QUASI-EQUIVALENCE OF COMPACTA AND SPACES OF COMPONENTS

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ABSTRACT.

Let X, Y be two compacta with $Sh(X) = Sh(Y)$. Then, the spaces of components of X, Y are homeomorphic. This does not happen, in general, when X, Y are quasi-equivalent.

In this paper we give a sufficient condition for the existence of a homeomorphism between the spaces of components of two quasi-equivalent compacta X, Y which maps each component in a quasi-equivalent component.

1. INTRODUCTION.

Two compacta X and Y, lying in the Hilbert cube Q, are *quasi-equivalent* [3] (notation $X \overset{q}{\simeq} Y$) if for every neighborhood (U, V) of (X, Y) in (Q, Q) there are two fundamental sequences (see [1] or [2] for the definition) $\mathbf{f} = \{f_k, X, Y\}_{Q,Q}$, $\mathbf{g} = \{g_k, Y, X\}_{Q,Q}$ such that $\mathbf{g} \cdot \mathbf{f}$ is U-homotopic to the fundamental identity sequence $\mathbf{i}_{X,Q}$ and $\mathbf{f} \cdot \mathbf{g}$ is V-homotopic to $\mathbf{i}_{Y,Q}$, i.e., if there exists a neighborhood (U', V') of (X, Y) in (Q, Q) such that

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$$g_k \cdot f_{k|U'} \simeq i_{|U'} \qquad \text{in} \quad U$$

and

$$f_k \cdot g_{k|V'} \simeq i_{|V'} \qquad \text{in} \quad V$$

for almost all k.

In [3] is proved that all 0-dimensional infinite compacta are quasi-equivalent. Thus, in general, it is not true that two quasi-equivalent compacta X, Y have homeomorphic spaces of components. This is in contrast with the situation that we have when $Sh(X) = Sh(Y)$. In this paper we give a sufficient condition for the existence of a homeomorphism between the spaces of components $\square(X)$ and $\square(Y)$, of two quasi-equivalent compacta X, Y, which maps each component in a quasi-equivalent component.

In $\square(X)$ we consider the metrizable topology induced by the upper-semicontinuous decomposition of the compactum X into components and we make use of the following well-known theorem (see [1] or [2]):

Theorem 1.

Let X, Y be compacta lying in the Hilbert cube Q. Then, for every fundamental sequence

$$\mathbf{f} = \{f_k, X, Y\}_{Q,Q}$$

there exists exactly one function

$$\Lambda_{\mathbf{f}} : \square(X) \longrightarrow \square(Y)$$

satisfying the following condition.

If $X_0 \in \square(X)$, then $\{f_k, X_0, \Lambda_{\mathbf{f}}(X_0)\}_{Q,Q}$ is a fundamental sequence.

Moreover, the function $\Lambda_{\mathbf{f}}$ is continuous and depends only on the fundamental class $[\mathbf{f}]$ of $\mathbf{f}$. This dependence is covariant, i.e., if $\mathbf{g} = \{g_k, Y, Z\}_{Q,Q}$ is a fundamental sequence then $\Lambda_{\mathbf{g}\cdot\mathbf{f}} = \Lambda_{\mathbf{g}} \cdot \Lambda_{\mathbf{f}}$.

2. Quasi-equivalence and spaces of components.

Let X, Y be quasi-equivalent compacta lying in the Hilbert cube Q. Let $((U_n)_{n\in N}, (V_n)_{n\in N})$ be a decreasing basis of open neighborhoods of (X, Y) in (Q, Q) and

$$\mathbf{f}^n = \{f_k^n, X, Y\}_{Q,Q} \quad , \quad \mathbf{g}^n = \{g_k^n, Y, X\}_{Q,Q}$$

fundamental sequences such that

$$\mathbf{g}^n \cdot \mathbf{f}^n \underset{U_n}{\simeq} \mathbf{i}_{X,Q}$$

(i.e. $\mathbf{g}^n \cdot \mathbf{f}^n$ and $\mathbf{i}_{X,Q}$ are U_n-homotopic) and

$$\mathbf{f}^n \cdot \mathbf{g}^n \underset{V_n}{\simeq} \mathbf{i}_{Y,Q},$$

for $n = 1, 2, \ldots$.. By theorem 1 the fundamental sequences $\mathbf{f}^n$, $\mathbf{g}^n$ induce maps

$$\Lambda_n : \square(X) \longrightarrow \square(Y),\ \Lambda_n' : \square(Y) \longrightarrow \square(X)$$

which fulfill the conditions of theorem 1. We prove the following theorem.

Theorem 2.

If the sequences of maps $(\Lambda_n)_{n\in N}$, $(\Lambda_n')_{n\in N}$ *converge continuously to* $\Lambda : \square(X) \longrightarrow \square(Y)$, $\Lambda' : \square(Y) \longrightarrow \square(X)$, *respectively, then*

i) Λ *and* Λ' *are mutually inverse homeomorphisms.*

ii) For every $X_0 \in \square(X)$ *and for every neighborhood* $\mathfrak{A}$ *of* X_0 *in* $\square(X)$ *there exists a closed neighborhood of* X_0, $\mathfrak{A}_0 \subset \mathfrak{A}$ *such that*

$$p^{-1}(\mathfrak{A}_0) \overset{q}{\simeq} \tilde{p}^{-1}(\Lambda(\mathfrak{A}_0))$$

where $p : X \longrightarrow \square(X)$ *and* $\tilde{p} : Y \longrightarrow \square(Y)$ *are the natural-projections.*
Moreover if for some pair of components $X_0 \in \square(X)$ $Y_0 \in \square(Y)$ *and for a subsequence of indices* $n_1 < n_2 <, \ldots \Lambda_{n_i}(X_0) = Y_0\, \Lambda'_{n_i}(Y_0) = X_0$ *then* $X_0 \overset{q}{\simeq} Y_0$.

Proof.

Part i). Let X_0 be a component of X. Let us set $Y_0 = \Lambda(X_0)$, $Y_n = \Lambda_n(X_0)$, $X_n = \Lambda'_n(Y_n)$. We must prove that $X_0 = \Lambda'(Y_0)$.
The map $\Lambda'_n \cdot \Lambda_n : \square(X) \longrightarrow \square(X)$ is induced (in the sense of theorem 1) by the fundamental sequence $\mathbf{g}^n \cdot \mathbf{f}^n$.

Thus,

$$\{g_k^n \cdot f_k^n,\ X_0, X_n\}_{Q,Q}$$

is a fundamental sequence. Since, by hypothesis,

$$\mathbf{g}^n \cdot \mathbf{f}^n \underset{U_n}{\simeq} \mathbf{i}_{X,Q}$$

then

$$g_k^n \cdot f_k^n|_{X_0} \simeq i|_{X_0}$$

in U_n^0, for almost all k, where U^0_n is the component of U_n containing X_0. Hence $X_n \subset U_n^0$.

The set $\mathfrak{A}_n$ of all the components of X contained in U_n^0 is an open neighborhood of X_0 in $\square(X)$. Let us prove that the family $\{\mathfrak{A}_n\}_{n\in N}$ is a basis of neighborhoods of X_0 in $\square(X)$. Let $\mathfrak{A}$ be an open neighborhood of X_0 in $\square(X)$ and $A = p^{-1}(\mathfrak{A})$. Take an open-closed of X, B, such that $X_0 \subset B \subset A$ and separate B and $X - B$ by two open sets of Q, V_0 and V_1. Take $U_n \subset V_0 \cup V_1$. Then $U_n^0 \subset V_0$ and

$$\mathfrak{A}_n \subset p(U_n^0 \cap X) \subset p(V_0 \cap X) = p(B) \subset \mathfrak{A}.$$

Therefore we have $\lim_{n\to\infty} X_n = X_0$, because $X_m \in \mathfrak{A}_m \subset \mathfrak{A}_n$, for every $m \geqslant n$. On the other hand, it is clear that $\lim_{n\to\infty} Y_n = Y_0$. Since Λ'_n is continuously convergent then $\lim_{n\to\infty} \Lambda'_n(Y_n) = \Lambda'(Y_0)$. Hence $X_0 = \Lambda'(Y_0)$. The proof that Λ is the left-inverse of Λ' is similar.

Part ii)

Let $\mathfrak{A}$ be a neighborhood of X_0 in $\square(X)$. Take an open-closed $\mathfrak{A}_0$ in $\square(X)$ such that $X_0 \in \mathfrak{A}_0 \subset \mathfrak{A}$. We claim that there is a n_0 such that

$$\Lambda_n(\mathfrak{A}_0) \subset \Lambda(\mathfrak{A}_0) \quad , \quad \Lambda_n'[\Lambda(\mathfrak{A}_0)] \subset \mathfrak{A}_0,$$

for all $n \geqslant n_0$. If, for instance, there is a sequence $n_1 < n_2 < \ldots$ such that $\Lambda_{n_i}(X_i) \notin \Lambda(\mathfrak{A}_0)$, for certain $X_i \in \mathfrak{A}_0$, $i = 1,2, \ldots$, then, since $\mathfrak{A}_0$ is compact metrizable, it must exist a subsequence of $(X_i)_{i\in N}$ converging in $\mathfrak{A}_0$. Without loss of generality we can assume that this subsequence is the original one $(X_i)_{i\in N}$ and we call X'_0 its limit. Since Λ_{n_i} converge-continuosly to Λ, then

$$\lim_{i\to\infty} [\Lambda_{n_i}\ (X_i)] = \Lambda\ (X'_0)$$

but this contradicts the fact that $A_{n_i}(X_i) \notin A(\mathfrak{A}_0)$ and $A(\mathfrak{A}_0)$ is open. A similar argument shows the existence of n_0 such that relation $A_n'[A(\mathfrak{A}_0)] \subset \mathfrak{A}_0$ holds for $n \geqslant n_0$.

Now, we claim that

$$\tilde{\mathfrak{f}}^n = \{f_k^n, p^{-1}(\mathfrak{A}_0), \tilde{p}^{-1}[A(\mathfrak{A}_0)]\}_{Q,Q}$$

and

$$\tilde{\mathfrak{g}}^n = \{g_k^n, \tilde{p}^{-1}[A(\mathfrak{A}_0)], p^{-1}(\mathfrak{A}_0)\}_{Q,Q}$$

are fundamental sequences for $n \geqslant n_0$. Let V^0 be a neighborhood of $\tilde{p}^{-1}[A(\mathfrak{A}_0)]$ in Q. Take an open neighborhood V of Y in Q, such that V^0 contains all the components of V which meet $\tilde{p}^{-1}[A(\mathfrak{A}_0)]$. There is a neighborhood U of X in Q such that

$$f_k^n|_U \simeq f_{k+1}^n|_U \text{ in } V$$

for almost all k. If we call U^0 the union of all the components of U meeting $p^{-1}(\mathfrak{A}_0)$, then

$$f_k^n|_{U^0} \simeq f_{k+1}^n|_{U^0} \text{ in } V^0,$$

for almost all k, because, in other case, there would be some $X_i \in \mathfrak{A}_0$ such that $A_n(Xi) \notin A(\mathfrak{A}_0)$. This proves our claim that $\tilde{\mathfrak{f}}^n$ is a fundamental sequence. The same argument is valid for $\tilde{\mathfrak{g}}^n$.

Furthermore, it is clear that

$$\tilde{\mathfrak{g}}^n \cdot \tilde{\mathfrak{f}}^n \underset{U^0_n}{\simeq} \mathfrak{i}_{p^{-1}(\mathfrak{A}_0), Q}$$

and

$$\tilde{\mathfrak{f}}^n \cdot \tilde{\mathfrak{g}}^n \underset{V^0_n}{\simeq} \mathfrak{i}_{\tilde{p}^{-1}[A(\mathfrak{A}_0)], Q},$$

where U_n^0 (resp. V_n^0) is the union of the components of U_n (resp. V_n) meeting $p^{-1}(\mathfrak{A}_0)$, (resp. $\tilde{p}^{-1}[A(\mathfrak{A}_0)]$). Moreover $(\{U_n^0\}_{n\in N}, \{V_n^0\}_{n\in N})$ is a basis of neighborhoods of $(p^{-1}(\mathfrak{A}_0), \tilde{p}^{-1}[A(\mathfrak{A}_0)])$ in (Q, Q). Thus

$$p^{-1}(\mathfrak{A}_0) \overset{q}{\simeq} \tilde{p}^{-1}[A(\mathfrak{A}_0)],$$

and part ii) is proved.

To prove the rest of the theorem, let us consider the fundamental sequences

$$\hat{\mathbf{f}}^{n_i} = \{f_k^{n_i},\ X_0,\ Y_0\}_{Q,Q}$$
$$\hat{\mathbf{g}}^{n_i} = \{g_k^{n_i},\ Y_0,\ X_0\}_{Q,Q},$$

for $n_1 < n_2 < \ldots.$, and let (U^0, V^0) be a neighborhood of (X^0, Y^0) in (Q, Q). Now let us consider a neighborhood (U_{n_i}, V_{n_i}), such that $U_{n_i}{}^0 \subset U^0$, $V_{n_i}{}^0 \subset V^0$, where $U_{n_i}{}^0$, $V_{n_i}{}^0$ are the components of U_{n_i}, V_{n_i} containing X_0, Y_0 respectively, By hypothesis, there is a neighborhood (U', V') of (X, Y) in (Q, Q) such that

$$g_k^{n_i} \cdot f_k^{n_i}|_{U'} \simeq i|_{U'} \text{ in } U_{n_i}$$

and

$$f_k^{n_i} \cdot g_k^{n_i}|_{V'} \simeq i|_{V'} \text{ in } V_{n_i}$$

for almost all k, If we take U_0' (resp. V_0') the component of U' (resp. V') containing X_0 (rep. Y_0), we have

$$g_k^{n_i} \cdot f_k^{n_i}|_{U'_0} \simeq i|_{U'_0} \text{ in } U_{n_i}{}^0 \subset U^0$$

and

$$f_k^{n_i} \cdot g_k^{n_i}|_{V'_0} \simeq i|_{V'_0} \text{ in } V_{n_i}{}^0 \subset V^0$$

for almost all k, Therefore X_0 and Y_0 are quasi-equivalent and the proof is complete.

Remark 1.

It is easy to prove that the families $(\mathbf{f}^n)_{n \in N}$, $(\mathbf{g}^n)_{n \in N}$ of fundamental sequences realizing the quasi-equivalence have the following property: If there exists a sequence of indices $n_1 < n_2 < \ldots$ such that $\mathbf{f}^{n_i} \simeq \mathbf{f}^{n_j}$ and $\mathbf{g}^{n_i} \simeq \mathbf{g}^{n_j}$ for every par n_i, n_j of indices, then $Sh(X) = Sh(Y)$.

Remark 2.

Let X, Y be compacta and let

$$A : \square(X) \longrightarrow \square(Y)$$

be a homeomorphism such that:

i) for every $X_0 \in \square(X)$ there is a closed neighborhood $\mathfrak{A}_0$ of X_0, with

$$p^{-1}(\mathfrak{A}_0) \overset{q}{\simeq} \widetilde{p}^{-1}[\Lambda(\mathfrak{A}_0)]$$

and ii) $\Lambda|_{\mathfrak{A}_0}$, $\Lambda^{-1}|_{\Lambda(\mathfrak{A}_0)}$ are continuous limits of the sequences of maps induced by a countable family of fundamental sequences defining this «local» quasi-equivalence. Then

$$X \overset{q}{\simeq} Y,$$

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