EXTENSION OF CROSS-SECTIONS AND A GENERALIZED DE RHAM THEOREM

by

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RESUMEN

En este artículo se estudia un problema de extensión de secciones diferenciables y se aplica el resultado obtenido en la demostración de un teorema de de Rham generalizado, que permite utilizar formas diferenciales para describir la cohomología de ciertos subespacios de variedades diferenciables.

INTRODUCTION

Let $\xi: E \xrightarrow{\pi} M$ be a smooth vector bundle and let $\mathcal{F} = \{F_i\}_{i\in I}$ be a finite family of smooth submanifolds of M. We study the following problem: which conditions should we impose on the family $\mathcal{F}$ so that any given family $\{\sigma_i\}_{i\in I}$ of smooth cross-sections σ_i of $\xi|_{F_i}$ (such that σ_i coincides with σ_j on the intersection $F_i \cap F_j$, $i, j \in I$) could be extended to a smooth cross-section σ of ξ?

We adopt, for convenience, the following *convention:* a manifold is understood to be a disjoint union of connected Hausdorff smooth manifolds with countable basis of open sets.

It is easy to see the necessity of the following conditions:

a) F_i is closed for all $i \in I$.

b) The intersection of any two manifolds of our family is again a submanifold of M.

Thus, we may assume that all manifolds in $\mathcal{F}$ are distinct and the intersection of any two members of $\mathcal{F}$ is either a member of $\mathcal{F}$ or the empty set. We say that $\mathcal{F}$ is an $\cap$*-family.*

c) If $F_i = F_j \cap F_k$ $(i, j, k \in I)$ and $z \in F_i$ then

$$T_z(F_i) = T_z(F_j) \cap T_z(F_k).$$

d) If $z \in \bigcup_{i \in I} F_i$, the family of subspaces $\{T_z(F_i)\}_{F_i \ni z}$ of $T_z(M)$ verifies:

$$T_z(F_i) \cap \sum_{j \in J} T_z(F_j) = \sum_{j \in J} T_z(F_i) \cap T_z(F_j), \quad i \in I, \quad J \subset I.$$

We prove in this article that the above conditions are also sufficient (theorem 2).

Theorem (2) is used mainly in the proof of a de Rham theorem (theorem 11) giving a description of the cohomology of $\bigcup_{i \in I} F_i$ with real coefficients in terms of differential forms.

Finally we show that an example of a family $\mathcal{F} = \{F_i\}_{i \in I}$ as above is provided by the «singular set» of a smooth toral action.

§ 1. Extension of crooss-sections.

(1) A family $\mathcal{F} = \{F_i\}_{i \in I}$, of submanifolds of a manifolds M, verifying the above properties a),b),c),d) is called in this paper a *net-work of submanifolds* of M.

(2) *Theorem*

Let $\xi \colon E \xrightarrow{\pi} M$ be a smooth real vector bundle and let $\mathcal{F} = \{F_i\}_{i \in I}$ be a net-work of submanifolds of M. Suppose that we are given smoth cross-sections $\sigma_i \in Sec\ \xi|_{F_i}$ for all $i \in I$, such that they agree on the intersections. There exists then a smooth cross-section $\sigma \in Sec\ \xi$ such that coincides with σ_i when restricted to F_i, for all $i \in I$.

The proof of this theorem is postponed to (8) after we have proved some preliminary lemmas.

Let $\zeta \colon N \xrightarrow{\varrho} B$ be a smooth real vector bundle and let $\zeta = \zeta_1 \oplus \ldots \oplus \zeta_n$ be a direct sum decomposition into subbundles $\zeta_i \colon N_i \xrightarrow{\varrho_i} B$.

If $I \subset \{1, \ldots, n\}$, we write $N_I = \bigoplus_{i \in I} N_i$.

$\varrho_I: N \to N_I$ is also a vector bundle, where ϱ_I is the projection given by $\varrho_I(z_1 \oplus \ldots \oplus z_n) = \bigoplus_{i \in I} z_i$.

The next lemma is taken essentially from the thesis of C-Watkiss ([8; lemma 1.5, page 103]).

Suppose that we have smooth maps $f_i: N_{I_i} \to \mathbf{R}^m$, $i = 1, \ldots, r$, where $\{I_1, \ldots, I_r\}$ is a family of subsets of $\{1, \ldots, n\}$.

Assume that $f_i|_{N_{I_i \cap I_j}} = f_j|_{N_{I_i \cap I_j}} \quad i, j = 1, \ldots, r$.

Define $f: N \to \mathbf{R}^m$ by

$$f(z) = \sum_{p=1}^{r} (-1)^{p+1} \Big(\sum_{1 \leq i_1 < \ldots < i_p \leq r} f_{i_1} (\varrho_{I_{i_1} \cap \ldots \cap I_{i_p}} (z)) \Big).$$

(3) *Lemma*

a) With the above hypotheses, f is a smooth extension of the functions f_i to all of N.

b) Assume further that we are given a smooth function $h: \varrho^{-1}(u) \to \mathbf{R}^m$ (U is an open set in B) such that $h|_{N_{I_i} \cap \varrho^{-1}(U)} = f_i|_{N_{I_i} \cap \varrho^{-1}(U)}$, $i = 1, \ldots, r$.

Then, for every open set V in B such that $\overline{V} \subset U$, there exists a smooth map $\widetilde{f}: N \to \mathbf{R}^m$ such that

$$\widetilde{f}_{|N_{I_i}} = f_i, \; i = 1, \ldots, r \text{ and } \widetilde{f}_{|\varrho^{-1}(V)} = h_{|\varrho^{-1}(V)}.$$

PROOF:

a) f is clearly smooth and it is an easy computation to check that $f(z) = f_k(z)$ for all $z \in N_{I_k}$.

b) We know by (a) that there exists a smooth extension f of the functions f_i to all of N. Let (λ, μ) be a smooth partition of unity subordinate to the open covering $(\varrho^{-1}(U), N - \varrho^{-1}(\overline{V}))$. Then $\widetilde{f} = \lambda h + \mu f$ has the required properties.

(4) *Lemma*

Suppose given a manifold M, a family $\{I_1, \ldots, I_r\}$ of subsets of $\{1, \ldots, n\}$ and smooth maps $f_i: N_{I_i} \to M$, $i = 1, \ldots, r$.

Assume that $f_i|_{N_{I_i \cap I_j} \cap W_0} = f_j|_{N_{I_i \cap I_j} \cap W_0} \quad i, j = 1, \ldots, r$, where W_0 is some open neighbourhood of B in N (we identify B with its image under the o-cross-section).

a) There exist then a smooth map $f: N \to M$ and an open neighbourhood W of B in N such that

$$f_{|N_{I_i} \cap W} = f_i|_{N_{I_i} \cap W}, \qquad i = 1, \dots r.$$

b) Assume further that we are given a smooth map $h: \varrho^{-1}(U) \to M$ (U is an open set in B) such that

$$f_i|_{N_{I_i} \cap W_0} = h_{|N_{I_i} \cap W_0}, \qquad i = 1, \dots, r.$$

Then, for every open set V in B with $\bar{V} \subset U$, there exists a smooth map $\tilde{f}: N \to M$ and a neighbourhood W of B in N such that

$$\tilde{f}_{|N_{I_i} \cap W} = f_i|_{N_{I_i} \cap W}, \qquad i = 1, \dots, r \text{ and}$$

$$\tilde{f}_{|\varrho^{-1}(V) \cap W} = h_{|\varrho^{-1}(V) \cap W}.$$

PROOF:

a) We may assume that M is a closed submanifold of some $\mathbf{R}^m$.

Construct a smooth function $\lambda: N \to \mathbf{R}^{>0}$ sot hat $q(z) = \lambda(z) \,.\, z$ satisfies:

1) $q(N) \subset W_0$

2) $q(z) = z$ for all z in some neighbourhood W' of B in N.

Define then smooth maps $f_i' = f_i \circ q$, $i = 1, \dots, r$ and observe that f_i' and f_j' coincide on $N_{I_j \cap I_i}$ $i, j = 1, \dots, r$.

Therefore, by lemma (3)a, there exists a smooth extension $\bar{f}: N \to \mathbf{R}^m$ of the f_i'.

Let $\varrho_T: T \to M$ be a tubular neighbourhood of M in $\mathbf{R}^m$. Construct $\lambda': N \to \mathbf{R}^{>0}$ smooth so that $q'(z) = \lambda'(z) \,.\, z$ satisfies:

1') $q'(N) \subset \bar{f}^{-1}(T)$

2') $q'(z) = z$ for all z in some neighbourhood $W \subset W'$ of B in N.

Then, $f = \varrho_T \circ \bar{f} \circ q'$ has the required properties.

b) Construct $q: N \to N$ as in (a) and define $h' = h \circ q$ ($h': \varrho^{-1}(U) \to M$). Then by lemma 3(b) there exists a smooth map $\hat{f}: N \to \mathbf{R}^m$ such that

$\hat{f}_{|N I_i} = f_i'$, $i = 1, \ldots, r$ and $\hat{f}_{|\varrho^{-1}(V)} = h'_{|\varrho^{-1}(V)}$, where $f_i' = f_i \circ q$.

Construct $q' : N \to N$ as in (a) but such that $q'(N) \subset \hat{f}^{-1}(T)$. Then $\tilde{f} = \varrho_T \circ \hat{f} \circ q'$ has the required properties.

Assume next that $\mathcal{F} = \{F_i\}_{i \in I}$ is a net-work of submanifolds of a manifold M with $\bigcap_{i \in I} F_i = F_0$ nonvoid, $M = F_\infty \in \mathcal{F}$. For each $i \in I$ choose N_i to be a subbundle of $\tau_{F_i|F_0}$ (restriction to F_0 of the tangent bundle of F_i, τ_{F_i}) such that

$$\Big(\sum_{k<i} \tau_{F_k|F_0}\Big) \oplus N_i = \tau_{F_i|F_0}$$

Then we have $\tau_{F_i|F_0} = \bigoplus_{k \leq i} N_k$. (We say that $i \leqslant j$; $i, j \in I$, if $F_i \subset F_j$).

Set $N^i = \bigoplus_{0<k\leq i} N_k$ for all $i \in I$.

We make the identification

$$T_x(M) = \bigoplus_{i \in I} N_i(x) \cong T_{0_x}(N^\infty), \qquad x \in F_0;$$

identifying the element $\bigoplus_{i \in I} v_i$ of $\bigoplus_{i \in I} N_i(x)$ to the element $(d\sigma_0)_x(v_0) \oplus \bigoplus_{i \in I - \{0\}} \omega_x(v_i)$ of $T_{0_x}(N^\infty)$, where $\sigma_o : F_o \to N^\infty$ is the o-cross-section and ω_x is the composition of the canonical isomorphism $N^\infty(x) \xrightarrow{\cong} T_{0_x}(N^\infty(x))$ and the injective linear map $T_{0_x}(N^\infty(x)) \to T_{0_x}(N^\infty)$ induced by the inclusion $N^\infty(x) \subset N^\infty$.

In particular, given a smooth map $f : 0 \to M$, where 0 is a neighbourhood of F_0 in N, such that $f(x) = x$ for $x \in F_0$, we consider $(df)_x$ as a linear map $(df)_x : T_x(N^\infty) \to T_x(N^\infty)$. For instance, it makes sense to say that f verifies $(df)_x =$ identify.

Fix an open set V in F_0 such that $\bar{V} \subset U$.

(5) *Lemma*

Assume that we are given smooth maps $f_i : N_i \to F_i$, $i \in I$, such that $f_i|_{F_0} =$ identity.

Then, there exist an open neighbourhood W of F_0 in N^∞ and a smooth map $f : N^\infty \to M$ such that

i) $f_{|W \cap N_i} = f_i|_{W \cap N_i}$, $i \in I$.

ii) $f(W \cap N^i) \subset F_i$

PROOF:

For each $i, j \in I - \{0\}$, let F_{ij} be the minimum submanifold in $\mathcal{F}$ containing both F_i and F_j.

By lemma (4) b there exist an open neighbourhood W_{ij} of F_0 in $N_i \oplus N_j$ and a smooth map $f_{ij}: N_j \oplus N_i \to F_{ij}$ such that

$$f_{ij}|_{W_{ij} \cap N_i} = f_i|_{W_{ij} \cap N_i}, \ f_{ij}|_{W_{ij} \cap N_j} = f_j|_{W_{ij} \cap N_j}.$$

For each triple i, j, k in $I - \{0\}$, let F_{ijk} be the minimum submanifold in $\mathcal{F}$ containing F_i, F_j, F_k.

By lemma (4) b there exist an open neighbourhood W_{ijk} of F_0 in $N_i \oplus N_j \oplus N_k$ and a smooth map

$$f_{ijk}: N_i \oplus N_j \oplus N_k k \to F_{ijk}$$

such that

$$f_{ijk}|_{W_{ijk} \cap (N_i \oplus N_j)} = f_{ij}|_{W_{ijk} \cap (N_i \oplus N_j)}$$

and similarly for ik, jk.

Continuing this process we finally obtain a smooth map $f: N^\infty \to M$ and an open neighbourhood W of F_0 in N^∞ having the required properties.

(6) *Lemma*

There exist an open neighbourhood U_0 of F_0 in M and a diffeomorphism $f: N^\infty \to U_0$ such that $f(N^i) = U_0 \cap F_i$ for all $i \in I$, $f_{|F_0} =$ identity, $(df)_x =$ identity for all $x \in F_0$.

PROOF:

Fix any Riemannian metric in M and denote by $exp^i: O_i \to F_i$ the exponential of the restriction of the metric to F_i, where O_i is an open neighbourhood of F_i in T_{F_i} (total space of τ_{F_i}).

Choose smooth maps $\lambda_i: N_i \to \mathbf{R}^{>0}$ and define $q_i: N_i \to N_i$ by $q_i(z) = \lambda_i(z) . z$ such that $q_i(N_i) \subset N_i \subset O_i$ and $q_i(z) = z$ for all z in some open neighbourhood $W_i \subset N_i \cap O_i$ of F_0 in N_i.

Define $f_i: N_i \to F_i$ by $f_i = exp^i \circ q_i$. Since $f_i|_{F_0} =$ identity, we deduce from lemma (5) that there exist an open neighbourhood W^0 of F_0 in N^∞ and a smooth map $\bar{f}: N^\infty \to M$ such that $\bar{f}_{|W^0 \cap N_i} = f_i|_{W^0 \cap N_i}$, $i \in I$, and $\bar{f}(W^0 \cap N_i) \subset F_i$, $i \in I$.

But $\bar{f}_{|F_0}$ = identity and if $x \in F_0$ and $v_i \in N_i(x)$, $i \in I$, $(d\bar{f})_x (\bigoplus_{i \in I} v_i) =$ $= v_0 \oplus \sum_{i \in I} (d\, exp^i)_x(v_i) = \bigoplus_{i \in I} v_i$.

In particular $\bar{f}$ is a local diffeomorphism on some neighbourhood of F_0 in N^∞, and then in view of [6; lemma 5.7, § 5] there exists an open neighbourhood $U^0 \subset W^0$ of F_0 in N^∞ such that $\bar{f}$ restricts to a diffeomorphism $\bar{f}: U^0 \to U'^0$.

Then $\bar{f}: U^0 \cap N^i \to F_i$ is a diffeomorphism onto an open set O'_i.

Choose an oepn neighbourhood O of F_0 in U'^0 such that $O \cap F_i \subset$ $\subset O_i'$ for all $i \in I$.

Choose open 1-disc-bundles D_i of N_i (with respect to some Riemannian bundle-metric on N_i) such that

$$\bigoplus_{i \in I - \{0\}} D_i \subset \bar{f}^{-1}(O) \cap U^0.$$

Construct diffeomorphisms $q_i'\, N_i \to D_i$, $q_i'(z) = \lambda_i'(z) \,.\, z$ for some smooth maps $\lambda_i' : N_i \to \mathbf{R}^{>0}$, which are the identity in some neighbourhood of F_0 in N_i.

Therefore we obtain we obtain a diffeomorphism

$$q' = \bigoplus_{i \in I - \{0\}} q_i' : N^\infty \to \bigoplus_{i \in I - \{0\}} D_i .$$

Finally, $f = \bar{f} \circ q'$, $U_0 = \bar{f}(\bigoplus_{i \in I - \{0\}} D_i)$ have the required properties.

(7) *Lemma*

Let $\zeta_i : E_i \xrightarrow{\pi_i} B$ $i = 1, \dots, n$ be real vector bundles. Denote its direct sum $\bigoplus_{i=1}^{\eta} \zeta_i$ by $\zeta : E \xrightarrow{\pi} B$, and let $\eta : N \xrightarrow{\varrho} E$ be a real vector bundle over E. Assume that we are given cross-sections $\sigma_i \in \operatorname{Sec} \eta_{|E_{I_i}}$ $i = 1, \dots, r$, where $\{I_1, \dots, I_r\}$ is a family of subsets of $\{1, \dots, n\}$, such that they agree on the intersections.

Then, there exists a smooth cross-section $\sigma \in \operatorname{Sec} \eta$ such that $\sigma_{|E_{I_i}} = \sigma_i$, $i = 1, \dots, r$.

PROOF:

Observe that if U is an open subset of B, then η trivializes on $\pi^{-1}(U)$ if and only if $\eta_{|U}$ trivializes. Therefore, there exists a locally finite open covering $\{U_\alpha\}_{\alpha \in A}$ of B such that we have trivializations.

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \times H \to \varrho^{-1}(\pi^{-1}(U_\alpha))$$

so that $\varrho\varphi_\alpha(z, h) = z$ for all $z \in \pi^{-1}(U_\alpha)$, $h \in H$ (H is the typical fibre of η).

Define $E_{I_i} \cap \pi^{-1}(U_\alpha) \xrightarrow{f_i^\alpha} H$ by the relation $\varphi_\alpha(z, f_i^\alpha(z)) = \sigma_i(z)$ for all $z \in E_{I_i}$ with $\pi z \in U_\alpha$.

If $z \in E_{I_i} \cap \pi^{-1}(U_\alpha) \cap E_{I_j} \cap \pi^{-1}(U_\alpha) = E_{I_i \cap I_j} \cap \pi^{-1}(U_\alpha)$ we have

$$\varphi_\alpha(z, f_i^\alpha(z)) = \sigma_i(z) = \sigma_j(z) = \varphi_\alpha(z, f_j^\alpha(z)).$$

Thus $f_j^\alpha(z) = f_j^\alpha(z)$. Therefore, by lemma (3) a, there exists f^α : $\pi^{-1}(U_\alpha) \to H$ such that $f^\alpha|_{E_{I_i} \cap \pi^{-1}(U_\alpha)} = f_i^\alpha$.

Define $\sigma^\alpha : \pi^{-1}(U_\alpha) \to \varrho^{-1}(\pi^{-1}(U_\alpha))$ by $\sigma^\alpha(z) = \varphi_\alpha(z, f^\alpha(z))$.

We have $\varrho\sigma^\alpha(z) = \varrho\varphi_\alpha(z, f^\alpha(z)) = z$ and hence $\sigma^\alpha \in \operatorname{Sec} \eta_{|\pi^{-1}(U_\alpha)}$.

If $z \in \pi^{-1}(U_\alpha) \cap E_{I_i}$ we have

$$\sigma^\alpha(z) = \varphi_\alpha(z, f^\alpha(z)) = \varphi_\alpha(z, f_i^\alpha(z)) = \sigma_i(z).$$

Choose now a smooth partition of unity $\{\lambda_\alpha\}_{\alpha \in A}$ subordinate to the open covering $\{U_\alpha\}_{\alpha \in A}$ and define $\sigma \in \operatorname{Sec} \eta$ by $\sigma = \sum_\alpha \lambda_\alpha \sigma_\alpha$. This σ has the required properties.

(8) *Proof of theorem* 2

By lemmas 6 and 7 there exist an open neighbourhood U^1 of $\bigcup_{i \in I_1} F_i$ (I_1 is the set of minimal elements of I) and a cross-section $\sigma^1 : U^1 \to E$, such that $\sigma^1|_{F_i \cap U^1} = \sigma_i|_{F_i \cap U^1}$ for all $i \in I$.

Choose V^1 open in M such that $\bigcup_{i \in I_1} F_i \subset V^1 \subset \overline{V}^1 \subset U^1$. Consider $M_1 = M - \overline{V}^1$ and the family $\{F_i \cap M_1\}_{i \in I - I_1}$.

By lemmas 6 and 7 there exist an open neighbourhood U^2 of $(\bigcup_{i \in I_2} F_i) \cap M_1$ in M_1 and a cross-section $\sigma^2 : U^2 \to E$ such that $\sigma^2|_{F_i \cap M_1 \cap U^2} = \sigma_i|_{F_i \cap M_1 \cap U^2}$, $i \in I - I_1$ (I_2 is the set of minimal elements of $I - I_1$).

Choose V^2 open in M_1 such that $(\bigcup F_i) \cap M_1 \subset V^2 \subset \overline{V}^2 \subset U^2$ (where $\overline{V}^2$ is now the closure of V^2 in M_1). Consider $M_2 = M_1 - \overline{V}^2$ (open submanifold of M_1 and hence of M) and the family $\{F_i \cap M_2\}_{i \in I - I_1 \cup I_2}$

Continuing this process we define inductively:

i) A family of open sets in $M : U^1, U^2, \ldots$

ii) A family of open submanifolds $M = M_0 \supset M_1 \supset M_2 \supset \ldots$

iii) A family of open sets $V^1, V^2, \ldots$

iv) A family of smooth cross-sections $\sigma^k : U^k \to E\,(k = 1, 2, \ldots)$ such that $(\bigcup_{i \in I_k} F_i) \cap M_{k-1} \subset V^k \subset \bar{V}^k$ (closure in M_{k-1}) $\subset U^k \subset M_{k-1}$, $M_k = M - \bar{V}^k$, $k = 1, 2, \ldots$ (I_k is defined inductively as the set of minimal elements of $I - \bigcup_{i<k} I_i$), $\sigma^k|_{F_i \cap M_{k-1} \cap U^k} = \sigma_i|_{F_i \cap M_{k-1} \cap U^k}$ for all $k \geqslant 1$ and all $i \in I - \bigcup_{t=0}^{k-1} I_t$ $(I_0 = \varnothing)$.

Observe that $\{U^k\}_{k \geq 1}$ is an open covering of $\bigcup_{i \in I} F_i$. Define $\bar{\sigma} : U = \bigcup_{k \geq 1} U^k \to E$ by $\bar{\sigma}(z) = \sum_{k \geq 1} \lambda^k(z)\, \sigma^k(z)$, where $\{\lambda^k\}$ is a smooth partition of unity subordinate to the open covering $\{U^k\}_{k \geq 1}$. It is clear that $\bar{\sigma}$ is a smooth cross-section on U and if $x \in F_i$ we have $\bar{\sigma}(x) = \sigma_i(x)$ for all $x \in F_i$.

Finally, choose an open set W such that $\bigcup_{i \in I} F_i \subset W \subset \overline{W} \subset U \subset M$ and let $\{\lambda, \mu\}$ be a smooth partition of unity subordinate to the open covering $\{W, M - \bigcup_{i \in I} F_i\}$. Then, $\sigma(x) = \lambda(x)\ \bar{\sigma}(x)$ if $x \in U$ and $\sigma(x) = O_x$ if $x \in M - \overline{W}$, has the required properties.

(9) *Corollary*

Let M, N be smooth manifolds and $\mathcal{F} = \{F_i\}_{i \in I}$ a network of submanifolds of M. Assume that we are given a family of smooth maps $f_i : F_i \to N$ such that they agree on the intersections.

Then, there exist an open neighbourhood U of $\bigcup_{i \in I} F_i$ and a smooth map $f : U \to N$ such that $f_{|F_i} = f_i$, for all $i \in I$.

Proof:

We may assume that N is a closed submanifold of some Euclidean space $\mathbf{R}^m$. Let $\varrho : T \to N$ be a tubular neighbourhood of N in $\mathbf{R}^m$ (cf. [5; chapter 4, theorem 5.2]). Then, by theorem 2, we can find a smooth map $\bar{f} : M \to \mathbf{R}^m$ such that $\bar{f}_{|F_i} = f_i$ for all $i \in I$.

The open set $U = \bar{f}^{-1}(T)$ and the map $f = \varrho \circ \bar{f} : U \to N$ have the required properties.

(10) *Remark*

Let M, N be smooth manifolds and A a subset of M. A map $f: A \to N$ is said to be smooth if and only if it is the restriction of a smooth map $\tilde{f}: U \to N$ defined on some open neighbourhood U of A in M.

The above corollary can be regarded as saying that in case $A = \bigcup_{i \in I} F_i$, where $\{F_i\}_{i \in I}$ is a net-work of submanifolds of M, then $f: A \to N$ is smooth if and only if $f_{|F_i}$ is smooth for all $i \in I$.

§ 2. De Rham theorem for networks of submanifolds.

Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a net-work of submanifolds of a manifold M.

Define $A^p(\mathcal{F}) \subset \bigoplus_{i \in I} (F_i)$ as follows

$(\Phi_i)_{i \in I} \in A^p(\mathcal{F})$ if and only if the p-forms Φ_i agree on the intersections.

Clearly $A^*(\mathcal{F}) = \bigoplus_p A^p(\mathcal{F})$ is a differential graded subalgebra of $\bigoplus_{i \in I} A^*(F_i)$.

Thus we have the cohomology $H^*_{dR}(\mathcal{F})$ of $(A^*(\mathcal{F}), d)$, wich is a commutative graded algebra.

(11) *Theorem* (Generalized de Rham isomorphism).

There is a natural isomorphism of graded algebras

$$H^*_{dR}(\mathcal{F}) \xrightarrow{\cong} H^*_{Sing.}(\bigcup_{i \in I} F_i; \mathbf{R})$$

Proof:

Denote by $S^\infty_p(F_i)$ the set of smooth p-simplexes of F_i and by $S_p(F_i)$ the set of (continuous) p-simplexes of F_i.

Define $C^p_\infty(\mathcal{F})$ (resp. $C^p(\mathcal{F})$) as the set of p-singular cochains with real coefficients of $S^\infty_p(\mathcal{F}) = \bigcup_{i \in I} S^\infty_p(F_i)$ (resp. $S_p(\mathcal{F}) = \bigcup_{i \in I} S_p(F_i)$).

Then $C^*_\infty(\mathcal{F})$ and $C^*(\mathcal{F})$, with the corresponding coboundary operators, are cochain algebras and we denote their cohomology by $H^*_\infty(\mathcal{F})$ and $H^*(\mathcal{F})$; with the cup product they are commutative graded algebras.

Moreover, restriction defines homomorphisms

$$C^*(\bigcup_{i\in I} F_i) \xrightarrow{\varphi_1} C^*(\mathcal{F}) \xrightarrow{\varphi_2} C^*_\infty(\mathcal{F})$$

of cochain algebras. They induce homomorphisms

$$H^*_{Sing.}(\bigcup_{i\in I} F_i;\, \mathbf{R}) \xrightarrow{\varphi^*_1} H^*(\mathcal{F}) \xrightarrow{\varphi^*_2} H^*_\infty(\mathcal{F})$$

On the other hand, define

$$\varphi_3 : A^q(\mathcal{F}) \longrightarrow C^p_\infty(\mathcal{F})$$

by

$$\varphi_3((\Phi_i)_{i\in I})\,(\sigma) = \int_\sigma \Phi_i \text{ if image } \sigma \subset F_i \text{ and } \sigma \text{ is smooth.}$$

Stokes theorem shows that φ_3 commutes with δ and thus φ_3 induces a linear map in cohomology.

We proof now theorem 11 by showing that we have isomorphisms

$$H^*_{Sing.}(\bigcup_{i\in I} F_i;\, \mathbf{R}) \underset{\cong}{\xrightarrow{\varphi^*_1}} H^*(\mathcal{F}) \underset{\cong}{\xrightarrow{\varphi^*_2}} H^*(\mathcal{F}) \underset{\cong}{\xleftarrow{\varphi^*_3}} H^*_{dR}(\mathcal{F})$$

i) φ^*_1 is a graded algebra isomorphism.

It is clear that φ^*_1 is a graded algebra homomorphism, so it remains to be shown that φ^*_1 is a linear isomorphism. We prove this fact through 3 steps labelled a), b) and c).

a) Let F_r be a maximal submanifold in the family $\mathcal{F}$, then the linear map $H^*_{Sing.}(\bigcup_{i\in I} F_i, \bigcup_{i\in I-\{r\}} F_i) \longrightarrow H^*_{Sing.}(F_r; \bigcup_{i<r} F_i)$ induced by the inclusion of pairs is an isomorphism.

We recall the following definition (cf. [2; page 285]):

Let X be a topological space and X_1, X_2 subspaces of X. We say that $X_1 \cap X_2$ *separates* X_1, X_2 if $X_2 - X_1$ and $X_1 - X_2$ are both open in $X_1 \cup X_2 - X_1 \cap X_2$.

Observe that $\bigcup_{i\in I-\{r\}} F_i$ and F_r are separated by $\bigcup_{i<r} F_i$. Set $X_1 = \bigcup_{i\in I-\{r\}} F_i$, $X_2 = F_r$, $X = \bigcup_{i\in I} F_i$ and consider the commutative diagram ($\check{H}$ means Čech cohomology)

$$\begin{array}{ccc} \check{H}(X, X_1) & \longrightarrow & \check{H}(X_2, X_1 \cap X_2) \\ \downarrow & & \downarrow \\ H^*_{Sing.}(X, X_1) & \longrightarrow & H^*_{Sing.}(X_2, X_1 \cap X_2) \end{array}$$

Since $\bigcup_{i\in I} F_i$ is an Euclidean neighbourhood retract (see proposition 13), the vertical arrows are isomorphisms by [2; proposition 6.12] and the upper horizontal arrow is an isomorphism by [2; VII (6.15)]. Therefore the lower horizontal arrow is also an isomorphism.

b) Let F_r be a maximal of $\mathcal{F}$ as before and denote by $\mathcal{U}$ the covering of $\bigcup_{i\in I} F_i$, $\mathcal{U} = (\bigcup_{i\in I-\{r\}} F_j, F_r)$. Denote by $H^*(\bigcup_{i\in I} F_i; \mathcal{U})$ the singular cohomology coming from cochains defined on simplexes that have image in one of the two sets of our covering. (It is not an open covering!). It follows from a) and [7; theorem 4, chapter 4] that the natural linear map

$$H^*_{Sing.}(\bigcup_{i\in I} F_i; \mathbf{R}) \underset{\Rightarrow}{\cong} H^*(\bigcup_{i\in I} F_i; \mathcal{U}) \tag{12}$$

is an isomorphism.

c) We prove finally that $\varphi^*{}_1$ is an isomorphism by induction on the number of elements of I. If I has only one element this is trivial since the map is the identity.

Assume that we know that c) holds for I with at most $r-1$ elements ($r \geqslant 2$) and assume now that I has r elements, and choose a maximal F_r.

We clearly have a row exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\bigcup_{i\in I} F_i; \mathcal{U}) & \longrightarrow & C^*(\bigcup_{i\in I-r} F_i) \oplus C^*(F_r) & \longrightarrow & C^*(\bigcup_{i<r} F_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^*(\mathcal{F}) & \longrightarrow & C^*(\mathcal{F}_1) \oplus C^*(F_r) & \longrightarrow & C^*(\mathcal{F}_2) & \longrightarrow & 0 \end{array}$$

where $\mathcal{F}_1 = \{F_i\}_{i\in}$, $\mathcal{F}_2 = \{F_i\}_{i<r}$.

Next, induction hypothesis and the 5-lemma applied to the corresponding cohomology diagram tell us that we have an isomorphism

$$H^*(\bigcup_{i\in I} F_i;\ \mathcal{U}) \underset{\rightarrow}{\cong} H^*(F).$$

Combine this with (12) to achieve the proof of i).

ii) $\varphi^*{}_2$ is a graded algebra isomorphism.

As before, it is clear that $\varphi^*{}_2$ is a graded algebra homomorphism. To skow that $\varphi^*{}_2$ is also a linear isomorphism we induct on the number of elements of I.

For I with only one element this is classical. Suppose that ii) is true for I with at most $r-1$ elements ($r \geqslant 2$) and I has now r elements.

Choose F_r maximal in the family $\mathcal{F}$.

We clearly have the row exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathcal{F}) & \longrightarrow & C^*(\mathcal{F}_1) \oplus C^*(F_r) & \longrightarrow & C^*(\mathcal{F}_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^*_\infty(\mathcal{F}) & \longrightarrow & C^*_\infty(\mathcal{F}_1) \oplus C^*_\infty(\mathcal{F}_r) & \longrightarrow & C^*_\infty(\mathcal{F}_2) & \longrightarrow & 0
\end{array}$$

where $\mathcal{F}_1 = \{F_i\}_{i\in I-\{r\}}$, $\mathcal{F}_2 = \{F_i\}_{i<r}$.

Finally, the 5-lemma and induction hypothesis applied to the corresponding cohomology diagram finish the proof of ii).

iii) $\varphi^*{}_3$ is a linear isomorphism.

We induct on the number of elements of I.

If I has only one elements F, we have $H^*_{dR}(\mathcal{F}) = H^*_{dR}(F)$, $H^*_\infty(\mathcal{F}) = H^*_\infty(F)$ and $\varphi^*{}_3$ is the de Rham isomorphism.

Suppose that iii) is true for I with at most $r-1$ elements ($r \geqslant 2$), and I has now r elements.

Choose F_r maximal in the family $\mathcal{F}$ and let $\mathcal{F}_1 = \{F_i\}_{i\in I-\{r\}}$, $\mathcal{F}_2 = \{F_i\}_{i<r}$ as before.

The following is a short exact sequence of cochain complexes:

$$0 \longrightarrow A^*(\mathcal{F}) \xrightarrow{f_1} A^*(\mathcal{F}_1) \oplus A^*(F_r) \xrightarrow{f_2} A^*(\mathcal{F}_2) \longrightarrow 0$$

where $f_1((\Phi_i)_{i\in I}) = ((\Phi_i)_{i\in I-\{r\}}, \Phi_r)$ and

$$f_2((\Phi_i)_{i\in I-\{r\}}, \Phi_r) = (\Phi_i - \Phi_r|_{F_i})_{i<r}.$$

In fact, the only non trivial part is the surjectivity of f_2 and this is consequence of theorem 2 applied to the net work $\mathcal{F}_2$ in F_r and the bundle $\Lambda r^*_{F_r}$.

Consider next the row exact commutative diagram of cochain complexes

$$\begin{tikzcd} 0 \arrow[r] & A^*(\mathcal{F}) \arrow[r] \arrow[d] & A^*(\mathcal{F}_1) \oplus A^*(F_r) \arrow[r] \arrow[d] & A^*(\mathcal{F}_2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^*_\infty(\mathcal{F}) \arrow[r] & C^*_\infty(\mathcal{F}_1) \oplus C^*(F_r) \arrow[r] & C^*(\mathcal{F}_2) \arrow[r] & 0 \end{tikzcd}$$

The 5-lemma and induction hypothesis applied to the corresponding cohomology diagram finish the proof of iii).

iv) $\varphi^*{}_3$ is an algebra homomorphism.

Consider the simplicial set $S_\infty(\mathcal{F})$ of smooth singular simplexes of $\bigcup_{i \in I} F_i$ with images contained in some F_i. Thus, we can form the commutative graded differential algebra $A(S_\infty(\mathcal{F}), \delta)$ (cf. [3; 13.5]).

The following diagram is commutative, where the non-labelled maps are the obvious ones:

$$\begin{tikzcd} A^*(S_\infty(\mathcal{F})) \arrow[r] & C^*_\infty(\mathcal{F}) \\ A^*(\mathcal{F}) \arrow[u] \arrow[ur, "\varphi_3"'] & \end{tikzcd}$$

From here we deduce the commutative triangle

$$\begin{tikzcd} H^*(A(S_\infty(\mathcal{F}), \delta) \arrow[r, "g", "\cong"'] & H^*_\infty(\mathcal{F}) \\ H^*_{dR}(\mathcal{F}) \arrow[u, "f"] \arrow[ur, "\cong \varphi^*{}_3"'] & \end{tikzcd}$$

but f is clearly an algebra homomorphism and g is an algebra isomorphism (cf. [3; property 15.6]), therefore $\varphi^*{}_3$ is an algebra isomorphism.

(13) *Proposition*

Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a finite $\cap$-family of closed submanifolds of a manifold M. Then $\bigcup_{i \in I} F_i$ is an Euclidean neighbourhood retract.

PROOF:

We recall that a topological space is called an Euclidean neighbourhood retract (ENR) if it is homeomorphic to a subspace Y of some $\mathbf{R}^n$, such that Y is a retract of some neighbourhood in $\mathbf{R}^n$ (cf. [2; page 81]).

We proof the proposition by induction on the number of elements of I.

If I has only one elements F, we may consider (by Whitney's embedding theorem) that F is a closed submanifold of some $\mathbf{R}^n$, and then we use the existence of tubular neighbourhoods to show that F is an ENR.

Suppose now that I has r elements and the proposition holds for I with less than r elements ($r \geqslant 2$).

Assume that F_r is maximal in the family $\mathcal{F}$. The two families $\mathcal{F}_1 = \{F_i\}_{i \in I - \{r\}}$, $\mathcal{F}_2 = \{F_i\}_{i<r}$ have less than r elements and hence they are ENR by induction hypothesis.

But

$$\Big(\bigcup_{i \in I - \{r\}} F_i\Big) \cap F_r = \bigcap_{i<r} F_i.$$

Thus $\bigcup_{i \in I} F_i = \Big(\bigcup_{i \in I - \{r\}} F_i\Big) \cup F_r$ is an ENR (cf. [1; chapter 4]).

3. Net-work of fixed points of toral actions

Let G be a compact abelian Lie group (in particular if G is connected it is a torus) acting amoothly on a manifold M and having finite orbit type (this happens for instance if M is compact or more generally if it has finite dimensional homology, (cf. [4; theorem 3]).

The purpose of this section is to prove the following theorem

(14) *Theorem*

$F = \{x \in M | G_x \neq e\}$ and $F^\circ = \{x \in M | G_x^\circ \neq e\}$ can be expressed as union of the members of networks of submanifolds of M.

Here, e denotes the unit element of G, G_x the isotropy subgroup at x of the action of G on M and G_x° the 1-component of G_x.

PROOF:

If H is a subgroup of G define $\widetilde{H} = \{a \in G | ax = x \text{ for all } x \in F_H\}$, where F_H denotes the fixed point set of the restricted action of H on M (i.e. $x \in F_H$ if and only if $ax = x$ for all $a \in H$).

It is clear that $H = \widetilde{H}$ if and only if $H = \bigcap_{x \in F_H} G_x$. In particular the family $\mathcal{G}$ of subgroups H of G such that $H = \widetilde{H}$, is finite. Let $\mathcal{G}^\circ$ be the 1-components of the subgroups of $\mathcal{G}$. Then we have $F^\circ \subset F$ and $F = \bigcup_{\substack{H \in \mathcal{G} \\ H \neq e}} F_H = \bigcup_{G_x \neq e} F_{G_x}$, $F^\circ = \bigcup_{\substack{H \in \mathcal{G}^\circ \\ H \neq e}} F_H = \bigcup_{G^\circ{}_x \neq e} F_{G^\circ{}_x}$.

We show now that the families $\{F_H\}_{\mathcal{G} \in H}$ and $\{F_H\}_{H \in \mathcal{G}^\circ}$ are networks.

It is well known that the sets F_H are closed submanifolds of M and it is easily checked that $\{F_H\}_{H \in \mathcal{G}}$ and $\{F_H\}_{H \in \mathcal{G}^\circ}$ are $\cap$-families.

To show that property c) of net-works is verified, we endow M with a G-invariant Riemannian metric. Observe that F_H is totally geodesic in M for all closed subgroups H of G (in particular for $H \in \mathcal{G}$ or $H \in \mathcal{G}^\circ$) and then use the following lemma.

(15) *Lemma*

If F_1 and F_2 are totally geodesic submanifolds of a Riemannian manifold M and $F_1 \cap F_2$ is also a submanifold of M, then $T_x(F_1 \cap F_2) = T_x(F_1) \cap T_x(F_2)$, $x \in F_1 \cap F_2$.

PROOF:

Trivially we have $T_x(F_1 \cap F_2) \subset T_x(F_1) \cap T_x(F_2)$. Conversely, if $h \in T_x(F_1) \cap T_x(F_2)$, let α be the geodesic in M:

$$\alpha(t) = exp_x(th).$$

By hypothesis $\alpha(t) \in F_1 \cap F_2$ and so $h = \alpha'(0) \in T_x(F_1 \cap F_2)$.

Finally, property d) of networks is easily verified since we may restrict ourselves to consider an Euclidean vector space with an orthogonal action of G.

(16) *Remark*

Since for a compact Riemannian manifold the zero-set of a Killing vector field coincides with the fixed point set of the action of a torus, we obtain from theorem 14 that the set of zeroes of the X_i (where $X_1, \ldots, X_r$ are killing vector fields with $[X_i, X_j] = 0$ for all i, j) is a net-work of submanifolds.

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