EXISTENCE OF SOLUTIONS BOUNDED AT INFINITY FOR A SEMILINEAR ELLIPTIC EQUATION ON $\mathbf{R}^N$

by

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ABSTRACT

We prove the existence of weak solutions for the equation $-\Delta u + \beta(u) \ni f$, where β is a maximal monotone graph in $\mathbf{R}$ with $0 \in \beta(0)$ and $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ is strictly bounded at infinity by the graph β.

These results continue those of Ph. Bénilan, H. Brézis and M. Crandall in the case $f \in L^1(\mathbf{R}^N)$.

0. INTRODUCTION

Let β be a maximal monotone graph (m. m. g.) in $\mathbf{R}$ with $0 \in \beta(0)$, for instance a continuous nondecreasing function on $\mathbf{R}$ such that $0 \in \beta(0)$. We shall be concerned with existence of solutions for the problem

$$(P) = (P_{\beta f}) \quad -\Delta u + \beta(u) \ni f \text{ on } \mathbf{R}^N$$

when $f \in L^1_{\text{loc}}(\mathbf{R}^N)$, $N \geqslant 1, f$ bounded at infinity.

For $f \in L^1(\mathbf{R}^N)$, (P) has been thoroughly studied by Ph. Bénilan, H. Brézis and M. Crandall in [2], their results being basic for our paper. Let us quote the existence part in [2] for reference:

$N \geqslant 3$. For every $f \in L^1(\mathbf{R}^N)$ there exists a unique $u \in M^{N/N-2}(\mathbf{R}^N)$ with $\Delta u \in L^1(\mathbf{R}^N)$, satisfying (P).

$N = 2$. Let $0 \in \text{Int}\, \beta(\mathbf{R})$. Then for every $f \in L^1(\mathbf{R}^2)$ there exists a $u \in W^{1,1}_{\text{loc}}(\mathbf{R}^2)$ with $|\text{grad}\ u| \in M^2(\mathbf{R}^2)$ and $\Delta u \in L^1(\mathbf{R}^2)$ satisfying

(P). If $\int f \neq 0$ or $\beta^{-1}(0) = \{0\}$ solutions are unique. In case of non-unicity two solutions in this class differ by a constant.

$N = 1$. Let $0 \in \operatorname{Int} \beta(\mathbf{R})$. Then for every $f \in L^1(\mathbf{R})$ there exists a $u \in W^{1,\infty}(\mathbf{R})$ with $u'' \in L^1(\mathbf{R})$ satisfying (P). Uniqueness as for $N = 2$.

Moreover the operator $T : f \to w = f + \Delta u$ on $L^1(\mathbf{R}^N)$ is an ordered contraction: with appropriate definition the composition $-\Delta \circ \beta^{-1}$ is T-accretive in $L^1(\mathbf{R}^N)$.

In the following we call these B. Br. Cr.'s solutions *usual* solutions for (P) on $\mathbf{R}^N$.

The paper begins by deriving an L^∞-estimate for large x for the usual solutions of (P) when $f \in L^1(\mathbf{R}^N)$ is «strictly bounded at infinity by β» in the following sense: let $\beta^+ = \lim_{r\to\infty} \beta(r)$, $\beta^- = \lim_{r\to-\infty} \beta(r)$ be the extrema of β. Then

$$(C)\ \beta^- < \liminf_{|x|\to\infty} f \leqslant \limsup_{|x|\to\infty} f < \beta^+$$

This allows for instance to prove that $u \to 0$ uniformly at infinity when $f \in L^1(\mathbf{R}^N)$ converges to 0 uniformly and we obtain an estimate of the convergence.

In a second section we apply these estimates to obtain weak solutions of (P) when $f \in L^1_{loc}(\mathbf{R}^N)$ and satisfies condition (C), as (repeated) monotone limits of usual solutions for approximate problems. The solutions thus obtained inherit the comparison properties of usual solutions and are themselves usual if $f \in L^1(\mathbf{R}^N)$ and are bounded at infinity.

In [3] Ph. Bénilan and M. Crandall prove that $-\Delta\circ\beta^{-1}$ suitably defined is accretive in $L^1[(1 + (x)^2)^{-\alpha}]$ for $0 < \alpha < \dfrac{N-2}{2}$, $N \geqslant 1$ (and so in particular solutions are unique in that functional setting). As constants do not belong to this space, our work deals on a more general situation.

We shall follow to a large extent the notations in [2]. Thus B will be a ball in $\mathbf{R}^N$, $B_R(x)$ the ball of radius $R > 0$ centered at x, $\Omega \subset \mathbf{R}^N$ an open set, $\partial\Omega$ its boundary, $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$ the Lebesgue spaces, $L^p_{loc}(\Omega)$ the corresponding local spaces and $L^p_0(\mathbf{R}^N) = \{f \in L^p(\Omega)$: f has compact support in $\bar{\Omega}\}$. $M^p(\Omega)$, $1 < p < \infty$ are Marcinkiewicz spaces (see for reference [2], Appendix), $W^{k,p}(\Omega)$, $k \geqslant 0$, $1 \leqslant p \leqslant \infty$ the Sobolev spaces and $W^{k,p}_0(\Omega)$ and $W^{k,p}_{loc}(\Omega)$ have the usual definitions. If $u : \mathbf{R}^N \to \mathbf{R}$ and $\lambda \in \mathbf{R}$, $[u > \lambda]$ denotes the set $\{x : u(x) > \lambda\}$.

We make use of several classes of functions on $\mathbf{R}$ as

$$\mathcal{P} = \{p \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) : p \text{ nondecreasing})\}$$
$$\text{and } \mathcal{P}_0 = \{p \in \mathcal{P} : p(0) = 0\}$$

β being a graph, for $r \in$ Domain $(\beta) = D(\beta)$, $\beta(r)$ is in general a set. We denote: $\beta^+(r) = \sup \beta(r)$ and $\beta^-(r) = \inf \beta(r)$. Also β^{-1}, the inverse graph, may be multivalued and we set for $s \in$ Range $(\beta) = R(\beta)$, $\beta_+^{-1}(s) = \sup \beta^{-1}(s)$ and $\beta_-^{-1}(s) = \inf \beta^{-1}(s)$.

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1. Usual solutions bounded at infinity

The main result of this section is

Theorem 1: Let β be a m. m. graph in $\mathbf{R}$ such that $0 \in \beta(0)$ and $0 \in$ Int $\beta(\mathbf{R})$ and let $f \in L^1(\mathbf{R}^N)$ satisfy condition (C):

$$\beta^- < \liminf_{|x| \to \infty} f \leqslant \limsup_{|x| \to \infty} f < \beta^+$$

Then the usual solutions of $(P_{\beta f})$ are bounded at infinity. Precisely if $w = \Delta u + f$ $(w(x) \in \beta(u(x))$ *a.e.*) we have

$$\limsup_{|x| \to \infty} w \leqslant \limsup_{|x| \to \infty} f \tag{1.1}$$

$$\liminf_{|x| \to \infty} w \geqslant \liminf_{|x| \to \infty} f \tag{1.1'}$$

Proof: Let us consider first estimate (1.1).

$$\text{Call } F^+ = \limsup_{|x| \to \infty} f \text{ and } F^- = \liminf_{|x| \to \infty} f$$

For $\varepsilon > 0$ such that $\beta^+ - F^+ > 2\varepsilon$, split $f = f' + f''$ with $f' = f'_\varepsilon = f|B_R(0)$ and $R = R_\varepsilon >> 0$ such that $f'' = f - f' \in L^\infty(\mathbf{R}^N)$ and $\sup f'' < F^+ + \varepsilon$.

Take now as $\beta_\varepsilon(s)$ the m.m. graph $\varepsilon.sign(s)$ and pose the problem

$$(\bar{P}) = (\bar{P}_\varepsilon) \quad - \Delta\bar{u} + \beta_\varepsilon(\bar{u}) \ni f'^+ = \max\ (f', 0)$$

From [2] we obtain a unique $\bar{u} = \bar{u}_\varepsilon \geqslant 0$ having compact support, say in $B_{R'}(0)$, $R' \geqslant R$, and $\bar{w} = \bar{w}_\varepsilon = \Delta\bar{u} + f'^{+} \in L_0^1(\mathbf{R}^N)$.

Now rewrite (P) as

$$\begin{cases} -\Delta(u - \bar{u}) + w = f'' + (f' - f'^{+}) + \bar{w}_\varepsilon \\ w \in \beta(u) \ a.e. \end{cases}$$

So

$$-\Delta(u - \bar{u}) + w \leqslant f'' + \varepsilon \tag{1.2}$$

In our situation for every $p \in \mathcal{P}_0$ we have $-\int_{\mathbf{R}^N} \Delta(u - \bar{u})\, p\,(u - \bar{u}) \geqslant 0$ (see [2], Appendix), so taking $p \geqslant 0$ we have

$$\int wp\,(u - \bar{u}) \leqslant \int (f'' + \varepsilon)\, p\,(u - \bar{u})$$

Define $E_r = [u - \bar{u} > r]$ for $r > 0$ and take a sequence p_n increasing to $g(s) = sign_0^{+}(s - r)$. Then in the limit

$$\int_{E_r} w \leqslant \int_{E_r} f'' + \varepsilon$$

Suppose now that for an r, $|E_r| > 0$. Then the latter inequality implies the existence of $\tilde{E}_r \subset E_r$, $|\tilde{E}_r| > 0$ such that a.e. on $\tilde{E}_r$: $\beta(u(y)) \leqslant f''(y) + \varepsilon \leqslant \sup f'' + \varepsilon \leqslant F^{+} + 2\varepsilon$, so $u(y) \leqslant \beta_{+}^{-1}(F^{+} + 2\varepsilon) = C_\varepsilon^{+}$. But $u(y) > r + \bar{u}(y) \geqslant r$ a.e. on E_r and thus $r < C_\varepsilon^{+}$. So we have

$$u \leqslant \bar{u}_\varepsilon + C_\varepsilon^{+} \tag{1.3}$$

As u_ε has compact support for every $\varepsilon > 0$ and $\lim_{\varepsilon \downarrow 0} C_\varepsilon^{+} = \beta_{+}^{-1}(F^{+}) \equiv C^{+}$ we finally obtain

$$\limsup_{|x| \to \infty} u \leqslant C^{+} \tag{1.4}$$

It remains to prove that $\limsup_{|x| \to \infty} w \leqslant \limsup_{|x| \to \infty} f$. This is obvious if β is continuous, i.e. single-valued, at $s = C^{+}$. Suppose now that β has a jump and that $\limsup w = B > F^{+}$. Then for $2\varepsilon < B - F^{+}$, $C_\varepsilon^{+} = \beta_{+}^{-1}(F^{+} + 2\varepsilon)$ equals C^{+} and so (1.3) gives $u \leqslant \bar{u} + C^{+}$. At almost every point where $u(x) < \bar{u}(x) + C^+$ if $|x| > R'$, $\bar{u}(x) = 0$ and so $w(x) < \beta^-(C^+) \leqslant F^+$. On the other hand, by virtue of Kato's inequality, we have $\Delta(u - \bar{u}) \leqslant 0$ on $[u - \bar{u} = C^+]$, the set where $u - \bar{u}$ attains its supremum and thus (1.2) gives $w \leqslant f'' + \varepsilon \leqslant F^+ + 2\varepsilon$ a.e. on this set. All together we have $w \leqslant F^+ + 2\varepsilon$ for $|x| >'R$ and, disposing of ε, $\limsup_{|x|\to\infty} w \leqslant F^+$.

The lower bound (1.1′) proceeds similarly. Only replace $f'^+, \bar{u}, \bar{w}$, C_ε^+, C^+ by the corresponding $f'^-, \tilde{u}, \tilde{w}, C_\varepsilon^-, C^- = \beta^{-1}(F^-)$ respectively and obtain analogous estimates (1.2′), (1.3′), (1.4′) $\neq$

REMARKS. 1. Observe that either bound, (1) or (1′), depends only on the corresponding half statement of condition (C).

2. In the situation of Th. 1 if $f \in L^1(\mathbf{R}^N)$ converges to zero at infinity, so does $w = \Delta u + f$ and so does u provided $\beta^{-1}(0) = \{0\}$.

The case where $\beta^{-1}(0) = [a, b] \supsetneq \{0\}$ can be reduced to the case $\beta^{-1}(0) = \{0\}$ by considering upper and lower estimates separately, translating then β along the x — axis ($\bar{\beta}(s) = \beta(s + b)$ and $\tilde{\beta}(s) = \beta(s + a)$ resp.) and translating correspondingly $u(\bar{u} = u - b$, $u = u - a$ resp). Then we obtain $\limsup_{|x|\to\infty} u \leqslant b$, $\limsup_{|x|\to\infty} u \geqslant a$.

3. The proof remains valid if we replace $f \in L^1(\mathbf{R}^N)$ by $f \in \mathcal{M}(\mathbf{R}^N)$, a bounded measure, in as far as β satisfies the appropriate condition for existence of solutions:

If $N \geqslant 3, \int_1^\infty [\beta(s) - \beta(-s)]\, s^{-\frac{2(N-1)}{N-2}} s < \infty$ (See Ph. Bénilan-H. Brézis |1|).

If $N = 2, \int_0^\infty [\beta(s) - \beta(-s)]\, e^{-as}\, ds < \infty$ for every $a > 0$ and

if $N = 1$ and $\sup D(\beta) = S^+$, $\inf D(\beta) = S^-$, $\int_0^{S^\pm} \beta(t)dt = +\infty$.

(for both see J. L. Vázquez [6]).

4. Due to the T-accretivity property of the operator $-\Delta + \beta$ in $L^p(\mathbf{R}^N)$, $1 \leqslant p \leqslant \infty$ (see [2]), for every $f \in L^\infty(\mathbf{R}^N)$ and $\varepsilon > 0$ the w_ε defined by

$$- \Delta u_\varepsilon + \beta(u_\varepsilon) + \varepsilon u_\varepsilon \ni f$$

$$w_\varepsilon = f + \Delta u_\varepsilon - \varepsilon u_\varepsilon, \quad w_\varepsilon \in \beta(u_\varepsilon)$$

satisfies $\|w_\varepsilon\|_\infty \leqslant \|f\|_\infty$, moreover $\sup f \geqslant \sup w_\varepsilon$, $\inf f \leqslant \inf w_\varepsilon$. In the limit the same is true for w in

$$-\Delta u + \beta(u) \ni f,\ w = f + \Delta u$$

Also if $N = 1$ and $f \in L^1(\mathbf{R})$ we know that the solution u of (P) verifies $u \in W^{1,\infty}(\mathbf{R})$, and $\|j(u)\|_\infty \leqslant 2\|f\|_1^2$, j being the primitive of β: $j(s) = \int_0^s \beta(t)\,dt$.

In that direction we obtain the following.

COROLLARY 1. In the situation of Th. 1, if moreover $f \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ with $p > \dfrac{N}{2}$, $N \geqslant 2$ or if $N = 1$ with no extra condition, $u \in L^\infty(\mathbf{R}^N)$ and the L^∞-bounds can be estimated in terms of f/B_R and $\|f\|_{L^\infty(\sim B_R)}$ for $R >> 0$.

PROOF: As $-\Delta\bar{u} + \beta_\varepsilon(\bar{u}) \ni f' + \in L^p_0(\mathbf{R}^N)$ we have $-\Delta\bar{u} \in L^p(\mathbf{R}^N)$, and $\bar{u} \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$, so $\bar{u} \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N)$ and by Sobolev $\bar{u}$ is a continuous function, and $\bar{u}$ has compact support.

But (1.3) gives $u \leqslant u + C^+_\varepsilon$ where $C^+_\varepsilon = \beta_+^{-1}(\|f\|_{L^\infty(\sim B_R)} + \varepsilon)$, and we are done. Similarly for the lower bound $\neq$

We finish this section by a modification of the proof of Theorem 1 in case $N \geqslant 3$ that allows for a better estimate of the behaviour of u at infinity and of Corollary 1.

In fact we replace $(\bar{P})$ in Th. 1 by

$$(\bar{P}^*) \qquad -\Delta\bar{u} = f'^{+}$$

and solve (P') by setting $u = f'^{+} * E_N$, where $E_N = C_N|x|^{2-N}$ is the fundamental solution for $-\Delta$. Thus $\bar{u} \geqslant 0$, $\bar{u} = 0(|x|^{2-N})$ as $|x| \to \infty$ and (1.2) takes the form

$$(1.2^*) \qquad -\Delta(u - \bar{u}) + w \leqslant f''$$

Arguing as in Th. 1 we arrive at

$$(1.3^*) \qquad u \leqslant \bar{u} + \beta_+^{-1}(\sup f'')$$

Now $\sup f'' = \sup\limits_{\sim B_R(0)} f$ and $\bar{u} \sim C_N|x|^{2-N} \int f'^{+}.\ dx$ for large x, so $u \leqslant \beta_+^{-1}(\sup f'') + 0(|x|^{2-N})$ for $|x| >> 0$.

Also, if $f \in L^p_{\text{loc}}(\mathbf{R}^N)$, $p > \dfrac{N}{2}$, we estimate

$$\|u\|_{L^\infty(B_{R_1})} \leqslant \|f\|_{L^p(B_R)} \cdot \|E_N\|_{L^{p'}(B_{R+R_1})}$$

where $R, R_1 > 0$, the balls are centered at 0 and $p' = \dfrac{p-1}{p}$. Thus the L^∞-norm of u on $\mathbf{R}^N$ depends only on $\|f\|_{L^p(B_R)}$ and $\|f\|_{L^\infty(\sim B_R)}$, $R > 0$.

2. Weak solutions for f locally integrable, bounded at infinity.

We study now the existence of solutions for $-\Delta u + \beta(u) \ni f$ under the weaker hypothesis on f that $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ and be strictly bounded at infinity by β.

We shall understand here by a weak solution for $-\Delta u + \beta(u) \ni f$ an $u \in L^1_{\text{loc}}(\mathbf{R}^N)$ such that $w \equiv f + \Delta u \in L^1_{\text{loc}}(\mathbf{R}^N)$ and $w \in \beta(u)$ a.e.

We obtain the following main result.

Theorem 2. Let β be a m.m.g. in $\mathbf{R}$ such that $0 \in \beta(0)$ and $0 \in \operatorname{Int} \beta(\mathbf{R})$ as in Th. 1 and let $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ be strictly bounded by β at infinity. Then there exists an $u \in L_{\text{loc}}(\mathbf{R}^N)$, weak solution for (P) and u and $w = \Delta u + f$ are bounded at infinity satisfying (1.1), (1.1′) of Th. 1. In case $f \in L^1(\mathbf{R}^N)$ these weak solutions coincide with the usual solutions.

Proof. Our solutions will be obtained as (perhaps repeated) monotone limits of usual solutions for approximate problems. We shall proceed by steps, beginning with the simpler cases.

Case 1: $f \geqslant 0$ for large x.

Let $R > 0$ be such that if $f = f' + f''$ with $f' = f|_{B_R(0)}$ we have $f'' \geqslant 0$, $\sup f'' < \beta^+$.

We pose the series of approximate problems.

$$(P_n) \qquad -\Delta u_n + \beta(u_n) \ni f_n$$

where $f_n = f \cdot \chi_n$ is the «cut» of f by means of $\chi_n(x) = \chi_0\left(\dfrac{|x|}{n}\right)$, with $\chi_0 \in C^\infty(\mathbf{R}_+)$, $0 \leqslant \chi_0 \leqslant 1$, $\chi_0 = 1$ on $[0, 1]$ and $\chi_0 = 0$ outside $[0, 2]$.

Then for $n > R$, $f_n \equiv f' + f''_n = f' + f'' \cdot \chi_n$. Now as in Th. 1 we construct functions $\bar{u}$ and $\tilde{u}$ depending on f' and constants C_ε^+, C_ε^- depending on β and f'', uniformly in n, such that, according to (1.3), (1.3'):

$$C_\varepsilon^- + \tilde{u} \leqslant u_n \leqslant \bar{u} + C_\varepsilon^+ \tag{2.1}$$

and we may take $C_\varepsilon^- = 0$.

Let us look at the convergence of $\{u_n\}$. Except in case $f \in L_0^1(\mathbf{R}^N)$, the sequence $\{f_n\}$ is strictly increasing and so is $\{u_n\}$ by the results of [2]. With this and the uniform domination of (2.1) we conclude the existence of $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ such that $u_n \uparrow u$ a.e. in $\mathbf{R}^N$ and in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and u satisfies (2.1). In particular

$$0 \leqslant \liminf_{|x|\to\infty} u \leqslant \limsup_{|x|\to\infty} u \leqslant \beta_+^{-1}(\limsup_{|x|\to\infty} f) \tag{2.2}$$

analogous to (1.4), (1.4').

(If $f \in L_0^1(\mathbf{R}^N)$, then f_n is constant for large n and we have usual solutions, that may not be unique when $N = 1{,}2$, $\beta^{-1}(0) \neq \{0\}$ and $\int f = 0$, see [2]).

As to the convergence of $\{w_n = \Delta u_n + f_n \in \beta(u_n)\}$, for $R_1 > 0$ u_n is solution of the problem

$$\left.\begin{aligned} -\Delta u_n + w_n = f_n,\ w_n \in \beta(u_n) \quad &\text{on } \Omega = B_{R_1}(0) \\ u_n = \hat{u}_n \quad &\text{on } S_{R_1}(0) = \partial\Omega \end{aligned}\right\}$$

where $\hat{u}_n$ is the trace of $u_n \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$. Observe that for large n f_n is constant on B_{R_1} and that $\hat{u}_n$ is nondecreasing convergent in $L^1(\partial\Omega)$. Then, using a result of Brézis [5] we have

$$\|\delta(w_n - w_{n'})\|_{L^1(\Omega)} \leqslant C\,(\|\hat{u}_n - \hat{u}_{n'}\|_{L^1(\partial\Omega)} + \|\delta(f_n - f_{n'})\|_{L^1(\Omega)})$$

where $\delta(x) =$ distance $(x, \partial\Omega)$, and so $\{w_n\}$ is $L^1(\Omega)$-Cauchy and arguing with an exhaustive sequence of balls we conclude that an appropriate subsequence converges in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$. But $\{f_n\}$ is nondecreasing and so $\{w_n\}$ is nondecreasing ([2]), whence there exists $w \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ such that $w_n \uparrow w$ a.e. and in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$.

Passing to the limit in (P_n) we obtain $-\Delta u + w = f$. It only remains to prove that $w(x) \in \beta(u(x))$ a.e. an this is consequence of Lemma 3 of [4], for $u_n \to u$ a.e., $w_n \to w$ in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and $w_n \in \beta(u_n)$ a.e.

Case 3: $f \leqslant 0$ for large x.

Same process. Now we may put $C_\varepsilon^+ = 0$ and instead of (2.2) we have

$$0 \geqslant \limsup_{|x|\to\infty} u \geqslant \liminf_{|x|\to\infty} u \geqslant \beta_-^{-1}(\liminf_{|x|\to\infty} f) \tag{2.2'}$$

Finally if $f \in L^1(\mathbf{R}^N)$ we conclude easily from [2] that the solutions u_n for the problems (P_n) converge to a usual solution of (P), by virtue of the estimates for $u - \hat{u}$ and grad $(u - \hat{u})$ in terms of $\|f - \hat{f}\|_1$.

So the Theorem is proved in both cases.

REMARK: The solutions here constructed are uniquely determined if $f \notin L_0^1(\mathbf{R}^N)$ as we have said before. Also $w = \Delta u + f$ is always uniquely determined.

Note that in passing to the limit the ordering property of (P) in $L^1(\mathbf{R}^N)$ is preserved, a.e. if u_i, $i = 1,2$ are solutions for f_i and $w_i = \Delta u_i + f_i$ we have

(2.3) $f_1 \leqslant f_2$ and $f_1 \neq f_2$ a.e. imply $u_1 \leqslant u_2$ and $w_1 \leqslant w_2$ a.e.

Case 3: general f.

In general we have to go through a double approximation process and that we shall do in two ways: i) taking positive part of f for large x and then cutting as in Case 1, ii) taking negative part and cutting. We arrive at two possible solutions.

i) IS SOLUTIONS

Approximate $(P) = (P_f)$ by $(P_{f_-^m}) = (P^m)$ where

$$f_-^m(x) = \begin{cases} f(x) & \text{if } |x| \leqslant m \\ f^+(x) & \text{if } |x| > m \end{cases}, \quad f^+(x) = \max\,(f(x), 0)$$

for $m > 0$. Then $f_-^m \downarrow f$ a.e. and in $L^1_{\text{loc}}(\mathbf{R}^N)$, and (P^m) falls in Case 1, so we replace (P^m) by

$$(P_n^m) \quad -\Delta u_n^m + \beta(u_n^m) \ni f_n^m = f_-^m \cdot \chi_n$$

and obtain solutions $u_n^m \in W^{1,1}_{\text{loc}}$, $W_n^m = \Delta u_n^m + f_n^m \in L^1(\mathbf{R}^N)$, $\{u_n^m\}$ satisfying (2.1) with $u, \tilde{u}, C_\varepsilon^+\, C_\varepsilon^-$ chosen independent of m, n being both large. Thus the solutions $u^m = \lim\limits_n \uparrow u_n^m$ for (P^m) satisfy (2.1) uniformly in m.

Convergence of u^m: except possibly in case $f \in L_0^1(\mathbf{R}^N)$ the sequence u^m has a limit a.e. and in $L^1_{\text{loc}}(\mathbf{R}_N)$, $\hat{u}$. Namely if $f \notin L_0^1(\mathbf{R}^N)$, f may be nonnegative for large $|x|$ or not. If $f \geqslant 0$ for large $|x|$, $f^{\underline{m}}$ is constant for large m and in that case the solution u^m constructed in Case 1 is unique, so $u = u^m$ for every $m >> 0$. If f is not positive for large x, $\{f^{\underline{m}}\}$ is decreasing nonconstant and it is not difficult to see that some subsequence of $\{u^m\}$ is decreasing, that there is only one possible limit $\hat{u} = \lim u^{m_k}$ and that $\hat{u} = \lim\limits_m u^m$.

Now the argument used before allows for the existence of $\hat{w} = \lim\limits_m w^m$ a.e. and in $L^1_{\text{loc}}(\mathbf{R}^N)$, w^m being a nonincreasing sequence, and we have $\hat{u} = \lim u^m$ a.e. and in $L^1_{\text{loc}}(\mathbf{R}^N)$, $\hat{w} \in \beta(\hat{u})$ a.e. and $-\Delta\hat{u} + \hat{w} = f$. $\hat{u}$ satisfies (2.1) and $\hat{w}$ (1.1), (1.1').

As before if $f \in L^1(\mathbf{R}^N)$, $\hat{u}$ is a usual solution and the comparison property (2.3) holds.

This solution we shall call inf-sup-solution or IS-solution for in general:

$$u = \inf_m \sup_n u_n^m = \lim_{m\to\infty} \lim_{n\to\infty} u_n^m$$

ii) SI SOLUTIONS:

Same process, except the first step: we approximate (P_f) by $(P_{f_{\underline{m}}})$, where

$$f_{\underline{m}}(x) = \begin{cases} f(x) & \text{if } |x| \leqslant m \\ f^-(x) & \text{if } |x| > m, \ f^-(x) = \min(f(x), 0) \end{cases}$$

and $m > 0$. $f_{\underline{m}} \uparrow f$ a.e. and in $L^1_{\text{loc}}(\mathbf{R}^N)$ and $(P_{f_{\underline{m}}})$ falls in Case 2. We obtain a solution u such that, with obvious notation:

$$u = \lim_m u_{\underline{m}} = \lim_m (\lim_n u_{m,n}) = \sup_m (\inf_n u_{m,n}) \qquad \text{and}$$

$$w = \lim_m w_m = \lim_m (\lim_n w_{m,n}) = \sup_m (\inf_n w_{m,n}),$$

$$w \in \beta(u) \text{ a.e. and } -\Delta u + w = f.$$

We call these solution SI- or sup-inf-solutions, and the properties of i) apply.

Also from $f^m_{=} \geqslant f_{\underline{m}}$ we deduce that $w^m_{=} \geqslant w_{\underline{m}}$ and in the limit $w \geqslant w$ a.e. If $f \notin L^1_0(\mathbf{R}^N)$, we also have $\hat{u} \geqslant u$, $\hat{u}$ and u being well defined. In case $f \geqslant 0$ for large x or $f \leqslant 0$ for large x, $\hat{u} = u$, $\hat{w} = w$ and we obtain the same solutions studied in cases 1 and 2 resp. $\neq$

3. Extension of Ths. 1 and 2

1) Both theorems may be adapted to the problem

$$(P_\Omega) = (P_{\beta, f, \Omega}) \begin{cases} -\Delta u + \beta(u) \ni f & \text{on } \Omega \subset \mathbf{R}^N \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

where Ω is as unbounded open set in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and β and f are similar to those in Ths 1 and 2. We shall develop this ideas in [7].

2) The proof of Theorems 1 and 2 apply to show the existence of weak solutions of (P) when $f = f_1 + f_2$ with $f_1 \in L^1(\mathbf{R}^N)$ and $f_2 \in L^\infty(\mathbf{R}^N)$ and is strictly bounded at infinity by β. In particular if Range $(\beta) = \mathbf{R}$, for $f \in L^1(\mathbf{R}^N) + L^\infty(\mathbf{R}^N)$.

Now the solutions are *not* bounded at infinity in general: the solutions $\bar{u} = \bar{u}_\varepsilon$ of $(\overline{P}_\varepsilon)$ $-\Delta\bar{u} + \beta_\varepsilon(\bar{u}) \ni f_1^+$ (see Th. 1) need not be bounded at infinity and the boundedness of weak solutions depended on the estimate (1.3) $u \leqslant \bar{u}_\varepsilon + c_\varepsilon^+$. However if f_1 is bounded by a radial integrable function near infinity, $u \to 0$ uniformly as $|x| \to \infty$ (cf. [1] for $N \geqslant 3$, [6] for $N = 2$) and we obtain solutions bounded at infinity.

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