrically. The set, call it S, of all vectors w satisfying $|\, w - c\,|^2 = R^2$ is a sphere with centre c and radius R, while the set, call it P, of all vectors w such that $(w, w_i) = a_i$ is a plane (« flat subset »). If the intersection of S and P is empty, then the theorem is true vacuously. The various geometrical possibilities can be easily visualized intuitively by means of a *schematic* two dimensional diagram. The set, call it L, of all linear combinations: $\sum_{i=1}^{n} c_i\, w_i$ plus a zero vector, where $c_1, \ldots, c_n$ are real numbers, is represented by a straight line through the origin; the plane P is represented by a straight line perpendicular to the straight line representing L (the point of intersection of these two straight lines represents the set of all vectors: $\sum_{i=1}^{n} a_i\, w_i$ plus a zero vector, which are common to P and L) ; and the sphere S is represented by a circle intersecting the straight line which represents P. If $\left| c = \sum_{i=1}^{n} (c, w_i)\, w_i \right| = 0$, i. e. c belongs to L, then all vectors on the intersection of the plane P and the sphere S are at the same distance from the zero vector. If $\left| c - \sum_{i=1}^{n} (c, w_i)\, w_i \right| > 0$, i. e. c does not belong to L, then a vector of the intersection which is nearest to the zero vector is

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i)\, w_i \right| - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right)^{\frac{1}{2}} \right] \frac{c - \sum_{i=1}^{n} (c, w_i)\, w_i}{\left| c - \sum_{i=1}^{n} (c, w_i)\, w_i \right|},$$

while a vector of the intersection which is farthest from the zero vector is

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i)\, w_i \right| + \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right)^{\frac{1}{2}} \right] \frac{c - \sum_{i=1}^{n} (c, w_i)\, w_i}{\left| c - \sum_{i=1}^{n} (c, w_i)\, w_i \right|}.$$

Since n may be taken to be zero in theorem 3, in which case the real numbers $a_1, \ldots, a_n$ and the vectors $w_1, \ldots, w_n$ are absent, one obtains

Corollary 3: If c is a vector, R is a non-negative number, and w is any vector such that $|w - c|^2 = R^2$, then

$$[|c| - R]^2 \leq |w|^2 \leq [|c| + R]^2. \tag{5}$$

If $|c| = 0$ then both equality signs hold in (5) for any w. If $|c| > 0$ sthen the left hand equality sign holds in (5) if and only if w equals

$$\frac{|c| - R}{|c|} \cdot c$$

plus a vector of zero length. If $|c| > 0$ then the right hand equality sign holds in (5) if and only if w equals

$$\frac{|c| + R}{|c|} \cdot c$$

plus a vector of zero length.

The following theorem will be useful in dealing with boundary value problems with «mixed» boundary conditions. For clarity, the results concerning upper and lower bounds are stated separately.

Theorem 4: (a) Suppose that c is a vector, R is a non-negative real number, n is a positive integer, $a_1, \ldots, a_n$ are real numbers, $w_1, \ldots, w_n$ are orthonormal vectors, and w is any vector such that both $|w - c|^2 \leq R^2$ and $(w, w_i) = a_i$ for $i = 1, \ldots, n$, then

$$|w|^2 \leq \sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} + \left(R^2 - \sum_{i=1}^{n} \{a_i - (c, w_i)\}^2\right)^{\frac{1}{2}}\right]^2. \tag{6}$$

If $\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| = 0$ then equality holds in (6) if and only if $|w - c|^2 = R^2$ and $(w, w_i) = a_i$ for $i = 1, \ldots, n$. If $\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| > 0$, then equality holds in (6) if and only if w equals the vector

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| + \right.$$

$$\left. + \left(R^2 - \sum_{i=1}^{n} \{a_i - (c, w_i)\}^2\right)^{\frac{1}{2}}\right] \frac{c - \sum_{i=1}^{n} (c, w_i) w_i}{\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|}$$

plus a vector of zero length.

(b) Furthermore, if, in addition

$$\left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2\right)^{\frac{1}{2}}\right] > 0, \tag{7}$$

then

$$\sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2\right)^{\frac{1}{2}}\right]^2 \leq |w|^2. \tag{8}$$

Equality holds in (8) if and only if w equals the vector

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| - \right.$$
$$\left. - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2\right)^{\frac{1}{2}}\right] \cdot \frac{c - \sum_{i=1}^{n} (c, w_i) w_i}{\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|},$$

plus a vector of zero length.

(c) If, instead of (7),

$$\left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2\right)^{\frac{1}{2}}\right] \leq 0, \tag{9}$$

then

$$\sum_{i=1}^{n} a_i^2 \leq |w|^2, \tag{10}$$

where equality holds if and only if $\left| w - \sum_{i=1}^{n} a_i w_i \right| = 0$.

Proof: (a) Given w such that $|w - c|^2 \leq R^2$, and $(w, w_i) = a_i$ for $i = 1, \ldots, n$ there is a number λ such that $0 \leq \lambda \leq 1$, for which $|w - c|^2 = \lambda R^2$. From the right hand side of inequality (3) of theorem 3 it follows that

$$|w|^2 \leq \sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} + \left(\lambda R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2\right)^{\frac{1}{2}}\right]^2,$$
$$\leq \sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n} (c, w_i)^2\right)^{\frac{1}{2}} + \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2\right)^{\frac{1}{2}}\right]^2,$$

and the equality conditions are easily obtained from theorem 3.

(Notice that, if $R > 0$, equality in (6) forces λ to equal 1, while if $R = 0$ the numerical value of λ does not matter).

(b) Still using the number λ introduced above, the left hand side of inequality (3) of theorem 3 implies that

$$\sum_{i=1}^{n} a_i^2 + \left[\left(|c|^2 - \sum_{i=1}^{n}(c, w_i)^2\right)^{\frac{1}{2}} - \left(\lambda R^2 - \sum_{i=1}^{n}\{a_i - (c, w_i)\}^2\right)^{\frac{1}{2}}\right]^2 \leq |w|^2,$$

and this last inequality, together with condition (7), yields the desired inequality (8). As for the condition of equality, notice that (7) implies that $\left| c - \sum_{i=1}^{n}(c, w_i)\, w_i \right| > 0$. The desired condition for equality in (8) then follows from theorem 3.

(c) Condition (9) implies that

$$\left| c - \sum_{i=1}^{n}(c, w_i)\, w_i \right|^2 = |c|^2 - \sum_{i=1}^{n}(c, w_i)^2 \leq R^2 - \sum_{i=1}^{n}\{a_i - (c, w_i)\}^2.$$

(Notice that, if the set of vectors w such that both $|w - c|^2 \leq R^2$ and $(w, w_i) = a_i$ for $i = 1, ..., n$ is not empty, then the right hand side of the last inequality is certainly ≥ 0, from (4). If the set of these vectors w is empty, then the theorem is true vacuously, anyway). On the other hand, using this last inequality

$$\begin{aligned}\left| c - \sum_{i=1}^{n} a_i\, w_i \right|^2 &= \left| c - \sum_{i=1}^{n}(c, w_i)\, w_i - \sum_{i=1}^{n}\{a_i - (c, w_i)\}\, w_i \right|^2, \\ &= \left| c - \sum_{i=1}^{n}(c, w_i)\, w_i \right|^2 + \sum_{i=1}^{n}\{a_i - (c, w_i)\}^2, \\ &\leq R^2.\end{aligned}$$

Thus the vector $\sum_{i=1}^{n} a_i\, w_i = \sum_{i=1}^{n}(w, w_i)\, w_i$ satisfies the same conditions as w, that is, $|w - c|^2 \leq R^2$ and $(w, w_i) = a_i$ for $i = 1, ..., n$. Hence inequality (10) is just Bessel's inequality again.

Theorem 4 may be easily interpreted geometrically. The set, call it S, of all vectors w satisfying $|w - c|^2 \leq R^2$ is a sphere, *plus* its interior, with center c and radius R, while the set, call it P, of all vectors w such that $(w, w_i) = a_i$ for $i = 1, ..., n$ is *a* plane («flat subset»). If the intersection of S and P is empty, the theorem is true vacuously. There are various geometrical possibilities, corresponding to cases (a), (b), and (c), each of which can be easily visualized by means of *a schematic* two dimensional diagram. The set, call it L, of all linear combinations : $\sum_{i=1}^{n} c_i w_i$ plus a zero vector, where $c_1, \ldots, c_n$ are real numbers, is represented by a straight line through the origin; the plane P is represented by a straight line perpendicular to that representing L (the point of intersection of these two straight lines represents the set of all vectors : $\sum_{i=1}^{n} a_i w_i$ plus a zero vector, which are common to P and L); and the «solid sphere» S is represented by a circle, *plus* its interior, intersecting the line representing P. Consider (a) first: if the center c belongs to L then the vectors w on the intersection of S and P (this intersection is a «solid circular disk») which are farthest from the zero vector are exactly all those vectors of the intersection which also lie *on* the sphere, i. e., such that $| w - c |^2 = R^2$. If the center c does not belong to L then a vector of the intersection of S and P which is farthest from the zero vector is the vector (*on* the sphere)

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right| + \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right)^{\frac{1}{2}} \right] \cdot \frac{c - \sum_{i=1}^{n} (c, w_i) w_i}{\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|} .$$

(b) states that if the vector $\sum_{i=1}^{n} a_i w_i$ does not belong to the intersection of S and P, then a vector of the intersection which is nearest to the zero vector is the vector (*on* the sphere)

$$\sum_{i=1}^{n} a_i w_i + \left[\left| c - \sum_{i=1}^{u} (c, w_i) w_i \right| - \left(R^2 - \sum_{i=1}^{n} \{ a_i - (c, w_i) \}^2 \right)^{\frac{1}{2}} \right] \cdot \frac{c - \sum_{i=1}^{n} (c, w_i) w_i}{\left| c - \sum_{i=1}^{n} (c, w_i) w_i \right|} .$$

Finally, (c) states that if the vector $\sum_{i=1}^{n} a_i w_i$ belongs to the intersection of S and P (notice that the vector $\sum_{i=1}^{n} a_i w_i$ is the orthogonal projection on L of any vector w of the intersection of S and P) then this projection, $\sum_{i=1}^{n} a_i w_i$, is a vector of the intersection of S and P which is nearest to the zero vector.

Since n may be taken to be zero in theorem 4, in which case the real numbers $a_1, \ldots, a_n$ are absent, one obtains

Corollary 4: (a) Suppose that c is a vector, R is a non-negative number and w is any vector such that $|w - c|^2 \leq R^2$, then

$$|w|^2 \leq [|c| + R]^2. \tag{11}$$

If $|c| = 0$ then equality holds in (11) for any w such that $|w|^2 \leq R^2$. If $|c| > 0$ then equality holds in (11) if and only if w equals the vector

$$\frac{|c| + R}{|c|} \cdot c$$

plus a vector of zero length.

(b) Furthermore, if, in addition

$$|c| - R > 0, \tag{12}$$

then

$$[|c| - R]^2 \leq |w|^2. \tag{13}$$

Equality holds in (13) if and only if w equals the vector

$$\frac{|c| - R}{|c|} \cdot c$$

plus a vector of zero length.

(c) If, instead of (12)

$$|c| - R \leq 0, \tag{14}$$

then

$$0 \leq |w|^2, \tag{15}$$

with equality if and only if $|w| = 0$.

Notice how (c) reduces to a triviality. Geometrically viewed, (c) states that if all that is known of a vector w is that it is on or inside of a sphere with center c and radius R, and the zero vector is also known to be on or inside the same sphere, then the only certain lawer bound for the length of w is the trivial bound, zero.

In applying theorems 3 and 4 to boundary value problems, a slightly different formulation is usually needed. Consider theorem 3 first. As mentioned earlier, the practical problem is to find known upper and lower bounds for a number $|v|^2$, where v is itself an a priori unknown vector, but about which some information (e. g., partial differential equation, boundary conditions) is available. In many problems, using only the information known about v, it is possible to choose known vectors y and z such that

$$(y - v,\ z - v) = 0, \tag{16}$$

and also to choose, for positive integers p and q, p known orthonormal vectors $y_1, \ldots, y_p$ and q known orthonormal vectors $z_1, \ldots, z_q$ satisfying the conditions

$$\begin{aligned} (y_i,\ z_j) &= 0, \\ (y_i,\ z - v) &= 0, \\ (y - v,\ z_j) &= 0, \end{aligned} \tag{17}$$

for i $= 1, \ldots, p$; $j = 1, \ldots, q$. Since equation (16) may be rewritten in the form

$$\left| v - \frac{y+z}{2} \right|^2 = \left| \frac{y-z}{2} \right|^2, \tag{18}$$

the desired reformulation of theorem 3 follows readily, using theorem 3 with $c = \dfrac{y+z}{2}$; $R = \left| \dfrac{y-z}{2} \right|$; $n = p + q$; $w_i = y_i$ for $i = 1, \ldots, p$; $w_{p+j} = z_j$ for $j = 1, \ldots, q$; and finally $a_k = (v, w_k)$ for $k = 1, \ldots, n$. It is to be noticed that, from (17), (v, w_k) is known for $k = 1, \ldots, n$, since $(v, y_i) = (z, y_i)$ for $i = 1, \ldots, p$, while $(v, z_j) = (y, z_j)$ for $j = 1, \ldots, q$.

Theorem 3′: If v is a vector, y and z are vectors such that $(y - v, z - v) = 0$, p and q are positive integers, $y_1, \ldots, y_p, z_1, \ldots, z_q$ are $p + q$ orthonormal vectors such that $(y_i, z - v) = 0$ for $i = 1, \ldots, p$ and $(y - v, z_j) = 0$ for $j = 1, \ldots, q$, then

$$\left.\begin{aligned} &\sum_{i=1}^{p} (z, y_i)^2 + \sum_{j=1}^{q} (y, z_j)^2 + \Bigg[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{\frac{1}{2}} - \\ &\qquad - \left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{\frac{1}{2}}\Bigg]^2 \\ &\qquad\qquad \leq |v|^2 \leq \\ &\sum_{i=1}^{p} (z, y_i)^2 + \sum_{j=1}^{q} (y, z_j)^2 + \Bigg[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{\frac{1}{2}} + \\ &\qquad + \left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{\frac{1}{2}}\Bigg]^2. \end{aligned}\right\} \tag{19}$$

If $\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2 = 0$, then both equality signs hold in (19). If $\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{i=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2 > 0$ then the left hand equality sign holds in (19) if and only if v equals the vector

$$\left.\begin{array}{l}\sum_{i=1}^{p}(z, y_i)\, y_i + \sum_{j=1}^{q}(y, z_j)\, z_j + \left[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{\frac{1}{2}} - \right. \\ \left. -\left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{\frac{1}{2}}\right] \cdot \dfrac{\frac{y+z}{2} - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right) y_i - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right) z_j}{\left|\frac{y+z}{2} - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right) y_i - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right) z_j\right|}\end{array}\right\} \quad (20)$$

plus a vector of zero length, while the right hand equality sign holds in (19) if and only if v differs by a vector of zero length from the vector obtained from (20) by replacing the square bracket by the sum instead of the difference of the two square roots.

If $p = q = 0$, and the orthonormal vectors are absent, one obtains

Corollary 3′: If v is a vector, z and y are vectors such that

$$(y - v, z - v) = 0,$$

then
$$\left(\left|\frac{y+z}{2}\right| - \left|\frac{y-z}{2}\right|\right)^2 \leq |v|^2 \leq \left(\left|\frac{y+z}{2}\right| + \left|\frac{y-z}{2}\right|\right)^2. \quad (21)$$

If $\left|\frac{y+z}{2}\right| = 0$ then both equality signs hold in (21). If $\left|\frac{y+z}{2}\right| > 0$ then the left equality sign holds in (21) if and only if v equals the vector

$$\left(\frac{\left|\frac{y+z}{2}\right| - \left|\frac{y-z}{2}\right|}{\left|\frac{y+z}{2}\right|}\right) \cdot \frac{y+z}{2}$$

plus a vector of zero length. If $\left|\frac{y+z}{2}\right| > 0$ then the right hand equality sign holds in (21) if and only if v equals the vector

$$\left(\frac{\left|\frac{y+z}{2}\right| + \left|\frac{y-z}{2}\right|}{\left|\frac{y+z}{2}\right|}\right) \cdot \frac{y+z}{2}$$

plus a vector of zero length.

Theorem 4 may be reformulated in a similar way. In some problems, using only the information known about v, it is possible to choose known vectors y and z such that

$$(y - v,\ z - v) \leqq 0, \tag{22}$$

and also to choose, for positive integers p and q, known orthonormal vectors $y_1, \ldots, y_p$ and q known orthonormal vectors $z_1, \ldots, z_q$ satisfying the conditions (17). Since equation (22) may be rewritten in the form

$$\left| v - \frac{y+z}{2} \right|^2 \leqq \left| \frac{y-z}{2} \right|^2,$$

the desired reformulation of theorem 4 follows readily, using theorem 4 with $c = \frac{y+z}{2}$; $R = \left|\frac{y-z}{2}\right|$; $n = p + q$; $w_i = y_i$ for $i = 1, \ldots, p$; $w_{p+j} = z_j$ for $j = 1, \ldots, q$ and finally $a_k = (v, w_k)$ for $k = 1, \ldots, n$. The final result is

Theorem 4′: (a) If v is a vector, y and z are vectors such that $(y - v,\ z - v) \leqq 0$, p and q are positive integers, $y_1, \ldots, y_p, z_1, \ldots, z_q$ are $p + q$ orthonormal vectors such that $(y_i,\ z - v) = 0$ for $i = 1, \ldots, p$ and $(y - v,\ z_j) = 0$ for $j = 1, \ldots, q$, then

$$\left.\begin{aligned} &|v|^2 \leqq \\ &\sum_{i=1}^{p} (z, y_i)^2 + \sum_{j=1}^{q} (y, z_j)^2 + \left[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{\frac{1}{2}} + \right. \\ &\left. + \left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{\frac{1}{2}}\right]^2. \end{aligned}\right\} \tag{23}$$

If $\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2 = 0$ then equality holds in (23) whenever $\left|v - \frac{y+z}{2}\right|^2 = \left|\frac{y-z}{2}\right|^2$. If $\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2 > 0$ then equality holds in (23) if and only if v differs by a vector of zero length from the vector obtained from (20) by replacing in the square bracket the difference of the square roots by their sum.

(b) If, in addition,

$$\left[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{\frac{1}{2}} - \left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{\frac{1}{2}}\right] > 0, \tag{24}$$

then

$$\left.\begin{aligned}&\sum_{i=1}^{p}(z, y_i)^2+\sum_{j=1}^{q}(y, z_j)^2\\&+\left[\left(\left|\frac{y+z}{2}\right|^2-\sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2-\sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{\frac{1}{2}}-\right.\\&\left.-\left(\left|\frac{y-z}{2}\right|^2-\sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2-\sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{\frac{1}{2}}\right]^2\leqq\\&\leqq|v|^2.\end{aligned}\right\}\tag{25}$$

Equality holds in (25) if and only if v differs from the vector (20) by a vector of zero length.

(c) If, instead of (24), the left hand side of (24) is $\leqq 0$, then

$$\sum_{i=1}^{p}(z, y_i)^2+\sum_{j=1}^{q}(y, z_j)^2\leqq|v|^2,\tag{26}$$

where equality holds if and only if $\left|v-\sum_{i=1}^{p}(z, y_i)\,y_i-\sum_{j=1}^{q}(y, z_j)\,z_j\right|=0.$

If $p=q=0$, and the orthonormal vectors are absent, one obtains

Corollary 4′: (a) If v is a vector, and z and y are vectors such that $(y-v,\ z-v)\leqq 0$, then

$$|v|^2\leqq\left[\left|\frac{y+z}{2}\right|+\left|\frac{y-z}{2}\right|\right]^2.\tag{27}$$

If $\left|\frac{y+z}{2}\right|=0$ then equality holds in (27) whenever $|v|^2=\left|\frac{y-z}{2}\right|^2$.

If $\left|\frac{y+z}{2}\right|>0$ then equality holds in (27) if and only if v equals the vector

$$\left(\frac{\left|\frac{y+z}{2}\right|+\left|\frac{y-z}{2}\right|}{\left|\frac{y+z}{2}\right|}\right)\cdot\frac{y+z}{2}$$

plus a vector of zero length.

(b) If, in addition

$$\left|\frac{y+z}{2}\right|-\left|\frac{y-z}{2}\right|>0,\tag{28}$$

then

$$\left[\left|\frac{y+z}{2}\right|-\left|\frac{y-z}{2}\right|\right]^2\leqq|v|^2,\tag{29}$$

with equality if and only if v equals the vector

$$\left(\frac{\left|\frac{y+z}{2}\right|-\left|\frac{y-z}{2}\right|}{\left|\frac{y+z}{2}\right|}\right)\cdot\frac{y+z}{2}$$

plus a vector of zero length.

(c) If, instead of (28),

$$\left|\frac{y+z}{2}\right|-\left|\frac{y-z}{2}\right| \leq 0, \tag{30}$$

then

$$0 \leq |v|^2, \tag{31}$$

with equality if and only if $|v| = 0$.

A few remarks concerning the «mean square approximation» of an unknown vector v and the equivalence of this approximation with the determination of upper and lower bounds for $|v|^2$ are perhaps appropriate. This equivalence is seen at once from Bessel's inequality. In the notation employed at the beginning of this section, let f be a vector, $g_1, ..., g_n$ be n orthonormal vectors, and $c_1, ..., c_n$ be real numbers. Since

$$\left|f-\sum_{i=1}^{n} c_i g_i\right|^2 = |f|^2 - \sum_{i=1}^{n} (f, g_i)^2 + \sum_{i=1}^{n} [c_i - (f, g_i)]^2,$$

it follows that the linear combination $\sum_{i=1}^{n} (f, g_i)\, g_i$ of $g_1, ..., g_n$, which gives the «best approximation to f in the sense of least squares» $\left(\text{i. e. which minimizes } \left|f-\sum_{i=1}^{n} c_i g_i\right|^2\right)$ is exactly the same linear combination of $g_1, ..., g_n$ which furnishes the «best lower bound» for $|f|^2$. Notice that if $g_1, ..., g_n$ and (f, g_i), $i = 1, ..., n$ are known, but f is not known, then the «least possible error», $\left|f-\sum_{i=1}^{n} (f, g_i)\, g_i\right|^2$ obtained by approximating f by $\sum_{i=1}^{n} (f, g_i)\, g_i$ is, in general, not known.

Consider theorem 1, regarding v as an unknown vector, $w_1, ..., w_p$ as known vectors, and (v, w_i) for $i = 1, ..., p$ as known numbers. Taking $f = v$, $p = n$, $g_i = w_i$, $i = 1, ..., p$, it follows that out of all linear combinations of $w_1, ..., w_p$, the projection $\sum_{i=1}^{n} (v, w_i)\, w_i$ is the best approximation to v the sense of least squares. The mean square error $\left| v - \sum_{i=1}^{n} (v, w_i)\, w_i \right|^2$ is in general not known.

Consider theorem 2, regarding v as an unknown vector, and $z, z_1, \ldots, z_q$ as known vectors. Let c_0 be a real number. Taking $f = v - c_0 z$, $n = q$, $g_i = z_i$, for $i = 1, \ldots, q$, it follows that (recall that $(z - v, v) = 0$ and $(v, z_i) = 0$, $i = 1, \ldots, q$) out of all vectors $c_0 z + \sum_{i=1}^{q} c_i z_i$, where $c_1, \ldots, c_q$ are real numbers, the best approximation to v in the sense of least squares is the vector

$$c_0 \left[z - \sum_{i=1}^{q} (z, z_i)\, z_i \right],$$

and the «mean square error» is

$$\left| v - c_0 \left[z - \sum_{i=1}^{q} (z, z_i)\, z_i \right] \right|^2 = (1 - 2 c_0) \, |v|^2 + c_0^2 \, |z|^2 - c_0^2 \sum_{i=1}^{q} (z, z_i)^2 .$$

The case $c_0 = \frac{1}{2}$ is of special interest, since then the error is known explicitly. The import of the factor $\frac{1}{2}$ is clearly seen from the fact that the condition $(z - v, v) = 0$ is equivalent to $\left| v - \frac{z}{2} \right|^2 = \left| \frac{z}{2} \right|^2$.

Consider theorem 3′, regarding v as an unknown vector, and $y, z, y_1, \ldots, y_p, z_1, \ldots, z_q$ as known vectors. Let c_0 and d_0 be real numbers. Taking $f = v - c_0 y - d_0 z$, $n = p + q$, $g_i = y_i$ for $i = 1, \ldots, p$, and $g_{p+j} = z_j$ for $j = 1, \ldots, q$, it follows that of all vectors

$$c_0 y + d_0 z + \sum_{i=1}^{p} c_i y_i + \sum_{j=1}^{q} d_j z_j,$$

where $c_1, \ldots, c_p, d_1, \ldots, d_q$ are real numbers, the best approximation to v in the sense of least squares is the vector

$$c_0 \left[y - \sum_{i=1}^{p} (y, y_i)\, y_i - \sum_{j=1}^{q} (y, z_j)\, z_j \right] +$$

$$+ d_0 \left[z - \sum_{i=1}^{p} (z, y_i)\, y_i - \sum_{j=1}^{q} (z, z_j)\, z_j \right] + \sum_{i=1}^{p} (z, y_i)\, y_i + \sum_{j=1}^{q} (y, z_j)\, z_j,$$

and the mean square error is

$$| v - (c_0 y + d_0 z) |^2 - \sum_{i=1}^{p} (- c_0 y + [1 - d_0]\, z, y_1)^2 - \sum_{j=1}^{q} ([1 - c_0]\, y - d_0 z, z_j)^2 .$$

The case $c_0 = d_0 = \frac{1}{2}$ is of special interest $\left(\text{recall that } \left| v - \frac{y+z}{2} \right|^2 = \right.$ $\left. = \left| \frac{y-z}{2} \right|^2 \right)$ for then the error is known explicitly to be

$$\left| \frac{y-z}{2} \right|^2 - \sum_{i=1}^{p} \left(\frac{y-z}{2}, y_i \right)^2 - \sum_{j=1}^{q} \left(\frac{y-z}{2}, z_j \right)^2 .$$

A similar statement applies to theorem 4′. Here $\left| v - \frac{y+z}{2} \right|^2 =$ $= \lambda \left| \frac{y-z}{2} \right|^2$, where $0 \leq \lambda \leq 1$, but the number λ is not necessarily known. In the (favorable) case, $c_0 = d_0 = \frac{1}{2}$, all that can be stated is that the mean square error is certainly not greater than

$$\left| \frac{y-z}{2} \right|^2 - \sum_{i=1}^{p} \left(\frac{y-z}{2}, y_i \right)^2 - \sum_{j=1}^{q} \left(\frac{y-z}{2}, z_j \right)^2 .$$

Finally, a remark concerning the consistent use of orthonormal sequences of vectors throughout in the formulatión of the various theorems. Consider n vectors (not necessarily orthonormal) $G_1, \ldots, G_n$, which are linearly independent, *i. e.* such that no linear combination of $G_1, \ldots, G_n$ has zero length, or, what is the same, such that the Gram determinant

$$\begin{vmatrix} (G_1, G_1) & \ldots & (G_1, G_n) \\ (G_n, G_1) & \ldots & (G_n, G_n) \end{vmatrix}$$

is positive. By the Gram - Schmidt method, say, one may construct (in principle, anyway) n of their linear combinations, call them $g_1, \ldots, g_n$, which are orthonormal, but this process may be awkward to carry out numerically in practice, given the linearly independent vectors $G_1, \ldots, G_n$ to start out with. If f is any vector, and $c_1, \ldots, c_n$ are real numbers, then

$$\left| f - \sum_{i=1}^{n} c_i G_i \right|^2 = |f|^2 - 2\left(f, \sum_{i=1}^{n} c_i G_i \right) + \left(\sum_{i=1}^{n} c_i G_i, \sum_{j=1}^{n} c_j G_j \right) =$$

$$= |f|^2 - \left| \sum_{i=1}^{n} c_i^* G_i \right|^2 + \left| \sum_{i=1}^{n} (c_i - c_i^*) G_i \right|^2 ,$$

where the real numbers $c_1^*, \ldots, c_n^*$ are the uniquely determined real numbers satisfying the system of linear equations

$$\sum_{j=1}^{n} c_j^* (G_j, G_i) = \left(\sum_{j=1}^{n} c_j^* G_j, G_i \right) = (f, G_i),$$

$i = 1, \ldots, n$. Thus

$$\sum_{i=1}^{n} c_i^* (f, G_i) = \left| \sum_{i=1}^{n} c_i^* G_i \right|^2 \leqq |f|^2,$$

with equality if and only if $\left| f - \sum_{i=1}^{n} c_i^* G_i \right| = 0$. The reformulations of the final inequalities of the various theorems in terms of linearly independent sequences of vectors, rather than in terms of orthonormal sequences of vectors, is now so obvious that it is not necessary to record them in detail. The resulting inequalities are not so symmetric as the ones in terms of orthonormal sequences, since they involve the solution of certain systems of linear equations (in the case of orthonormal sequences of vectors the solution of the corresponding systems of linear equations is immediate, since then the systems of linear equations have matrices of coefficients of diagonal form). However, for numerical purposes, starting with a given sequence of linearly independent vectors, it is sometimes easier to apply the inequalities in terms of linearly independent sequences of vectors, solving the necessary systems of linear equations as they arise instead of first orthonormalizing the given linearly independent vectors.

3. **Applications.** In Diaz [1], a simple example was given of the application of theorems 1 and 2 and the corresponding corollaries. Let v be a solution of a linear boundary value problem consisting of a partial differential equation plus linear boundary conditions. In order that the upper and lower bounds given by theorems 1 and 2 be of practical use, one must be able to evaluate them explicitly, without knowing the vector v. Using Green's identity, in a form suitable for the boundary value problem under consideration, one must be able to choose known vectors w_i such that the scalar products (v, w_i) are known and also choose known vectors z_i and z which fulfill the conditions $(z_i, v) = 0$ and $(z - v, v) = 0$. Once these vectors are chosen, theorems 1 and 2 furnish monotone sequences of upper and lower bounds for the number $|v|^2$. Consequently, in a particular problem, once is it shown how to

choose vectors w_i such that (v, w_i) is known, and vectors z_i and z such that $(z_i, v) = 0$ and $(z - v, v) = 0$, the detailed statement of the upper and lower bounds for $|v|^2$ is superfluous, since the result can be easily written down directly from theorems 1 and 2. However, the maximum and minimum principles obtained from corollaries 1 and 2 are usually of interest in themselves, since, so to speak, they furnish the initial upper and lower bounds of the monotone sequences for $|v|^2$, and at the same time, once they are stated, it is usually clear just how the vectors w_i and z_i may be chosen.

Theorem 3′ can be used to deal directly with problems in which both the differential equation and the boundary conditions are non-homogeneous, but it is usually more convenient to reduce such a problem to a semi-homogeneous one by means of a particular solution of the non-homogeneous equation or by means of a function satisfying the non-homogeneous boundary condition. Theorem 4′ can be used to deal directly with problems with «mixed boundary conditions».

Rather than treating, in a general form, a boundary value problem involving a system of partial differential equations together with linear boundary conditions, the general procedure to be followed will be illustrated by means of a few typical important special cases. The reader will have no difficulty in finding a host of new examples by himself. Further, only the simplest differential equations will be considered in each case. It is perfectly clear, for example, how to modify the results for Laplace's equation

$$v_{xx} + v_{yy} = 0,$$

in order to obtain results for the self adjoint elliptic equation

$$(av_x + bv_y)_x + (bv_x + cv_y)_y = 0,$$

where $b^2 - ac < 0$. The extension to any finite number of dimensions is also immediate. For simplicity, to avoid a detailed statement of conditions insuring the convergence of the integrals involved, only bounded domains will be considered.

(I) *Neumann's problem.* The boundary value problem consists in finding v such that

$$\left.\begin{aligned} \Delta v &= 0, \quad \text{on} \quad D, \\ \frac{\partial v}{\partial n} &= f, \quad \text{on} \quad C, \end{aligned}\right\} \tag{32}$$

where D is a bounded, plane, connected, open set with a smooth boundary C, and f is a given function on C. The quadratic integral to be estimated is the Dirichlet integral $\int_D (v_x^2 + v_y^2)\, dx\, dy$.

In dealing with this problem, at least two different choices for the elements of the linear vector space are possible. These two cases will be treated separately:

(a) The elements of the auxiliary vector space are sufficiently smooth real valued functions defined on $D + C$. Vector addition and multiplication of vectors by scalars is defined in the usual way. The scalar product (φ, ψ) is defined by the bilinear Dirichlet's integral

$$(\varphi, \psi) = \int_D (\varphi_x \psi_x + \varphi_y \psi_y)\, dx\, dy = \left\{ \begin{array}{l} -\int_D \varphi \Delta \psi\, dx\, dy + \int_C \varphi \frac{\partial \psi}{\partial n}\, ds, \\ -\int_D \psi \Delta \varphi\, dx\, dy + \int_C \psi \frac{\partial \varphi}{\partial n}\, ds, \end{array} \right\} \quad (33)$$

where the last equality is just Green's identity, and $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the outer normal to C. Since D is connected, $(\varphi, \varphi) = 0$ implies that φ is a constant function.

Given any function w, the scalar product (v, w) is known, since, from (33)

$$(v, w) = \int_C w \frac{\partial v}{\partial n}\, ds.$$

Corollary 1 then yields the following maximum principle for v: If w is any non-constant function, then

$$\frac{\left(\int_C w \frac{\partial v}{\partial n}\, ds \right)^2}{\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy, \quad (34)$$

with equality if and only if $v = \lambda w + \mu$ for some real numbers λ and μ.

However, corollary 2, although still true, does not yield a useful principle, since, as can be easily seen from (33), the information known about v is not sufficient to determine non-trivial known functions z which fulfill the condition $(z, v) = (v, v)$.

(b) The elements of the auxiliary vector space are ordered pairs $[p, q]$ of sufficiently smooth real valued functions defined on $D + C$. Vector addition is defined in the usual way:

$$[p_1, q_1] + [p_2, q_2] = [p_1 + q_1,\ p_2 + q_2],$$

while multiplication of a vector $[p, q]$ by a real number α is defined by

$$\alpha\,[p, q] = [\alpha\, p,\ \alpha\, q].$$

The scalar product $([p_1, q_1], [p_2, q_2])$ is defined by the bilinear integral

$$([p_1, q_1], [p_2, q_2]) = \int_D \{p_1 q_1 + p_2 q_2\}\, dx\, dy. \tag{35}$$

Clearly, $([p, q], [p, q]) = 0$ implies $p = q = 0$. Using Green's theorem, and the definition, (35), of the scalar product, it follows that

$$([p, q], [\varphi_x, \varphi_y]) = -\int_D \varphi \cdot (p_x + q_y)\, dx\, dy + \int_C \varphi\, (pn_x + qn_y)\, dr, \tag{36}$$

where n_x and n_y are the x and y components of the outer normal to C. Notice that

$$([v_x, v_y], [v_x, v_y]) = \int_D (v_x^2 + v_y^2)\, dx\, dy.$$

Corollary 2 then yields (taking $\varphi = v$ in (36) the following minimum principle for v (or perhaps more precisely, for the derivatives v_x and v_y): If p and q satisfy $p_x + q_y = 0$ on D, and $pn_x + qn_y = \dfrac{\partial v}{\partial n}$ on C, then

$$\int_D (v_x^2 + v_y^2)\, dx\, dy \leq \int_D (p^2 + q^2)\, dx\, dy, \tag{36'}$$

with equality if and only if both $v_x = p$ and $v_y = q$. (This is precisely Kelvin's minimum kinetic energy theorem, see Lamb [I], pages 47 and 57, as has been remarked in Diaz and Weinstein [2], page 109).

Since the condition $p_x + q_y = 0$, on D, can always be replaced by $p = w_y$, $q = -w_x$, on D, where w is a suitable function (not necessarily single-valued) the last minimum principle for v may be restated as follows: If w is any function (not necessarily single-valued, but such that w_x and w_y are single-valued) such that

$$\frac{\partial w}{\partial s} = w_x \frac{dx}{ds} + w_y \frac{dy}{ds} = - w_x n_y + w_y n_x = \frac{\partial v}{\partial n}, \text{ on } C,$$

then

$$\int_D (v_x^2 + v_y^2)\, dx\, dy \leq \int_D (w_x^2 + w_y^2)\, dx\, dy, \tag{37}$$

with equality if and only if both $v_x = w_y$ and $v_y = - w_x$. (This is precisely the new formulation of Kelvin's theorem referred to by Diaz and Weinstein [2], page 109, and Weinstein [1], page 148).

Corollary 1 (taking $[p, q] = [v_x, v_y]$ and $[\varphi_x, \varphi_y] = [w_x, w_y]$ in (36) yields exactly the same maximum principle for v that was obtained in (34) above.

(II) *Dirichlet problem.* The boundary value problem consists in finding v such that

$$\left.\begin{aligned} \Delta v &= 0, \quad \text{on } D, \\ v &= f, \quad \text{on } C, \end{aligned}\right\} \tag{38}$$

where f is a given function en C. Again it is required to find upper and lower bounds for the Dirichlet integral $\int_D (v_x^2 + v_y^2)\, dx\, dy$. The cases of auxiliary vector space consisting of single functions and of pairs of functions will again be considered separately.

(a) The elements of the vector space are single functions and the scalar product is defined by (33).

Given any function w such that $\Delta w = 0$ on D, the scalar product (v, w) is known, since, from (33)

$$(v, w) = \int_C v \frac{\partial w}{\partial n}\, dr.$$

Corollary 1 then yields the following maximum principle for v: If w is any non-constant function such that $\Delta w = 0$ on D, then

$$\frac{\left(\int_C v \frac{\partial w}{\partial n}\, ds\right)^2}{\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy, \tag{39}$$

with equality if and only if $v = \lambda\, w + \mu$ for some real numbers λ and μ.

Given any function z such that $z = v$ on C, it follows from (33) that $(z, v) = (v, v)$. Corollary 2 then yields the familiar Dirichlet's principle: If z is any function such that $z = v$ on C then

$$\int_D (v_x^2 + v_y^2)\,dx\,dy \leq \int_D (z_x^2 + z_y^2)\,dx\,dy, \tag{40}$$

with equality if and only if $v = z$.

(b) The elements of the auxiliary vector space are ordered pairs, $[p, q]$ of functions, and the scalar product is defined by (35).

Given any vector $[p, q]$ such that $p_x + q_y = 0$ on D, it follows from (36) that the scalar product $([p, q], [v_x, v_y])$ is known, since

$$([p, q], [v_x, v_y]) = \int_C v\,(pn_x + qn_y)\,ds.$$

Corollary 1 then yields the following maximum principle for v (or, perhaps more precisely, for the derivatives v_x and v_y): If p and q satisfy $p_x + q_y = 0$, on D, and $\int_D (p^2 + q^2)\,dx\,dy > 0$, then

$$\frac{\left[\int_C v\,(pn_x + qn_y)\,ds\right]^2}{\int_D (p^2 + q^2)\,dx\,dy} \leq \int_D (v_x^2 + v_y^2)\,dx\,dy, \tag{41}$$

with equality if and only if both $v_x = \lambda p$ and $v_y = \lambda q$ for some real number λ.

Since the condition $p_x + q_y = 0$, on D, can always be replaced by $p = w_y, q = - w_x$, on D, where w is a suitable function (not necessarily single-valued), the last maximum principle for v may be restated as follows: If w is any non-constant function (not necessarily single-valued but such that w_x and w_y are single-valued) then

$$\frac{\left[\int_C v\,\frac{\partial w}{\partial s}\,ds\right]^2}{\int_D (w_x^2 + w_y^2)\,dx\,dy} \leq \int_D (v_x^2 + v_y^2)\,dx\,dy, \tag{42}$$

$\left(\text{where } \frac{\partial w}{\partial s} = w_y n_x - w_x n_y\right)$ with equality if and only if $v_x = \lambda w_y$ and $v_y = - \lambda w_x$ for some real number λ. The maximum principle given by (42) includes the maximum principle for v given by (39). For, let w be a harmonic function, and w^* be a harmonic conjugate of w on D, that is, $w_x = w_y^*$, $w_y = - w_x^*$, on D. Then, since $\frac{\partial w}{\partial n} = \frac{\partial w^*}{\partial s}$ on C, the lower bound for $\int_D (v_x^2 + v_y^2)\, dx\, dy$ furnished by w in (39) is equal to the lower bound furnished by w^* in (42), and vice versa.

Corollary 2 (taking $[p, q] = [v_x, v_y]$ and $[\varphi_x, \varphi_y] = [z_x, z_y]$ in (36)) yields exactly the same minimum principle for v that was obtained in (40) above.

As a matter of fact, (42) is just Schwarz' inequality:

$$\frac{\left[\int_D (w_x v_x^* + w_y v_y^*)\, dx\, dy\right]^2}{\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^{*2} + v_y^{*2})\, dx\, dy,$$

where v^* denotes a harmonic conjugate of v on D, that is, $v_x = v_y^*$, $v_y = - v_x^*$, on D; because

$$\int_D (w_x v_x^* + w_y v_y^*)\, dx\, dy = \int_C w \frac{\partial v^*}{\partial n}\, ds = - \int_C w \frac{\partial v}{\partial s}\, ds = \int_C v \frac{\partial w}{\partial s}\, ds.$$

Or, even more directly, without introducing the harmonic conjugate function v^*, (42) is just Schwarz' inequality

$$\frac{\left[\int_D (v_x w_y - x_y w_x)\, dx\, dy\right]^2}{\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

because

$$\begin{aligned} \int_C v \frac{\partial w}{\partial s}\, ds &= \int_C v\, (w_y n_x - w_x n_y)\, ds, \\ &= \int_D [(v w_y)_x - (v w_x)_y]\, dx\, dy, \\ &= \int_D (v_x w_y - v_y w_x)\, dx\, dy. \end{aligned}$$

(III) *Mixed problem.* The boundary value problem consist in finding v such that

$$\left.\begin{aligned} \Delta v = 0, \quad &\text{on} \quad D, \\ \alpha v + \beta \frac{\partial v}{\partial n} = f, \quad &\text{on} \quad C, \end{aligned}\right\} \tag{43}$$

where the functions f, α and β are given real valued functions on C, such that both $\alpha^2 + \beta^2 > 0$ and $\alpha\beta \geqq 0$ on C. Again it is required to find upper and lower bounds for the Dirichlet integral of a solution v of (43). The first condition on α and β merely insures that a boundary condition on v is actually imposed at each point of C, by requiring that α and β do not vanish simultaneously on C. The reason for the second condition on α and β is apparent from equation (46′). The present boundary value problem includes Neumann's ($\alpha = 0$, $\beta = 1$) and Dirichlet's ($\alpha = 1$, $\beta = 0$) as special cases.

Corollary 4′, as will be seen below, is always available for this problem. However, as in the special cases discussed above, corollaries 1, 2 and 3′ may yield useful results more readily in particular cases.

Let C_α denote the subset of C on which α vanishes, C_β denote the subset of C on which β vanishes. The boundary condition to be satisfied by v is then

$$\left.\begin{aligned} \beta \frac{\partial v}{\partial n} = f, \quad &\text{on} \quad C_\alpha, \\ v = f, \quad &\text{on} \quad C_\beta, \\ \alpha v + \beta \frac{\partial v}{\partial n} = f, \quad &\text{on} \quad C - (C_\alpha + C_\beta). \end{aligned}\right\} \tag{44}$$

Consider first the application of corollary 4′:

(a) The elements of the vector space are single functions, and the scalar product is defined by the Dirichlet integral (33). Now

$$(y-v, z-v) = \left\{\begin{aligned} &-\int_D (y-v)\, \Delta\, (z-v)\, dx\, dy + \int_{C_\alpha + C_\beta + (C - C_\alpha - C_\beta)} (y-v) \cdot \frac{\partial\, (z-v)}{\partial n}\, ds, \\ &-\int_D (z-v)\, \Delta\, (y-v)\, dx\, dy + \int_{C_\alpha + C_\beta + (C - C_\alpha - C_\beta)} (z-v) \cdot \frac{\partial\, (y-v)}{\partial n}\, ds, \end{aligned}\right\} \tag{45}$$

and since the roles of y and z are perfectly symmetric, only the top equation need be used. In order that corollary 4′ be of practical use, it must be possible, once functions α, β and f are given (satisfying $\alpha^2 + \beta^2 > 0$ and $\alpha\beta \geqq 0$ on C) to choose a non-trivial set of functions y and z such that any harmonic function v in D which fulfills the boundary conditions (44) also satisfies $(y - v, z - v) \leqq 0$. From (45), this last inequality certainly holds if y and z are such that

$$\left.\begin{aligned} &\Delta(z - v) = 0, \quad \text{on} \quad D, \\ \text{and} \\ &(y - v)\cdot\frac{\partial}{\partial n}(z - v) \leqq 0, \quad \text{on} \quad C. \end{aligned}\right\} \tag{46}$$

Since $\alpha\beta \geqq 0$ on C, and

$$\left.\begin{aligned} &2\alpha\beta\cdot(y - v)\cdot\frac{\partial}{\partial n}(z - v) = \\ &= \left[\alpha\cdot(y - v) + \beta\frac{\partial}{\partial n}(z - v)\right]^2 - [\alpha(y - v)]^2 - \left[\beta\cdot\frac{\partial}{\partial n}(z - v)\right]^2, \end{aligned}\right\} \tag{46'}$$

it follows that the last condition of (46) clearly holds if

$$\alpha y + \beta\frac{\partial z}{\partial n} = \alpha v + \beta\frac{\partial v}{\partial n} = f, \quad \text{on} \quad C. \tag{47}$$

$\Big($Notice that (47) implies that $\frac{\partial z}{\partial n} = \frac{\partial v}{\partial n} = \frac{f}{\beta}$, on C_α, and that $y = v = \frac{f}{\alpha}$, on $C_\beta\Big)$. Thus corollary 4′ may be applied, provided that y and z satisfy

$$\left.\begin{aligned} \Delta z = 0, \quad \text{on} \quad D, \\ \alpha y + \beta\frac{\partial z}{\partial n} = f, \quad \text{on} \quad C. \end{aligned}\right\} \tag{48}$$

Notice that for special boundary value problems, the determination of y and z satisfying (48) may be equivalent to the solution of the original problem, and then corolary 4′ does not yield *a* useful result. For example, if $C = C_\alpha$ (Neumann's problem) then z is required by (48) to be a solution of the same Neumann's problem again. On the other hand, if $C = C_\beta$ (Dirichlet problem) then y is required by (48) to satisfy $y = v$ on C, and choosing z to be zero furnishes again the familiar Dirichlet principle which was obtained earlier. Another non-trivial result is obtained if none of the three sets C_α, C_β, and $C - (C_\alpha + C_\beta)$ is empty. Finally, in order to apply theorem 4′, it is readily seen that, since

$$(y_i, z - v) = -\int_D y_i \Delta(z - v)\,dx\,dy + \int_C y_i\frac{\partial}{\partial n}(z - v)\,ds,$$

the functions y_i may simply be chosen to vanish on $C - C_\alpha$, and since

$$(y - v, z_j) = -\int_D (y - v)\, \Delta z_j\, dx\, dy + \int_C (y - v) \cdot \frac{\partial z_j}{\partial n}\, ds,$$

the functions z_j may simply be chosen to satisfy

$$\left.\begin{aligned} \Delta z_j &= 0, \quad \text{on} \quad D, \\ \frac{\partial z_j}{\partial n} &= 0, \quad \text{on} \quad C - C_\beta. \end{aligned}\right\} \tag{49}$$

It is not necessary to repeat in detail the result of theorem 4'.

(b) The elements of the vector space are pairs of functions, and the scalar product is defined by (35). Now, by taking $y = [y_1, y_2]$ and $z = [z_x, z_y]$,

$$\left.\begin{aligned} ([y_1, y_2] - [v_x, v_y], [z_x, z_y] - [v_x, v_y]) = -\int_D (z - v) \left\{ (y_1 - v_x)_x + (y_2 - v_y)_y \right\} dx\, dy \\ + \int_C (z - v) \left\{ (y_1 - v_x) n_x + (y_2 - v_y) n_y \right\} ds, \end{aligned}\right\} \tag{50}$$

so that the condition $(y - v, z - v) \leq 0$ is certainly satisfied if

$$\left.\begin{aligned} y_{1x} + y_{2y} &= 0, \quad \text{on} \quad D, \\ \alpha z + \beta (y_1 n_x + y_2 n_y) &= \alpha v + \beta \frac{\partial v}{\partial n} = f, \quad \text{on} \quad C. \end{aligned}\right\} \tag{51}$$

(The last condition of (51) implies that $z = v = \frac{f}{\alpha}$ on C_β, and $y_1 n_x + y_2 n_y = \frac{\partial v}{\partial n} = \frac{f}{\beta}$ on C_α). Thus corollary 4' is useful in many mixed problems. In order to apply theorem 4', one may simply choose vectors $y_i = [y_{i1}, y_{i2}]$ and $[z_{ix}, z_{iy}]$ such that

$$\left.\begin{aligned} y_{i1x} + y_{i2y} &= 0, \quad \text{on} \quad D, \\ y_{i1} n_x + y_{i2} n_y &= 0, \quad \text{on} \quad C - C_\beta, \\ z_i &= 0, \quad \text{on} \quad C - C_\alpha. \end{aligned}\right\} \tag{52}$$

Since in (51) and (52) the condition to be fulfilled in D is that a certain divergence is zero, this condition may be formulated in terms of single functions (as was done in the case of the Dirichlet and Neumann problems). Letting w be such that $y_1 = w_y$, $y_2 = - w_x$; and w_i be such that

$y_{1i} = w_{iy}$, $y_{2i} = -w_{ix}$, and recalling that $\dfrac{\partial w}{\partial s} = w_y n_x - w_x n_y$ the conditions (51) and (52) may be rewritten simply

$$\alpha z + \beta \frac{\partial w}{\partial s} = \alpha v + \beta \frac{\partial v}{\partial n} = f, \quad \text{on} \quad C, \tag{51'}$$

and

$$\left.\begin{aligned} \frac{\partial w_i}{\partial s} &= 0, \quad \text{on} \quad C - C_\beta, \\ z_i &= 0, \quad \text{on} \quad C - C_\alpha. \end{aligned}\right\} \tag{52'}$$

The detailed statement of the results of theorem 4′ and corollary 4′ as applied to this special case, is not needed.

Perhaps the simplest mixed boundary value problem is that in which $C - (C_\alpha + C_\beta)$ is empty, that is, v is prescribed on a subset C_β, of C, and $\dfrac{\partial v}{\partial n}$ is prescribed on the remainder, C_α, of C. The results either become trivial or reduce to those of Neumann's problem when $C = C_\alpha$ and to those of Dirichlet's problem when $C = C_\beta$. Corollary 2 does not yield useful principles either when the auxiliary vector space consists of single functions of when it consists of pairs of functions. However, corollary 1, when the auxiliary vector space consists of single functions, yields the following two principles:

(1) If w is any non-constant function such that $w = 0$ on C_β, then

$$\frac{\left(\displaystyle\int_{C_\alpha} w \frac{\partial v}{\partial n}\, ds\right)^2}{\displaystyle\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

with equality if and only if $v = \lambda w + \mu$ for some real numbers λ and μ.

(2) If w is any non-constant function such that $\Delta w = 0$, on D, and $\dfrac{\partial w}{\partial n} = 0$ on C_α, then

$$\frac{\left(\displaystyle\int_{C_\beta} v \cdot \frac{\partial w}{\partial n}\, ds\right)^2}{\displaystyle\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

with equality if and only if $v = \lambda w + \mu$ for some real numbers λ and μ.

Corollary 1, when the auxiliary vector space consist of pairs of functions, yields the following principle (taking $\varphi = v$ in (36)):

(3) If p and q satisfy $p_x + q_y = 0$, on D, and $pn_x + qn_y = 0$, on $C - C_\beta$ and

$$\int_D (p^2 + q^2)\, dx\, dy > 0,$$

then

$$\frac{\left(\int_{C_\beta} v\,(pn_x + qn_y)\, ds\right)^2}{\int_D (p^2 + q^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

with equality if and only if $v_x = \lambda p$ and $v_y = \lambda q$ for some real number λ.

Replacing the condition $p_x + q_y = 0$ on D, by $p = w_y$, $q = - w_x$ on D, the last principle can be phrased in terms of a single function:

(4) If w is any non-constant function (not necessarily single valued, but such that w_x and w_y are single-valued) such that $\frac{\partial w}{\partial s} = 0$ on $C - C_\beta$, then

$$\frac{\left(\int_{C_\beta} v\, \frac{\partial w}{\partial s}\, ds\right)^2}{\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

with equality if and only if $v_x = \lambda\, w_y$ and $v_y = - \lambda\, w_x$ for some real number λ.

(IV) *A problem in the theory of thin elastic plates.*

Although various mixed problems can be discussed, only a particular problem, which occupies a place similar to that occupied by Neumann's problem in the theory of Laplace's equation, will be considered. For the theory of plates, reference is made to K. O. Friedrichs [2], and S. Timoshenko [1]. The boundary value problem consists in finding v (the deflection of the plate) such that

$$\left.\begin{aligned} \Delta\Delta v &= f, \quad \text{on } D, \\ V_n = (1 - \mu)\frac{\partial^3 \Delta v}{\partial s^2 \partial n} + \frac{\partial \Delta v}{\partial n} &= 0, \quad \text{on } C, \\ M_n = - \mu\,\Delta v - (1 - \mu)\frac{\partial^2 v}{\partial n^2} &= 0, \quad \text{on } C, \end{aligned}\right\} \tag{54}$$

where f is a given function on D, μ is an elastic constant, called Poisson's ratio, such that $0 \leq \mu < 1$; and V_n and M_n denote the transverse force and the bending moment on C, respectively. The quadratic integral to be estimated is the energy integral

$$\left.\begin{aligned}&\int_D [(\Delta v)^2 - 2(1-\mu)(v_{xx}v_{yy} - v_{xy}^2)]\,dx\,dy = \\ &= \int_D [\mu(\Delta v)^2 + (1-\mu)(v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2)]\,dx\,dy.\end{aligned}\right\} \tag{55}$$

The elements of the vector space are single functions. The scalar product is defined by

$$\left.\begin{aligned}(\varphi, \psi) &= \int_D [(\Delta\varphi)(\Delta\psi) - (1-\mu)(\varphi_{xx}\psi_{yy} + \varphi_{yy}\psi_{xx} - 2\varphi_{xy}\psi_{xy})]\,dx\,dy, \\ &= \int_D [\mu(\Delta\varphi)(\Delta\psi) + (1-\mu)(\varphi_{xx}\psi_{xx} + \varphi_{yy}\psi_{yy} + 2\varphi_{xy}\psi_{xy})]\,dx\,dy, \\ &= \begin{cases}\displaystyle\int_D \psi\,\Delta\Delta\varphi\,dx\,dy + \int_C\left\{\frac{\partial\psi}{\partial n}\cdot\left[\mu\Delta\varphi + (1-\mu)\frac{\partial^2\varphi}{\partial n^2}\right] - \psi\left[(1-\mu)\frac{\partial^3\varphi}{\partial s^2\partial n} + \frac{\partial\Delta\varphi}{\partial n}\right]\right\}ds, \\ \displaystyle\int_D \varphi\,\Delta\Delta\psi\,dx\,dy + \int_C\left\{\frac{\partial\varphi}{\partial n}\cdot\left[\mu\Delta\psi + (1-\mu)\frac{\partial^2\psi}{\partial n^2}\right] - \varphi\left[(1-\mu)\frac{\partial^3\psi}{\partial s^2\partial n} + \frac{\partial\Delta\psi}{\partial n}\right]\right\}ds.\end{cases}\end{aligned}\right\} \tag{56}$$

Corollary 1 yields the following maximum principle for v: If w is any function such that $(w, w) > 0$, then

$$\left.\begin{aligned}&\frac{\left(\int_D w\cdot\Delta\Delta v\,dx\,dy\right)^2}{\int_D [\mu(\Delta w)^2 + (1-\mu)(w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2)]\,dx\,dy} \leq \\ &\leq \int_D [\mu(\Delta v)^2 + (1-\mu)(v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2)]\,dx\,dy.\end{aligned}\right\} \tag{57}$$

Equality holds in (57) if and only if $v(x, y) = aw(x, y) + bx + dy + c$ for some real numbers a, b, c, and d.

(V) *Three dimensional elasticity.* Suppose body forces are absent, i. e., the differential equations are homogeneous, and consider only the boundary value problem which is the analogue of the Dirichlet problem in the theory of Laplace's equation. For brevity, the usual sumation convention will be employed, where a repeated subscript implies summation over 1, 2, 3. Also, $(x_1, x_2, x_3) = (x, y, z)$ and a comma denotes partial differentiation, e. g., $v_{p,q} = \dfrac{\partial v_p}{\partial x_q}$. The boundary value problem consists in determining the displacement $v = [v_1, v_2, v_3]$ in an open set B with a smooth boundary S, where v_i, $i = 1, 2, 3$, are real valued functions defined on $B + S$, such that

$$\text{for } i = 1, 2, 3, \text{ and} \qquad \left.\begin{aligned} (d_{ijpq} v_{p,q})_{,j} &= 0, \quad \text{on} \quad B, \\ v_i &= f_i, \quad \text{on} \quad S, \end{aligned}\right\} \tag{58}$$

where f_1, f_2, f_3 are known functions on S. The functions d_{ijpq} are known real valued functions on $B + S$ (in the usual «two constant» elasticity,

$$d_{ijpq} v_{p,q} = \mu\, v_{i,j} + (\lambda + \mu)\, v_{j,i}, \tag{59}$$

where λ and μ are Lamé's constants of elasticity, see Love [1]). The quadratic integral to be estimated is the energy integral

$$\int_B d_{ijpq} v_{p,q} v_{i,j}\, dB, \tag{60}$$

where the known functions d_{ijpq} are, of course, assumed to be such that, for any point P of B and any real numbers ξ_{ij}, $i, j = 1, 2, 3$,

$$d_{ijpq}(P) \cdot \xi_{ij}\, \xi_{pq} \geqq 0,$$

which implies that the integral (60) is positive semi definite.

(a) The elements of the vector space are triples of functions $\varphi = [\varphi_1, \varphi_2, \varphi_3]$ and the scalar product is defined by (assume further that $d_{ijpq} = d_{pqij}$, so that the scalar product will be symmetric)

$$\left.\begin{aligned} ([\varphi_1, \varphi_2, \varphi_3], [\psi_1, \psi_2, \psi_3]) &= \int_B d_{ijpq}\, \varphi_{p,q}\, \psi_{i,j}\, dB, \\ &= \begin{cases} -\displaystyle\int_B (d_{ijpq}\, \psi_{p,q})_{,j}\, \varphi_i\, dB + \int_S (d_{ijpq}\, \psi_{p,q} n_j)\, \varphi_i\, dS, \\ -\displaystyle\int_B (d_{ijpq}\, \varphi_{p,q})_{,i}\, \psi_j\, dB + \int_S (d_{ijpq}\, \varphi_{p,q} n_j)\, \psi_i\, dS, \end{cases} \end{aligned}\right\} \tag{61}$$

where n_1, n_2, n_3 are the components of the outer normal to S.

Corollary 1 then yields the following maximum principle for v: If $([w_1, w_2, w_3], [w_1, w_2, w_3]) > 0$, and

$$d_{ijpq} w_{p,q} = 0, \quad \text{on} \quad V,$$

for $i = 1, 2, 3$, then

$$\frac{\left(\int_S (d_{ijpq} w_{p,q} n_j) \cdot v_i \, dS\right)^2}{\int_B d_{ijpq} w_{p,q} w_{i,j} \, dB} \leq \int_B d_{ijpq} v_{p,q} v_{i,j} \, dB, \tag{62}$$

with equality if and only if

$$\int_B d_{ijpq} (v_p - aw_p)_{,q} \, (v_i - aw_i)_{,j} \, dB = 0,$$

for some real number s.

Corollary 2 yields the following minimum principle for $v = [v_1, v_2, v_3]$ If the functions z_1, z_2, z_3 are such that

$$z_i = v_i, \quad \text{on} \quad S,$$

for $i = 1, 2, 3$, then

$$\int_B d_{ijpq} v_{p,q} v_{i,j} \, dB \leq \int_B d_{ijpq} z_{p,q} z_{i,j} \, dB, \tag{63}$$

with equality if and only if

$$\int_B d_{ijpq} \, (v_p - z_p)_{,q} \, (v_i - z_i)_{,j} \, dB = 0.$$

(b) The elements of the auxiliary vector space are sequences $[P_{ij}]$ of nine functions each, the scalar product being defined by

$$([P_{ij}], [Q_{ij}]) = \int_B d_{ijpq} P_{pq} Q_{ij} \, dB. \tag{64}$$

Clearly, by Green's theorem

$$\left.\begin{aligned} ([P_{ij}], [v_{i,j}]), = \int_B d_{ijpq} P_{pq} v_{i,j} \, dB = - \int_B (d_{ijpq} P_{pq})_{,j} \, v_i \, dB + \\ + \int_S (d_{ijpq} P_{pq} n_j) \, v_i \, dS. \end{aligned}\right\} \tag{65}$$

Corollary 1 then yields the following maximum principle for v (or, perhaps more precisely, for the derivatives $v_{i,j}$): If the functions P_{ij}, $i, j = 1, 2, 3$, satisfy

$$(d_{ijpq} P_{pq})_{,j} = 0, \quad \text{on} \quad B, \tag{66}$$

for $i = 1, 2, 3$, and $\int_B d_{ijpq} P_{pq} P_{ij} \, dB > 0$, then

$$\frac{\left(\int_S (d_{ijpq} P_{pq} n_j) \, v_i \, dS\right)^2}{\int_B d_{ijpq} P_{pq} P_{ij} \, dB} \leq \int_B d_{ijpq} v_{p,q} v_{i,j} \, dB, \tag{67}$$

with equality if and only if $\int_B d_{ijpq} (v_{p,q} - aP_{pq}) (v_{i,j} - aP_{ij}) \, dB = 0$ for some real number a.

Without entering into a detailed statement, it will merely be mentioned that a lower bound for the integral (60) may be obtained from the last result (see the procedure followed in II b above) in terms of three arbitrary functions, using either the Maxwell or Morera stress functions for three dimensional elasticity (which are described, for example, in LOVE [1], page 88) in order to satisfy condition (66) above.

4. **Concluding remarks.** The starting point for the present paper were the papers by DIAZ and WEINSTEIN [1], [2], which dealt with upper and lower bounds for the Dirichlet integral in Dirichlet's and Neumann's problem. The second paper quoted contains the following formula for the torsional rigidity, or stiffness, S:

$$S = P - \int_D (\varphi_x^2 + \varphi_y^2) \, dx \, dy,$$

where P is the polar moment of inertia of the cross section with respect to the centroid of the section, and φ is the warping function. The above formula for S, together with the results on upper and lower bounds for the Dirichlet integral in Neumann's problem, furnish a method for determining practically upper and lower bounds for the torsional rigidity, *regardless* of whether the cross-section is simply or multiply connected. Numerical applications of this method will be published elsewhere. In [1] it was mentioned that the « direct method of obtaining upper and lower bounds for the Dirichlet integral by using Schwarz' inequality as a starting point» could be applied in the case of other quadratic functionals. This was formulated explicitly, in a general form, in Diaz [1].

Related questions have been discussed in many recent papers, and it seems worthwhile to review briefly the relation of some of these results to the conclusions of the present paper.

The initial paper along these lines seems to have been that of Trefftz [1] who showed how to obtain a lower bound for the Dirichlet integral of a solution of Dirichlet's problem, starting with an arbitrary non-constant harmonic function, thus complementing Dirichlet principle, which states that any function having the prescribed values on the boundary will furnish an upper bound for the Dirichlet integral of the solution. Trefftz stated that his method was applicable to other boundary value problems, but does not seem to have considered any other examples. In order to improve the initial upper and lower bounds, Trefftz proceeds by solving certain linear equations, a procedure which is in practice equivalent to that of orthonormalizing certain functions first. The advantage that is gained by using orthonormal functions is that the upper and lower bounds can be explicitly written down, and the exact role of Bessel's inequality is then apparent.

K. O. Friedrichs [1], page 20 (see also Courant - Hilbert [1], vol. I, pages 208 and 209, «Problem III» and «Problem IV»), uses a formal transformation of a given variational problem into other related variational problems. Starting with Dirichlet's principle in the case of the Dirichlet problem, Friedrichs obtains two minimum principles given in equations (31) and (32) of his paper, which can be shown to be closely related to (41) and (42) respectively, of the present paper. Consider (32) of Friedrichs' paper. Restated in the notation of section 2 of the present paper, this minimum principle says that

$$\frac{1}{2}\int_D (w_x^2 + w_y^2)\, dx\, dy + \int_C w \frac{\partial v}{\partial s}\, ds \tag{*}$$

is minimized, over the class of all sufficiently smooth functions w defined on $D + C$, by a harmonic conjugate v^* ot the solution v of the Dirichlet problem:

$$\Delta v = 0, \quad \text{on} \quad D,$$
$$v = \text{a given function}, \quad \text{on} \quad C.$$

Since $v_x = v_y^*$, $v_y = - v_x^*$, and $\dfrac{\partial v}{\partial s} = - \dfrac{\partial v^*}{\partial n}$, the minimum value of (*) is minus one half the Dirichlet integral of v^*, or, what is the same, minus one half the Dirichlet integral of v. Thus, for any w:

$$\frac{1}{2}\int_D (w_x^2 + w_y^2)\, dx\, dy + \int_C w \frac{\partial v}{\partial s} ds \geq -\frac{1}{2}\int_D (v_x^2 + v_y^2)\, dx\, dy.$$

Now, suppose w is a non-constant function, and let λ be any real number. The last inequality continues to hold when w is replaced by λw. In particular, the inequality holds for $\lambda_{\min}\cdot w$ where

$$\lambda_{\min} = -\frac{\int_C w \frac{\partial v}{\partial s} ds}{\int_D (w_x^2 + w_y^2)\, dx\, dy}$$

minimizes the quadratic form in λ. The resulting inequality obtained upon replacing w by $\lambda_{\min}\cdot w$ is exactly (42) of the present paper. A similar procedure can be shown to lead from (31) of Friedrichs [1] to (41) of the present paper.

W. Prager and J. L. Synge [1] consider three dimensional elasticity, body forces being absent. In their notation, their main result seems to be the following inequalities:

$$(S^*\cdot I'')^2 \leq S^2 \leq S^{*2}, \qquad (5.8), \text{ page } 249,$$

and

$$\sum_{q=1}^{n} (S^*\cdot I_q'')^2 \leq S^2 \leq S^{*2} - \sum_{p=1}^{m} (S^*\cdot I_p')^2, \qquad (10.21), \text{ page } 258,$$

where $S^2 = S\cdot S$ and $S^{*2} = S^*\cdot S^*$, the scalar product being denoted with a dot. However,

$$I'' = \frac{S''}{(S''\cdot S'')^{\frac{1}{2}}}, \quad \text{page } 247,$$

$$S^*\cdot S'' = S\cdot S'', \qquad (4.7), \text{ page } 247,$$

and

$$I_q''\cdot I_s'' = \begin{cases} 0, & \text{if } q \neq s \\ 1, & \text{if } q = s, \end{cases} \qquad (7.1), \text{ page } 252,$$

so that the left hand inequalities of (5.8) and (10.21) are merely

$$\frac{(S\cdot S'')^2}{S''\cdot S''} \leq S\cdot S,$$

and

$$\sum_{q=1}^{n} (S\cdot I_q'')^2 \leq S\cdot S,$$

i. e., Schwarz' and Bessel's inequalities, respectively. Moreover, since

$$S^* \cdot S = S \cdot S, \qquad (4.2), \text{ page } 246,$$

$$S \cdot I'_p = 0, \qquad (9.5), \text{ page } 255,$$

and

$$I'_p \cdot I'_r = \begin{cases} 0, & \text{if } p \neq r, \\ 1, & \text{if } p = r, \end{cases} \qquad (7.1), \text{ page } 252,$$

the right hand inequalities of (5,8) and (10.21) are merely

$$(S \cdot S)^2 = (S^* \cdot S)^2 \leq (S^* \cdot S^*)\,(S \cdot S),$$

and

$$\left[\sum_{p=1}^{m} (S^* \cdot I'_p)^2\right] + (S \cdot S) = \left[\sum_{p=1}^{m} (S^* \cdot I'_p)^2\right] + \left(S^* \cdot \frac{S}{(S \cdot S)^{\frac{1}{2}}}\right)^2 \leq (S^* \cdot S^*),$$

i. e., Schwarz' and Bessel's inequalities respectively (notice that the $m + 1$ vectors $I'_1, \ldots, I'_m, \dfrac{S}{(S \cdot S)^{\frac{1}{2}}}$ are orthonormal). Thus the simple and immediate analytic origin of inequalities (5.8) and (10.21) is hidden by the geometrical approach systematically employed and advocated in Prager and Synge [1], where no mention of Bessel's inequality is to be found, and Schwarz' inequality is only mentioned in passing on page 249.

J. L. Synge [2], in dealing with the special case of elasticity, body forces being present, arrives at an inequality ((35) of page 19) which can be seen to coincide with and may be said to have suggested the inequality (19) of theorem 3 of the present paper, although it is not stated in exactly the symmetrical form of (19). The argument leading to (34) in Synge [2], employs the method of Lagrange's multipliers and hence would seem to involve some justification, especially since the linear space dealt with is infinite dimensional.

J. L. Synge [1] deals with Neumann's Dirichlet's and the mixed problem, for the sake of generality, in a Riemannian-N-space. The mixed problem for Laplace's equation, as dealt with in [1] (pages 454-457) may be interpreted as an application of theorem 4′ to the particular problem considered there, and may be said to have suggested theorem 4′, but neither the final inequalities of theorem 4′ nor a detailed account of the various possibilities is given. For the two dimensional Neumann's problem for Laplace's equation, the following inequality is given for the Dirichlet integral of a solution v:

$$\frac{\left(\int_D [p_1^* w_x'' + p_2^* w_y''] \, dx \, dy\right)^2}{\int_D [(w_x'')^2 + (w_y'')^2] \, dx \, dy} \leq$$

$$\leq \int_D (v_x^2 + v_y) \, dx \, dy \leq \int_D [(p_1^*)^2 + (p_2^*)^2] \, dx \, dy, \qquad \text{(28), page 452,}$$

where w'' is a non-constant function, and the functions p_1^* and p_2^* satisfy the conditions

$$\frac{\partial p_1^*}{\partial x} + \frac{\partial p_2^*}{\partial y} = 0, \qquad \text{on } D,$$

$$p_1^* n_x + p_2^* n_y = \frac{\partial v}{\partial n}, \qquad \text{on} \quad C$$

where $\frac{\partial v}{\partial n}$ is a given function on C. The upper bound in this last inequality is just Kelvin's minimum energy theorem, as remarked both by Synge [1] and Diaz and Weinstein [1], [2]. This upper bound was first formulated in terms of a single function (a «stream function») by Diaz and Weinstein [2]. As to the lower bound, it can be shown at once that it is independent of the functions p_1^*, p^2 employed. Because, by Green's theorem and the conditions satisfied by p_1^*, p_2^*, it follows that

$$\int_D [p_1^* w_x'' + p_2^* w_y''] \, dx \, dy = \int_C w'' \frac{\partial v}{\partial n} \, ds,$$

and the lower bound then coincides with the lower bound for Dirichlet's integral in Neumann's problem, in terms of a single arbitrary non-constant function, which was already given in Diaz and Weinstein [1]. A similar remark applies to the inequality for the Dirichlet integral of a solution v of Dirichlet's problem, which follows from table I on page 455 of Synge [1]. The inequality is

$$\frac{\left(\int_D [w_x^* p_1'' + w_y^* p_2''] \, dx \, dy\right)^2}{\int_D [(p_1'')^2 + (p_2'')^2] \, dx \, dy} \leq \int_D (v_x^2 + v_y^2) \, dx \, dy \leq \int_D [(w_x^*)^2 + (w_y^*)^2] \, dx \, dy$$

where the function w^* satisfies

$$w^* = v, \quad \text{on} \quad C,$$

v being a given function on C, while the functions p_1'' and p_2'' satisfy

$$\frac{\partial p_1''}{\partial x} + \frac{\partial p_2''}{\partial y} = 0, \quad \text{on} \quad D,$$

and $\int_D [(p_1'')^2 + (p_2'')^2]\, dx\, dy > 0$. The upper bound is Dirichlet's principle, while it can be shown at once that the lower bound is independent of w^*. Because, by Green's theorem, and the conditions satisfied by w^*, p_1'' and p_2'', it follows that

$$\int_D [w_x^* p_1'' + w_y^* p_2'']\, dx\, dy = \int_C v \cdot [p_1'' n_x + p_2'' n_y]\, ds,$$

and the lower bound then coincides with that given by inequality (41) of the present paper.

In a later paper, J. L. Synge [3] presents his general geometrical scheme.

L. Broglio [1] has obtained maximum and minimum principles for solutions of various linear boundary value problems, all of which can be obtained readily from the results of the present paper, and some of them can be improved. Only an example need be mentioned. Consider Neumann's problem

$$\Delta v = 0, \quad \text{on} \quad D,$$

$$\frac{\partial v}{\partial n} = f, \quad \text{on} \quad C,$$

where f is a given function on C. On page 6 (teorema A) of Broglio [1], an inequality is given, which, for the two dimensional case, reads:

$$\int_D [(w_x^*)^2 + (w_y^*)^2]\, dx\, dy \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

where the function w^* satisfies

$$\Delta w^* = 0, \quad \text{on} \quad D,$$

$$\int_D [(w_x^*)^2 + (w_y^*)^2]\, dx\, dy = \int_C w^* \frac{\partial v}{\partial n}\, ds.$$

Since, given any non-constant harmonic function w, the function λw, where

$$\lambda = \frac{\int_C w \frac{\partial v}{\partial n} ds}{\int_D (w_x^2 + w_y^2)\, dx\, dy},$$

may be taken as w^*, the lower bound for the Dirichlet integral of v given by BROGLIO is seen to be just

$$\frac{\left(\int_C w \frac{\partial v}{\partial n} ds\right)^2}{\int_D (w_x^2 + w_y^2)\, dx\, dy} \leq \int_D (v_x^2 + v_y^2)\, dx\, dy,$$

which is thus a special case of the lower bound for $\int_D (v_x^2 + v_y^2)\, dx\, dy$ given by DIAZ and WEINSTEIN [1] (see also (34) of the present paper), where the function w is an arbitrary non-constant function, and not required to be harmonic.

The results of Topolyanskii [2], which are patterned exactly after TREFFTZ [1], also follow readily.

The bibliography contains references to papers, dealing with related questions, which have come to the writer's attention. In some instances, their results can be obtained rapidly by following the unified approach of the present paper.

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