REFLEXIVITY OF PROJECTIVE TENSOR PRODUCTS OF ECHELON AND COECHELON KÖTHE SPACES

by

J. A. LÓPEZ MOLINA*

ABSTRACT.

Let λ^p, μ^r, $p \geqslant 1$, $r \geqslant 1$, be sequence echelon Köthe spaces. We show: a) $\lambda^1 \hat{\otimes}_\pi \mu^1$, $\lambda^1 \hat{\otimes}_\pi (\mu^1)^\alpha$ and $(\lambda^1)^\alpha \hat{\otimes}_\pi (\mu^1)^\alpha$ are reflexive iff λ^1 and μ^1 are reflexive. b) If $r > 1$, $\lambda^1 \hat{\otimes}_\pi \mu^r$, $\lambda^1 \hat{\otimes}_\pi (\mu^r)^\alpha$ and $(\lambda^1)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ are reflexive iff λ^1 is reflexive. c) If $p > 1$, $r > 1$, $p > r/(r-1)$, then $\lambda^p \hat{\otimes}_\pi \mu^r$ is reflexive; if $p > r$, then $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$ is reflexive; if $p/(p-1) > r$, then $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ is reflexive. d) In all other cases, $\lambda^p \hat{\otimes}_\pi \mu^r$, $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$ and $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ are reflexive iff λ^p or μ^r is a Montel space.

1. INTRODUCTION.

In [2], Holub has characterized the spaces $\ell^p \hat{\otimes}_\pi \ell^r$ which are reflexive. Here we characterize the reflexivity of the projective tensor product $E \hat{\otimes}_\pi F$ when E and F are sequence echelon or coechelon Köthe spaces as consequence of a previous study of the canonical Schauder basis on these spaces.

A separated locally convex space $E[\mathcal{T}_E]$ on the field $\mathbb{K}$ of real or complex numbers, will be called, simply, a space. Notation and concepts not explicitly defined are standard in topological vector spaces theory (see [3]). In particular, $\overline{\text{aco}}$ (A) will be the closed absolutely convex hull of the set A of the space

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$E[\mathcal{T}_E]$ and $\mathcal{B}(E, F)$ will be the set of all continuous bilinear forms on the product $E\times F$ of the spaces $E[\mathcal{T}_E]$ and $F[\mathcal{T}_F]$. If it is neccessary, each $f \in \mathcal{B}(E, F)$ will be identified in the canonical form, without previous advertence and without change of notation, with an element of the set $\mathcal{L}(E, F'_\sigma)$ of all continuous linear maps from $E[\mathcal{T}_E]$ into the topological weak dual $[F', \sigma(F', F)]$ of $F[\mathcal{T}_F]$.

The Minkowski functional of a neighbourhood U of zero in the space $E[\mathcal{T}_E]$, will be denoted by p_U. The completion of the projective tensor product $E \underset{\pi}{\otimes} F$ of the spaces $E[\mathcal{T}_E]$ and $F[\mathcal{T}_F]$ will be represented by $E \underset{\pi}{\hat{\otimes}} F$. The seminorm on $E \underset{\pi}{\hat{\otimes}} F$ assocciated with the continuous seminorms α and β of $E[\mathcal{T}_E]$ and $F[\mathcal{T}_F]$ respectively, will be denoted by $\alpha \otimes \beta$.

$\mathbb{N}$ will be the set of non zero natural numbers. If there is not a risk of confusion, the topological projective limit of a projective system of spaces

$$\{ E_n [\mathcal{T}_n], I_{nm} \} \quad n \geqslant m, \; n, m \text{ in a directed set } D,$$

will be written $\varprojlim E_n$. Analogously, given an increasing sequence

$$\{ E_n \}_{n=1}^{\infty}$$

of vector spaces and a Hausdorff locally convex topology $\mathcal{T}_n$ on each E_n, $n \in \mathbb{N}$, such that the canonical inclusion $I_{nm} : E_n[\mathcal{T}_n] \to E_m[\mathcal{T}_m]$ is continuous for every $n \leqslant m$ in $\mathbb{N}$, the vector space

$$E = \bigcup_{n=1}^{\infty} E_n$$

with the locally convex inductive limit topology of the inductive system

$$\{ E_n [\mathcal{T}_n], I_{nm} \}, \quad n \leqslant m, \; n, m \in \mathbb{N},$$

will be denoted by $\varinjlim E_n$.

Concerning sequence spaces, we shall use the notations and definitions of [4]. By example: ω will be the set of all sequences $x = (x_i) = (x_i)_{i=1}^{\infty}$ of elements of $\mathbb{K}$ and e_n, $n \in \mathbb{N}$, will be the sequence with all its components equal to 0, except the number n component, which is equal to 1. A notation such as (x_{ij}^{nk}) or similar. will represent a family of sequences indexed by the parameters n, k, j, each sequence of this family having x_{ij}^{nk} as i-component. In some cases we shall write $(x_{ij}^{nk})_{i=1}^{\infty}$ for remark the index i of the components of each sequence.

Let $a^k = (a_i^k)$ a sequence of sequences of non negative real numbers such that $a_i^k \leq a_i^{k+1}$ for each i, $k \in \mathbb{N}$ and such that for each $i \in \mathbb{N}$, there is $k \in \mathbb{N}$ with $a_i^k \neq 0$. For each real number $p \geq 1$, we define the echelon Köthe space of order p

$$\lambda^p = \{(x_i) \in \omega \,/\, N_k((x_i)) = (\sum_{n=1}^{\infty} |x_i|^p a_i^k)^{1/p} < \infty, \forall k \in \mathbb{N}\} \qquad (1)$$

The coechelon Köthe space of order p is the α-dual of λ^p:

$$(\lambda^p)^\alpha = \{(x_i) \in \omega \,/\, \sum_{n=1}^{\infty} |x_i y_i| < \infty, \forall (y_i) \in \lambda^p\}$$

Unless it is otherwise clearly stated, we shall always consider on λ^p the topology defined by the family of seminorms $\{N_k, k \in \mathbb{N}\}$. Then λ^p is a Frechet space whose topological dual is $(\lambda^p)^\alpha$; with the exceptions before cited, we shall always consider on $(\lambda^p)^\alpha$ the strong topology $\beta((\lambda^p)^\alpha, \lambda^p)$. When $a_i^k = 1$ for every k, $i \in \mathbb{N}$, we obtain the classical space ℓ^p.

If λ is a locally convex sequence space and $J \subset \mathbb{N}$, the subspace of λ

$$\lambda_J = \{(x_i) \in \lambda \,/\, x_i = 0 \text{ if } i \notin J\}$$

is called a sectional subspace of λ. If $J' = \mathbb{N} \setminus J$ it is clear that we have

$$\lambda^p = \lambda^p_J \oplus \lambda^p_{J'} \text{ and } (\lambda^p)^\alpha = (\lambda^p)^\alpha_J \oplus (\lambda^p)^\alpha_{J'}$$

for every echelon space λ^p and coechelon space $(\lambda^p)^\alpha$. The sequence of echelons $a^k = (a_i^k)$ is called strongly increasing if there is no infinite set $J \subset \mathbb{N}$ such that there is k_0 with the property that, for every $k \geq k_0$ in $\mathbb{N}$, there is $M_k > 0$ such that

$$\forall i \in J \ , \ a_i^k \leq M_k a_i^{k_0}$$

It is known that the echelon space λ^p, $p \geq 1$, determined by $\{a^k\}_{k=1}^{\infty}$ is Montel if and only if the sequence $\{a^k\}_{k=1}^{\infty}$ is strongly increasing, that is, if λ^p has no sectional subspace isomorphic to ℓ^p. If $p > 1$, every λ^p is reflexive, but λ^1 is reflexive if and only if λ^1 is a Montel space.

Given the echelon space (1), the family of sets $\{U_k(\lambda^p), k \in \mathbb{N}\}$ or simply, $\{U_k, k \in \mathbb{N}\}$ if there is no risk of confusion, where

$$U_k = \{(x_i) \in \lambda^p \,/\, N_k((x_i)) \leq 1/k\}, \qquad (2)$$

is a 0-neighbourhoods basis in λ^p. Then, if $\mathcal{U}(F)$ is the filter of 0-neighbourhoods in the space $F\,[\mathcal{T}_F]$, the family of sets $\{E_{k,V},\ k \in \mathbb{N},\ V \in \mathcal{U}(F)\}$, where

$$E_{k,V} = \{ u \in \lambda^p \otimes F \,/\, (p_{U_k} \otimes p_V)(u) \leqslant 1 \}, \tag{3}$$

is a 0-neighbourhoods basis in $\lambda^p \underset{\pi}{\otimes} F$.

Let λ^p, $p \geqslant 1$, be the echelon space (1). For every $k \in \mathbb{N}$ we define the vector space

$$\lambda^p_k = \{ (x_i) \in \omega \,/\, x_i = 0 \ \text{ if } \ a^k_i = 0 \ \text{ and } \ N_k((x_i)) < \infty \},$$

its α-dual

$$(\lambda^p_k)^\alpha = \{ (x_i) \in \omega \,/\, \sum_{i=1}^{\infty} |x_i y_i| < \infty \quad \forall (y_i) \in \lambda^p_k \},$$

and the mapping $I_k : \omega \to \omega$ such that $I_k((x_i))$ is the sequence whose i-component is x_i if $a^k_i \neq 0$ and equal to 0 if $a^k_i = 0$. If no other topology is explicitly defined, we shall always consider on λ^p_k the topology generated by the norm N_k on λ^p_k. Then λ^p_k is isomorphic to ℓ^p and its topological dual $(\lambda^p_k)'$ is $I_k((\lambda^p_k)^\alpha)$. Unless it is otherwise clearly stated, we will always consider on $(\lambda^p_k)'$ the strong topology $\beta((\lambda^p_k)', \lambda^p_k)$.

Let I_{nm} be, for $m \leqslant n$ in $\mathbb{N}$, the restriction of I_n to λ^p_m. Then the echelon space λ^p is the reduced projective limit

$$\lambda^p = \lim_{\leftarrow} \lambda^p_n \tag{4}$$

of the projective system $\{\lambda^p_n, I_{nm}\}$, $m \leqslant n$ in $\mathbb{N}$. Further

$$(\lambda^p)^\alpha = \bigcup_{n=1}^{\infty} I_n((\lambda^p_n)^\alpha)$$

and for every $\sigma((\lambda^p)^\alpha, \lambda^p)$-bounded set M, there is $n \in \mathbb{N}$ such that

$$M \subset I_n((\lambda^p_n)^\alpha)$$

and M is $\sigma((\lambda^p_n)', \lambda^p_n)$- bounded.

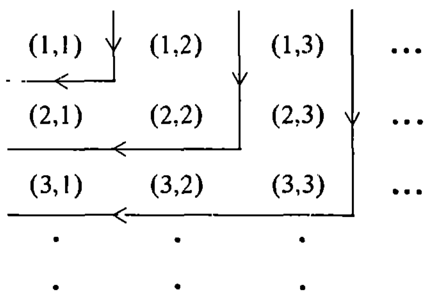
2. SCHAUDER BASIS ON $\lambda^p \hat{\otimes}_\pi \mu^r$ AND $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$

A Schauder basis in the space $E[\mathcal{T}_E]$ will be represented by

$$\{u_n\}_{n=1}^{\infty} \text{ or } \{u_n, u_n'\}_{n=1}^{\infty},$$

$\{u_n'\}_{n=1}^{\infty}$ being the sequence of coefficient functionals.

We enumerate $\mathbb{N} \times \mathbb{N}$ by means of the bijective map $\psi: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\psi(n, m) = (m-1)^2 + n$ if $n \leqslant m$ and $\psi(n, m) = (n-1)^2 + 2n - m$ if $n > m$, whose diagram is



Then, it is easy to see that, if $\{u_n, u_n'\}_{n=1}^{\infty}$ and $\{v_n, v_n'\}_{n=1}^{\infty}$ are Schauder bases in the spaces $E[\mathcal{T}_E]$ and $F[\mathcal{T}_F]$ respectively, the sequence

$$\{u_n \otimes v_m, u_n' \otimes v_m'\}_{\psi(n,m)=1}^{\infty}$$

is a Schauder basis in $E \otimes_\pi F$, which is called the tensor product basis of the given bases.

If $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis in a locally convex perfect sequence space λ, the sequence of coefficient functionals is $\{e_n\}_{n=1}^{\infty}$, considering now every e_n, $n \in \mathbb{N}$, as element of λ'. Then we have $\lambda' = \lambda^\alpha$ because $\{e_n\}_{n=1}^{\infty}$ must be a Schauder basis in $[\lambda', \sigma(\lambda', \lambda)]$. If μ is another sequence space with the same properties, $\{e_n \otimes e_m, e_n \otimes e_m\}_{\psi(n,m)=1}^{\infty}$ is a Schauder basis in $\lambda \otimes_\pi \mu$. Hence $\{e_n \otimes e_m\}_{\psi(n,m)=1}^{\infty}$ is a Shauder basis in

$$[\mathcal{B}(\lambda, \mu), \sigma(\mathcal{B}(\lambda, \mu), \lambda \otimes_\pi \mu)]$$

and every $f \in \mathcal{B}(\lambda, \mu)$ is represented by an infinite matrix (c_{ij}) such that

$$c_{ij} = \langle e_i \otimes e_j, f \rangle = \langle e_j \ , \ f(e_i) \rangle \tag{5}$$

Then, for each $i \in \mathbb{N}$, $f(e_i)$ is the sequence $(c_{ij})_{j=1}^{\infty}$ and we have

$$\forall (x_i) \in \lambda \ , \ \forall (y_j) \in \mu \ , \ \langle (x_i) \otimes (y_j), f \rangle = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij} x_i y_j. \tag{6}$$

In this section, we shall use the following observation: given the perfect spaces λ and μ as before, if $T \subset \mathbb{N}$ and $f = (c_{ij}) \in \mathcal{B}(\lambda, \mu')$, the matrix $(\overline{c}_{ij})$ such that $\overline{c}_{ij} = 0$ if $i \notin T$ and $\overline{c}_{ij} = c_{ij}$ if $i \in T$, also defines an element $\overline{T} \in \mathcal{B}(\lambda, \mu)$ as consequence of normality of λ. Then, if for each $(x_i) \in \lambda$ we denote $(\overline{x}_i)$ the sequence with $\overline{x}_i = x_i$ if $i \in T$ and $\overline{x}_i = 0$ if $i \notin T$, given two representations of $z \in \lambda \otimes \mu$

$$z = \sum_{h=1}^{t} (x_i^h) \otimes (y_j^h) = \sum_{h=1}^{t'} (x_i'^h) \otimes (y_j'^h) \tag{7}$$

we also have

$$\sum_{h=1}^{t} (\overline{x}_i^h) \otimes (y_j^h) = \sum_{h=1}^{t} (\overline{x}_i'^h) \otimes (y_j'^h) \tag{8}$$

Now, we will make a detailed study of the Schauder basis

$$\{ e_i \otimes e_j \}_{\psi(i,j)=1}^{\infty}$$

in the product $\lambda^p \hat{\otimes}_\pi \mu^r$ and $\lambda^p \otimes_\pi (\mu^r)^\alpha$, λ^p and μ^r being echelon spaces.

Lemma 1. *Let* λ^p, μ^r, $p \geq 1$, $r \geq 1$, *be echelon spaces such that* λ^1 *is reflexive. Then*

$$\lambda^p \times (\mu^r)^\alpha = \varinjlim \lambda^p \times I_k((\mu_k^r)^\alpha)$$

Proof. Let $\mathcal{T}$ be the locally convex inductive limit topology of the spaces $\lambda^p \times I_k((\mu_k^r)^\alpha)$, $k \in \mathbb{N}$ on

$$\lambda^p \times (\mu^r)^\alpha = \bigcup_{k=1}^{\infty} \lambda^p \times I_k((\mu_k^r)^\alpha)$$

The identity map I: $\lambda^p \times (\mu^r)^\alpha \to [\lambda^p \times (\mu^r)^\alpha, \mathcal{T}]$ has sequentially closed graph because I^{-1} is continuous. As $\lambda^p \times (\mu^r)^\alpha$ is ultrabornological, by the closed graph theorem, I is an isomorphism. q.e.d.

Lemma 2. *Let λ^p, μ^r, $p \geq 1$, $r \geq 1$ be echelon spaces such that λ^1 is reflexive. If F is any space, every separately continuous bilinear map $f: \lambda^p \times (\mu^r)^\alpha \to F$ is continuous.*

Proof. It si easy to see that, for every $k \in \mathbb{N}$, the restriction f_k of f to the product of metrizable barrelled spaces $\lambda^p \times I_k\ ((\mu^r_k)^\alpha)$ is separately continuous. Hence f_k is continuous and by lemma 1, f is continuous. q.e.d.

Proposition 1. *Let λ^p, μ^r, $p \geq 1$, $r \geq 1$, be echelon spaces, such that λ^1 and μ^1 are reflexive. Then*

$$\lambda^p \underset{\pi}{\otimes} \mu^r \quad , \quad \lambda^p \underset{\pi}{\otimes} (\mu^r)^\alpha \text{ and } (\lambda^p)^\alpha \underset{\pi}{\otimes} (\mu^r)^\alpha$$

are barrelled spaces.

Proof. The four spaces λ^p, λ^r, $(\lambda^p)^\alpha$ and $(\mu^r)^\alpha$ are barrelled because λ^p and μ^r are reflexive. The proof follows from lemma 2 and a well known result (see page 337 of [3]). q.e.d.

Proposition 2. *Let λ^p, μ^r, $p \geq 1$, $r \geq 1$ be echelon spaces such that λ^1 and μ^1 are reflexive. Then $\{e_i \otimes e_j, e_i \otimes e_j\}_{\psi(i,j)=1}^{\infty}$ is a Schauder basis in the spaces*

$$\lambda^p \underset{\pi}{\hat{\otimes}} \mu^r \quad , \quad \lambda^p \underset{\pi}{\hat{\otimes}} (\mu^r)^\alpha \quad \text{and} \quad (\lambda^p)^\alpha \underset{\pi}{\hat{\otimes}} (\mu^r)^\alpha$$

Proof. Let E be any of the spaces $\lambda^p \underset{\pi}{\otimes} \mu^r$, $\lambda^p \underset{\pi}{\otimes} (\mu^r)^\alpha$, $(\lambda^p)^\alpha \underset{\pi}{\otimes} (\mu^r)^\alpha$ and $\hat{E}$ its completion. As $\{e_i \otimes e_j\}_{\psi(i,j)=1}^{\infty}$ is a Schauder basis in E, the sequence

$$\{e_i \otimes e_j\}_{\psi(i,j)=1}^{\infty}$$

is a Schauder basis in $[E', \sigma(E', E)]$. Then, if $z \in E$ and $f \in E' = \hat{E}'$, the set

$$\left\{ \sum_{\psi(i,j)=1}^{k} \langle e_i \otimes e_j, f \rangle e_i \otimes e_j \ , \ k \in \mathbb{N} \right\}$$

is equicontinuous in E'; hence it is also equicontinuous in $\hat{E}'$ and $\sigma(\hat{E}', \hat{E})$-bounded. Then

$$\sup_{k \in \mathbb{N}} \left| \sum_{\psi(i,j)=1}^{k} \langle e_i \otimes e_j, f \rangle \langle z, e_i \otimes e_j \rangle \right| < \infty$$

and the proof follows from proposition 1 and a well known result on biorthogonal sequences (see [3], page 295). q.e.d.

In the proof of theorem 1, we shall need the following theorem of Grothendieck (1):

Theorem A: *Let* $F\,[\mathcal{T}_F]$ *be a Frechet space and* $E\,[\mathcal{T}_E]$ *a complete barrelled DF-space with the approximation property. Then the topologies*

$$\beta(\mathcal{B}(F, E), F \underset{\pi}{\otimes} E) \textit{ and } \beta(\mathcal{B}(F.E), F \underset{\pi}{\hat{\otimes}} E)$$

on $\mathcal{B}(F, E)$ *are identical.*

In the following, also we shall use the concepts of shrinking and boundedly complete Shauder basis. A Schauder basis $\{u_n, u_n'\}_{n=1}^{\infty}$ in the space $E\,[\mathcal{T}_E]$ is a shrinking basis if $\{u_n'\}_{n=1}^{\infty}$ is a Schauder basis in $[E', \beta(E', E)]$. The Schauder basis $\{u_n, u_n'\}_{n=1}^{\infty}$ is a boundedly complete basis if for every sequence

$$\{\alpha_n\}_{n=1}^{\infty}$$

in $\mathbb{K}$ such that the set

$$\left\{ \sum_{n=1}^{k} \alpha_n u_n \ , \ k \in \mathbb{N} \right\}$$

is bounded in E, the series

$$\sum_{n=1}^{\infty} \alpha_n u_n$$

converges in E. If $\{u_n, u_n'\}_{n=1}^{\infty}$ is a Schauder basis in the semireflexive space $E\,[\mathcal{T}_E]$ then $\{u_n, u_n'\}_{n=1}^{\infty}$ is shrinking and boundedly complete. Conversely:

if a space $E[\mathcal{T}_E]$ has a shrinking and boundedly complete Schauder basis, then $E[\mathcal{T}_E]$ is semireflexive.

Theorem 1. *Let λ^p, $p \geq 1$, be a Montel echelon space. Let μ^r, $r \geq 1$, be an echelon space such that μ^1 is reflexive. Then $\{e_i \otimes e_j, e_i \otimes e_j\}_{\psi(i,j)=1}^{\infty}$ is a shrinking basis in the spaces $\lambda^p \hat{\otimes}_\pi \mu^r$ and $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$.*

Proof. Let H be μ^r or $(\mu^r)^\alpha$. Let E be the space $\lambda^p \otimes_\pi H$ and $\hat{E}$ be its completion. By proposition 2, it is enough to show that, given $f = (c_{ij}) \in \mathcal{B}(\lambda^p, H) = \hat{E}'$, the sequence

$$f_n = f - \sum_{\psi(i,j)=1}^{n} c_{ij}\, e_i \otimes e_j \quad , \quad n \in \mathbb{N} \tag{9}$$

converges to zero in $\beta(\hat{E}', \hat{E})$.

Let us suppose that this is not true. Then, noting that in the case

$$E = \lambda^p \otimes_\pi \mu^r,$$

E is dense in the metrizable space $\hat{E}$, and using theorem A in the case

$$E = \lambda^p \otimes_\pi (\mu^r)^\alpha,$$

always exist a real number $\epsilon > 0$, a bounded sequence $\{z_n\}_{n=1}^{\infty}$ in E and a subsequence of $\{f_n\}_{n=1}^{\infty}$ (again denoted by $\{f_n\}_{n=1}^{\infty}$) such that

$$\forall n \in \mathbb{N}\ |\langle z_n, f_n\rangle| > \epsilon \tag{10}$$

Let $f_n = (c_{ij}^n)$ be the infinite matrix representation of f_n, $n \in \mathbb{N}$, and z_n, $n \in \mathbb{N}$, be

$$z_n = \sum_{h=1}^{h_n} (x_{ih}^n) \otimes (y_{jh}^n) \tag{11}$$

The proof of theorem 1 will be realized in the following steps:

1) *For every $i \in \mathbb{N}$, we have*

$$\lim_{n \to \infty} (c_{ij}^n)_{j=1}^{\infty} = 0 \quad \textit{in } H^\alpha. \tag{12}$$

Proof: Let $i \in \mathbb{N}$ be fixed. Given $h \in \mathbb{N}$, if $n > h^2$, there is $j_0 \geqslant h + 1$ such that

$$c_{ij}^n = 0 \text{ if } 1 \leqslant j < j_0 \text{ and } c_{ij}^n = c_{ij} \text{ if } j \geqslant j_0.$$

As H is reflexive, $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis in H^{α}. Then (12) follows because $(c_{ij})_{j=1}^{\infty} \in H^{\alpha}$.

2) *For every* $i \in \mathbb{N}$ *we have*

$$\lim_{v \to \infty} \sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} |c_{ij}^v| \left| \sum_{h=1}^{h_n} x_{ih}^n y_{jh}^n \right| = 0 \tag{13}$$

Proof: In the perfect space H, the normal hull of a $\sigma(H, H^{\alpha})$-bounded set is $\sigma(H, H^{\alpha})$-bounded. Then, by (12), it is enough to see that

$$L = \left\{ \left(\sum_{h=1}^{h_n} x_{ih}^n y_{jh}^n \right)_{j=1}^{\infty},\ n \in \mathbb{N} \right\} \subset H$$

is $\sigma(H, H^{\alpha})$-bounded. Let (w_j) be in H^{α}. Then $e_i \otimes (w_j) \in E'$. As $\{z_n\}_{n=1}^{\infty}$ is a bounded sequence in E, we have

$$\sup_{n \in \mathbb{N}} \left| \sum_{j=1}^{\infty} w_j \sum_{h=1}^{h_n} x_{ih}^n y_{jh}^n \right| = \sup_{n \in \mathbb{N}} \left| \sum_{h=1}^{h_n} x_{ih}^n \sum_{j=1}^{\infty} y_{jh}^n w_j \right| =$$

$$= \sup_{n \in \mathbb{N}} |<z_n, e_i \otimes (w_j)>| < \infty$$

and L is $\sigma(H, H^{\alpha})$-bounded.

3) *For each* $n \in \mathbb{N}$, *there is* $m_n \in \mathbb{N}$ *such that*

$$\epsilon < \left| \sum_{h=1}^{h_n} \sum_{j=1}^{\infty} \sum_{i=1}^{m_n} c_{ij}^n x_{ih}^n y_{jh}^n \right| \tag{14}$$

Proof: By (10), (11) and (6), given $n \in \mathbb{N}$ we have

$$\epsilon < \left| \sum_{h=1}^{h_n} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}^n x_{ih}^n y_{jh}^n \right| = \delta$$

Let T^n_{uh} be the sequence obtained from $(x^n_{ih})_{i=1}^{\infty}$ making zero its u first components. Then $\lim_{u\to\infty} T^n_{uh} = 0$ in λ^p. As $f_n \in \mathcal{L}(\lambda^p, H^\alpha_\sigma)$ and $(y^n_{jh}) \in H$, we have

$$\lim_{u\to\infty} |<(y^n_{jh}), f_n(T^n_{uh})>| = \lim_{u\to\infty} \left| \sum_{j=1}^{\infty} \sum_{i=u}^{\infty} c^n_{ij} x^n_{ih} y^n_{jh} \right| = 0;$$

Hence there is $m_n \in \mathbb{N}$ such that

$$\left| \sum_{h=1}^{h_n} \sum_{j=1}^{\infty} \sum_{i=m_n+1}^{\infty} c^n_{ij} x^n_{ih} y^n_{jh} \right| < \delta - \epsilon$$

and then (14) holds.

4) *There are strictly increasing sequences* $\{v(s), s \in \mathbb{N}\}$ *and* $\{m(s), s \in \mathbb{N}\}$ *in* $\mathbb{N}$ *such that*

$$\forall s \in \mathbb{N}, \sum_{i=1}^{v(s)} \sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} \left| c^{m(s)}_{ij} \right| \left| \sum_{h=1}^{h_n} x^n_{ih} y^n_{jh} \right| < \frac{\epsilon}{2} \tag{15}$$

and

$$\forall s \in \mathbb{N}, \frac{\epsilon}{2} < \left| \sum_{i=v(s)+1}^{v(s+1)} \sum_{j=1}^{\infty} \sum_{h=1}^{h_{m(s)}} c^{m(s)}_{ij} x^{m(s)}_{ih} y^{m(s)}_{jh} \right| \tag{16}$$

Proof: By (14) there is $v(1) \in \mathbb{N}$ such that

$$\epsilon < \left| \sum_{i=1}^{v(1)} \sum_{j=1}^{\infty} \sum_{h=1}^{h_1} c^1_{ij} x^1_{ih} y^1_{jh} \right|$$

and by (13) there is m (1) such that

$$\sum_{i=1}^{v(1)} \sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} \left| c^{m(1)}_{ij} \right| \left| \sum_{h=1}^{h_n} x^n_{ih} y^n_{jh} \right| < \frac{\epsilon}{2}$$

Again by (14), there is $v(2) > v(1)$ such that

$$\epsilon < \left| \sum_{i=1}^{v(2)} \sum_{j=1}^{\infty} \sum_{h=1}^{h_{m(1)}} c^{m(1)}_{ij} x^{m(1)}_{ih} y^{m(1)}_{jh} \right|$$

Hence, for $s = 1$, (15) and (16) hold. Let us suppose that

$$v(1) < v(2) < \ldots < v(s+1) \text{ and } m(1) < m(2) < \ldots < m(s)$$

are defined satisfaying (15) and (16). By (13) there is $m(s+1) > m(s)$ such that

$$\sum_{i=1}^{v(s+1)} \sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} \left| c_{ij}^{m(s+1)} \right| \left| \sum_{h=1}^{h_n} x_{ih}^n \, y_{jh}^n \right| < \frac{\epsilon}{2}$$

and by (14) there is $v(s+2) > v(s+1)$ such that

$$\epsilon < \left| \sum_{i=1}^{v(s+2)} \sum_{j=1}^{\infty} \sum_{h=1}^{h_{m(s+1)}} c_{ij}^{m(s+1)} \, x_{ih}^{m(s+1)} \, y_{jh}^{m(s+1)} \right|$$

Then, (15) and (16) hold for $s+1$.

5) By proposition 2, $\{ e_i \otimes e_j \}_{\psi(i,j)=1}^{\infty}$ is a Schauder basis in $[E', \sigma(E', E)]$ and hence $\lim_{n \to \infty} f_n = 0$ in $\sigma(E', E)$. E being barrelled (proposition 1), the sequence $\{ f_n \}_{n=1}^{\infty}$ must be equicontinuous. Then there are $k_0 \in \mathbb{N}$ and a 0-neighbourhood V in H such that

$$\forall n \in \mathbb{N} \quad f_n \in E_{k_0, V}^{\circ} \tag{17}$$

Let us suppose that $a^k = (a_i^k)$ and $b^k = (b_i^k)$, $k \in \mathbb{N}$, are the echelon sequences which defines λ^p and μ^r respectively. As $\{ z_n \}_{n=1}^{\infty}$ is a bounded sequence in E, for each $k \in \mathbb{N}$ there is a real number $M_k > 0$ such that

$$\forall n \in \mathbb{N}, \quad (N_k \otimes p_V)(z_n) < M_k$$

Then, for every k and n in $\mathbb{N}$, there is a representation of z_n

$$z_n = \sum_{h=1}^{h_{nk}} (x_{ih}^{nk}) \otimes (y_{jh}^{nk}) \tag{18}$$

such that

$$\forall n \in \mathbb{N}, \quad \sum_{h=1}^{h_{nk}} \left(\sum_{i=1}^{\infty} \left| x_{ih}^{nk} \right|^p a_i^k \right)^{1/p} p_V((y_{jh}^{nk})) < M_k \tag{19}$$

Let now $\{d_k\}_{k=k_0}^{\infty}$ be a sequence of positive real numbers such that

$$\sum_{k=k_0}^{\infty} \frac{1}{d_k} = \frac{1}{2\,k_0} \tag{20}$$

and let us define for every $k \in \mathbb{N}$ and $s \in \mathbb{N}$

$$T_{ks} = \left\{ i \in \mathbb{N} \;/\; v(s) + 1 \leqslant i \leqslant v(s+1) \text{ and } a_i^k > \left(\frac{M_k\, d_k}{\epsilon}\right)^p \cdot a_i^{k_0} \right\} \tag{21}$$

Given a set $D \subset T_{ks}$, for every $(x_i) \in \omega$ we consider the sequence $(\bar{x}_i)$ such that $\bar{x}_i = x_i$ if $i \in D$ and $\bar{x}_i = 0$ if $i \notin D$. Now we define the sequence in E

$$J_D^n = \sum_{h=1}^{h_n} (\bar{x}_{ih}^n) \otimes (y_{jh}^n) \quad , \quad n \in \mathbb{N}$$

Then we have:

6) *For* $k \geqslant k_0$, $s \in \mathbb{N}$ *and* $D \subset T_{ks}$ *it is valid the inequality*

$$\forall n \in \mathbb{N}. \;,\quad (N_{k_0} \otimes p_V)(J_D^n) < \frac{\epsilon}{d_k} \tag{22}$$

Proof: By (7), (8) and (18) one has

$$\forall n \in \mathbb{N} \quad , \quad J_D^n = \sum_{h=1}^{h_{nk}} (\bar{x}_{ih}^{nk}) \otimes (y_{jh}^{nk})$$

and by (19)

$$M_k > \sum_{h=1}^{h_{nk}} \Big(\sum_{i \in D} \left| x_{ih}^{nk} \right|^p a_i^k \Big)^{1/p} \cdot p_V\big((y_{jh}^{nk})\big) >$$

$$> \frac{M_k\, d_k}{\epsilon} \sum_{h=1}^{h_{nk}} \Big(\sum_{i \in D} \left| x_{ih}^{nk} \right|^p a_i^{k_0} \Big)^{1/p} \cdot p_V\big((y_{jh}^{nk})\big) \geqslant \frac{M_k\, d_k}{\epsilon} (N_{k_0} \otimes p_V)(J_D^n)$$

and hence (22) follows.

7) *For each* $s \in \mathbb{N}$, *there is* i_s *such that* $v(s) + 1 \leqslant i_s \leqslant v(s+1)$ *and for all* $k \geqslant k_0$, $i_s \notin T_{ks}$.

Proof: If 7) were not true, there would be natural numbers $k_1, k_2, \ldots, k_t$ greater than $k_0 - 1$ such that

$$\{ i \,/\, v(s) + 1 \leqslant i \leqslant v(s+1) \} = \bigcup_{u=1}^{t} T_{k_u s} = \bigcup_{u=1}^{t} D_{us}$$

where the sets $D_{1s}, D_{2s}, \ldots, D_{ts}$ are pairwise disjoints and each $D_{us} \subset T_{k_u s}$, $u = 1, 2, \ldots, t$. Then, by (16), (17) and (22)

$$\frac{\epsilon}{2} < \sum_{u=1}^{t} \left| \sum_{i \in D_{us}} \sum_{j=1}^{\infty} \sum_{h=1}^{h_{m(s)}} c_{ij}^{m(s)} x_{ih}^{m(s)} y_{jh}^{m(s)} \right| \leqslant$$

$$\leqslant \sum_{u=1}^{t} \sup_{(c_{ij}) \in E^{\circ}_{k_0,V}} \left| \sum_{i \in D_{us}} \sum_{j=1}^{\infty} \sum_{h=1}^{h_{m(s)}} c_{ij} x_{ij}^{m(s)} y_{jh}^{m(s)} \right| =$$

$$= \sum_{u=1}^{t} \sup_{(c_{ij}) \in E^{\circ}_{k_0,V}} \left| < J_{D_{us}}^{m(s)}, (c_{ij}) > \right| = \sum_{u=1}^{t} k_0 \, (N_{k_0} \otimes pv) \, (J_{D_{us}}^{m(s)}) <$$

$$< \sum_{u=1}^{t} k_0 \frac{\epsilon}{d_{k_u}} < k_0 \frac{\epsilon}{2 k_0} = \frac{\epsilon}{2}$$

which is a contradiction.

8) *End of the proof of theorem 1:* By the step 7) we construct an infinite sequence $\{ i_s \}_{s=1}^{\infty}$ such that

$$\forall s \in \mathbb{N} \quad , \quad \forall k \geqslant k_0 \quad , \quad a_{i_s}^{k} \leqslant \left(\frac{M_k \, d_k}{\epsilon} \right)^{p} a_{i_s}^{k_0}$$

which is a contradiction with the fact that λ^p is a Montel space and hence the sequence $a^k = (a_i^k)$, $k \in \mathbb{N}$, is strongly increasing. q.e.d.

Theorem 2. Let λ^p, $p \geqslant 1$ *be a Montel echelon space and* μ^r, $r \geqslant 1$, *be an echelon space such that* μ^1 *is reflexive. Then* $\{ e_i \otimes e_j, e_i \otimes e_j \}_{\psi(i,j)=1}^{\infty}$ *is a boundedly complete Schauder basis in* $\lambda^p \hat{\otimes}_{\pi} \mu^r$ *and* $\lambda^p \hat{\otimes}_{\pi} (\mu^r)^{\alpha}$.

Proof. Let H be the space μ^I or the space $(\mu^I)^\alpha$ and $E = \lambda^p \hat{\otimes}_\pi H$. Let us suppose that $\{\alpha_{ij}\}_{\psi(i,j)=1}^{\infty}$ is a sequence in $\mathbb{K}$ such that the sequence of partial sums

$$S_n = \sum_{\psi(i,j)=1}^{n} \alpha_{ij} e_i \otimes e_j \quad , \quad n \in \mathbb{N}$$

is bounded in E. The proof will be complete if we show that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in E.

Let us suppose that $\{S_n\}_{n=1}^{\infty}$ is not a Cauchy sequence in E. Then there are a real number $\epsilon > 0$, an equicontinuos sequence $\{f_n\}_{n=1}^{\infty}$ in E', and two strictly increasing sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that

$$n_k < m_k < m_k^2 < n_{k+1} \qquad \forall k \in \mathbb{N} \tag{23}$$

and

$$|< S_{m_k} \quad S_{n_k}, f_k > | > \epsilon \qquad \forall k \in \mathbb{N} \tag{24}$$

By theorem 1, $\{e_i \otimes e_j, e_i \otimes e_j\}_{\psi(i,j)=1}^{\infty}$ is a shrinking basis in E. Then, for each $f = (c_{ij}) \in E'$, the sequence

$$\sum_{\psi(i,j)=1}^{n} c_{ij} e_i \otimes e_j \qquad \forall n \in \mathbb{N}$$

is $\beta(E', E)$–Cauchy in E'. Then $\{S_n\}_{n=1}^{\infty}$ is $\sigma(E, E')$–Cauchy in E because

$$\left| < S_m \quad S_n, f > \right| = \left| \sum_{\psi(i,j)=n+1}^{m} \alpha_{ij} c_{ij} \right| \leqslant \sup_{k \in \mathbb{N}} \left| < S_k, \sum_{\psi(i,j)=n+1}^{m} c_{ij} e_i \otimes e_j > \right|$$

Hence, if we put

$$z_k = S_{m_k} - S_{n_k} = \sum_{\psi(i,j)=n_k+1}^{m_k} \alpha_{ij} e_i \otimes e_j \quad , \quad k \in \mathbb{N}$$

we obtain

$$\lim_{k \to \infty} z_k = 0 \text{ in } \sigma(E, E') \tag{25}$$

As H is a reflexive space with Schauder basis, H^{α} has a Schauder basis. Then E is separable and $\sigma(E', E)$ is metrizable on the equicontinuous subsets of E'. In consequence, we can take a subsequence

$$\{ f_{n_t} \}_{t=1}^{\infty}$$

of $\{ f_n \}_{n=1}^{\infty}$ which $\sigma(E', E)$ -converges to $f \in E'$. By (25), there is $t_0 \in \mathbb{N}$ such that

$$\forall t \geqslant t_0,\ | < z_{n_t}, f_{n_t} - f > | \geqslant | < z_{n_t}, f_{n_t} > | - | < z_{n_t}, f > | \geqslant \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Then, putting $g_n = f_n - f$, $n \in \mathbb{N}$, and taking a subsequence if it is neccesary, we can suppose that

$$\lim_{n \to \infty} g_n = 0 \quad \text{in} \quad \sigma(E', E) \tag{26}$$

$$\lim_{n \to \infty} z_n = 0 \quad \text{in} \quad \sigma(E, E') \tag{27}$$

$$\forall n \in \mathbb{N}\ | < z_n, g_n > | > \frac{\epsilon}{2} \tag{28}$$

Let $g_n = (c_{ij}^n)$ be the infinite matrix representation of g_n, $n \in \mathbb{N}$, and let us define for every $i, s \in \mathbb{N}$ the set

$$F_{is} = \{ j \in \mathbb{N} \ / \ n_s + 1 \leqslant \psi(i, j) \leqslant m_s \}$$

The proof will be completed in the following steps:

1) *For each* $i \in \mathbb{N}$

$$\lim_{n \to \infty} \sup_{s \in \mathbb{N}} \left| \sum_{j \in F_{is}} \alpha_{ij} c_{ij}^n \right| = 0 \tag{29}$$

Proof: Fixed $i \in \mathbb{N}$, $g_n(e_i) = (c_{ij}^n)_{j=1}^{\infty} \in H^{\alpha}$. Given $(w_j) \in H$, using (26)

$$0 = \lim_{n \to \infty} | < e_i \otimes (w_j), g_n > | = \lim_{n \to \infty} \left| \sum_{j=1}^{\infty} w_j c_{ij}^n \right|$$

and hence

$$\lim_{n\to\infty} (c_{ij}^n)_{j=1}^{\infty} = 0 \quad \text{in} \quad [H^\alpha, \sigma(H^\alpha, H)]. \tag{30}$$

H being perfect, for each $(w_j) \in H^\alpha$, there is $(w_j') \in H^\alpha$ such that $\alpha_{ij} w_j' = |\alpha_{ij} w_j|$; as $e_i \otimes (w_j') \in F'$ and $\{S_n\}_{n=1}^{\infty}$ s a bounded sequence in E, we have

$$\sup_{m \geqslant i^2} \sum_{j=1}^{m} |\alpha_{ij} w_j| \leqslant \sup_{n\in\mathbb{N}} |< S_n, e_i \otimes (w_j') >| < \infty$$

Then $(\alpha_{ij})_{j=1}^{\infty} \in H$; as (30) also holds in the normal topology of the perfect space H^α, we have

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |\alpha_{ij} c_{ij}^n| = 0$$

and (29) follows from (23) and the definition of F_{is}.

2) *There are two strictly increasing sequences* $\{v(s)\}_{s=1}^{\infty}$ *and* $\{t(s)\}_{s=1}^{\infty}$ *in* $\mathbb{N}$ *such that*

$$\frac{\epsilon}{4} < \left| \sum_{i=t(s)+1}^{t(s+1)} \sum_{j\in F_{i,v(s)}} \alpha_{ij} c_{ij}^{v(s)} \right| \tag{31}$$

Proof: By (28)

$$\frac{\epsilon}{2} < |< z_1, g_1 >| = \left| \sum_{\psi(i,j)=n_1+1}^{m_1} \alpha_{ij} c_{ij}^1 \right|$$

Let $t(1) = \max\{ i \in \mathbb{N} / \text{there is } j \in \mathbb{N} \text{ such that } n_1 + 1 \leqslant \psi(i,j) \leqslant m_1 \}$ and $v(1) = 1$. Let us suppose that we have defined $t(1) < t(2) < \ldots < t(s)$ and $v(1) < v(2) < \ldots < v(s-1)$ for $s \geqslant 1$ in such a way that (31) holds. Using (30), there is $v(s) > v(s-1)$ such that

$$\sum_{i=1}^{t(s)} \sup_{h\in\mathbb{N}} \left| \sum_{j\in F_{ih}} \alpha_{ij} c_{ij}^{v(s)} \right| < \frac{\epsilon}{4} \tag{32}$$

Now, we define

$t(s+1) = \max \{ i \in \mathbb{N} / \text{ there is } j \in \mathbb{N} \text{ such that } n_{v(s)} + 1 \leqslant \psi(i,j) \leqslant m_{v(s)} \}$.

Then, by (28)

$$\frac{\epsilon}{2} < | < z_{v(s)}, g_{v(s)} > | = \left| \sum_{\psi(i,j) = n_{v(s)}+1}^{m_{v(s)}} \alpha_{ij} c_{ij}^{v(s)} \right| \leqslant$$

$$\leqslant \left| \sum_{i=1}^{t(s)} \sum_{j \in F_{i,v(s)}} \alpha_{ij} c_{ij}^{v(s)} \right| + \left| \sum_{i=t(s)+1}^{t(s+1)} \sum_{j \in F_{i,v(s)}} \alpha_{ij} c_{ij}^{v(s)} \right|$$

and by (32), (31) holds for v (s) and t (s + 1). From (31) we also obtain

$$t(s+1) > t(s).$$

3) As E is barrelled (proposition 1), by (26) the sequence $\{g_n\}_{n=1}^{\infty}$ is equicontinuous. Then there is $k_0 \in \mathbb{N}$ and a 0-neighbourhoood V in H such that

$$\forall n \in \mathbb{N} \qquad g_n \in E^{\circ}_{k_0,V} \tag{33}$$

Let $a^k = (a_i^k)$ be the sequence of echelons which defines λ^p. From boundedness of sequence $\{z_n\}_{n=1}^{\infty}$ in E, we obtain a sequence $\{M_k\}_{k=1}^{\infty}$ of positive real numbers such that

$$\forall n \in \mathbb{N} \quad , \quad \forall k \in \mathbb{N} \qquad (N_k \otimes p_V)(z_n) < M_k \tag{34}$$

and from (34), for every k, n in $\mathbb{N}$ we obtain a representation of z_n

$$z_n = \sum_{h=1}^{h_{nk}} (x_{ih}^{nk}) \otimes (y_{jh}^{nk}) \tag{35}$$

such that

$$\forall k \in \mathbb{N}, \ \forall n \in \mathbb{N} \ \sum_{h=1}^{h_{nk}} \Big(\sum_{i=1}^{\infty} |x_{ij}^{nk}|^p a_i^k \Big)^{1/p} \cdot p_V((y_{jh}^{nk})) < M_k \tag{36}$$

Now, we choose a sequence $\{d_k\}_{k=k_0}^{\infty}$ of positive real numbers such that

$$\sum_{k=k_0}^{\infty} \frac{1}{d_k} = \frac{1}{8 k_0}$$

and we define for $k \geqslant k_0$ and $s \in \mathbb{N}$ the set

$$T_{ks} = \left\{ i \in \mathbb{N} \,/\, t(s) + 1 \leqslant i \leqslant t(s+1) \text{ and } a_i^k > \left(\frac{M_k\, d_k}{\epsilon}\right)^p \cdot a_i^{k_0} \right\}$$

If, for $D \subset T_{ks}$, we define, equal as in theorem 1, the elements

$$J_D^n = \sum_{h=1}^{h_{nk}} (\bar{x}_{ih}^{nk}) \otimes (y_{jh}^{nk}) = \sum_{i \in D} \sum_{j \in F_{in}} \alpha_{ij}\, e_i \otimes e_j \quad , \quad n \in \mathbb{N}$$

(this equality holds by (7), (8) and the original definition of z_n), we obtain, exactly with the same reasoning

$$\forall n \in \mathbb{N} \qquad (N_{k_0} \otimes p_V)(J_D^n) < \frac{\epsilon}{d_k} \tag{37}$$

4) *For each* $s \in \mathbb{N}$, *there is* $i_s \in \mathbb{N}$ *such that* $t(s) + 1 \leqslant i_s \leqslant t(s+1)$ *and* i_s *belongs to no* T_{ks} *with* $k \geqslant k_0$.

Proof.: In another case, and as in theorem 1, there would be in $\mathbb{N}$ $k_1, k_2, .., k_t$ higher or equal to k_0 and pairwise disjoints sets $D_{us} \subset T_{k_u s}$, $u = 1, 2, \ldots, t$ such that

$$\{ i \,/\, t(s) + 1 \leqslant i \leqslant t(s+1) \} = \bigcup_{u=1}^{t} T_{k_u s} = \bigcup_{u=1}^{t} D_{us}$$

Then, by (31), (33) and (37)

$$\frac{\epsilon}{4} < \left| \sum_{i=t(s)+1}^{t(s+1)} \sum_{j \in F_{i,v(s)}} \alpha_{ij}\, c_{ij}^{v(s)} \right| \leqslant \sum_{u=1}^{t} \left| \sum_{i \in D_{us}} \sum_{j \in F_{i,v(s)}} \alpha_{ij}\, c_{ij}^{v(s)} \right| =$$

$$= \sum_{u=1}^{t} \left| < J_{D_{us}}^{v(s)}, c_{ij}^{v(s)} > \right| \leqslant \sum_{u=1}^{t} \sup_{(c_{ij}) \in E_{k_0,V}^{o}} \left| < J_{D_{us}}^{v(s)}, (c_{ij}) > \right| =$$

$$= \sum_{u=1}^{t} k_0 (N_{k_0} \otimes p_V)(J_{D_{us}}^{v(s)}) < k_0 \sum_{u=1}^{t} \frac{\epsilon}{d_{k_u}} < k_0 \frac{\epsilon}{8\, k_0} = \frac{\epsilon}{8}$$

which is a contradiction.

5) *End of the proof of theorem 2:* By 4) we can construct an infinite sequence $\{i_s\}_{s=1}^{\infty}$ in $\mathbb{N}$ such that

$$\forall k \geqslant k_0, \ \forall s \in \mathbb{N} \quad , \quad a_{i_s}^k \leqslant \left(\frac{M_k\, d_k}{\epsilon}\right)^p \cdot a_{i_s}^{k_0}$$

which is impossible because, λ^p being a Montel space, the sequence of echelons $a^k = (a_i^k)$ is strongly increasing. q.e.d.

3. CHARACTERIZATION OF REFLEXIVITY.

We began with a new proof of the classical result of Holub ([2]) with a method which we shall use later.

Theorem 3. *(Holub,* [2]) *If* $p > 1, r > 1$, *the tensor product* $\ell^p \hat{\otimes}_\pi \ell^r$ *is reflexive if and only if* $p > r/(r-1)$.

Proof. Sufficiency: If A and B are the closed unit balls of ℓ^p and ℓ^r, $\overline{\text{aco}}\,(A \otimes B)$ is the closed unit ball of $\ell^p \hat{\otimes}_\pi \ell^r$. By the theorems of Krein and Eberlein, the reflexivity of this space will be proved if we show that every sequence in $A \otimes B$ has a weakly convergent subsequence. If p' and r' are the conjugated numbers of p and r and $\{x^n \otimes y^n\}_{n=1}^{\infty}$ is a sequence in $A \otimes B$, by reflexivity or ℓ^p and ℓ^r, we can suppose, passing to a subsequence if it is neccesary, that $\{x^n\}_{n=1}^{\infty}$ converges to x in $\sigma(\ell^p, \ell^{p'})$ and $\{y^n\}_{n=1}^{\infty}$ converges to y in $\sigma(\ell^r, \ell^{r'})$. Let f be an element of $\mathcal{B}(\ell^p, \ell^r) = \mathcal{L}(\ell^p, \ell^{r'})$ (by closed graph theorem). As

$$p > r/(r-1) = r',$$

by Pitt theorem, $f \in \mathcal{L}(\ell^p, \ell^{r'})$ is compact. As $\ell^{p'}$ is separable and ℓ^p is reflexive, by Schauder theorem on compactness of the adjoint f' of f, we have that

$$\{f(x^n)\}_{n=1}^{\infty}$$

converges to $f(x)$ in $\ell^{r'}$. As $\{y^n\}_{n=1}^{\infty}$ is bounded in the space ℓ^r, we obtain that $\{x^n \otimes y^n\}_{n=1}^{\infty}$ converges to $x \otimes y$ in $\sigma(\ell^p \hat{\otimes}_\pi \ell^r, \mathcal{B}(\ell^p, \ell^r))$ because

$$|<x^n \otimes y^n - x \otimes y, f>| \leqslant |<(x^n - x) \otimes y^n, f>| + |<x \otimes (y^n - y), f>| =$$

$$= |<y^n, f(x^n - x)>| + |<y^n - y, f(x)>|$$

Necessity: Let us suppose that $p \leqslant r' = r / (r - 1)$. We follow the previous notations. Now the identity map I from ℓ^p into $\ell^{r'}$ is not compact. Hence and by reflexivity of ℓ^p there is in A a $\sigma(\ell^p, \ell^{p'})$-convergent sequence $x^n = (x_i^n)$, $n \in \mathbb{N}$, to $x = (x_i) \in A$ which has no convergent subsequence in $\ell^{r'}$. In particular, as $x \in \ell^{r'}$, there is a real number $\epsilon > 0$, a subsequence of $\{x^n\}_{n=1}^{\infty}$ (again denoted by $\{x^n\}_{n=1}^{\infty}$) and a sequence $z^n = (z_i^n)$, $n \in \mathbb{N}$ in B such that

$$\forall n \in \mathbb{N} \qquad |< z^n, x^n - x >| > \epsilon \tag{38}$$

As ℓ^r is reflexive, we can suppose, passing to a subsequence if it is necessary, that $\{z^n\}_{n=1}^{\infty}$ $\sigma(\ell^r, \ell^{r'})$-converges to $z = (z_i) \in B$.

If $\ell^p \hat{\otimes}_\pi \ell^r$ were reflexive, there would be a $\sigma(\ell^p \hat{\otimes}_\pi \ell^r, \mathcal{B}(\ell^p, \ell^r))$-convergent subsequence of $\{x^n \otimes z^n\}_{n=1}^{\infty} \subset A \otimes B$ to the limit $w \in \ell^p \hat{\otimes}_\pi \ell^r$. We shall denote again this subsequence by $\{x^n \otimes z^n\}_{n=1}^{\infty}$. As $\{e_i \otimes e_j, e_i \otimes e_j\}_{\psi(i,j)=1}^{\infty}$ is a Schauder basis in $\ell^p \hat{\otimes}_\pi \ell^r$ and for every i, j in $\mathbb{N}$ we have

$$< w, e_i \otimes e_j > = \lim_{n \to \infty} < x^n \otimes z^n, e_i \otimes e_j > = \lim_{n \to \infty} x_i^n z_j^n = x_i z_j$$

we obtain $w = x \otimes z$. But $I \in \mathcal{L}(\ell^p, \ell^{r'}) = \mathcal{B}(\ell^p, \ell^r)$ (closed graph theorem). Then

$$< x^n \otimes z^n - x \otimes z, I > = < z^n, x^n > - < z, x > = < z^n, x^n - x > + < z^n - z, x >$$

and hence, $< z^n, x^n - x >$ must be arbitrarily small with n, which contradicts (38). Then $\ell^p \hat{\otimes}_\pi \ell^r$ is not reflexive. q.e.d.

Theorem 4. *Let λ^p, μ^r, $p \geqslant 1$, $r \geqslant 1$, be echelon spaces. Then:*

1) *$\lambda^1 \hat{\otimes}_\pi \mu^1$ is reflexive if and only if λ^1 and μ^1 are reflexive.*

2) *If $r > 1$, $\lambda^1 \hat{\otimes}_\pi \mu^r$ is reflexive if and only if λ^1 is reflexive.*

3) *If $p > 1$, $r > 1$ and $p > r/(r-1)$, $\lambda^p \hat{\otimes}_\pi \mu^r$ is always reflexive.*

4) *If $p > 1$, $r > 1$ and $p \leqslant r/(r-1)$, $\lambda^p \hat{\otimes}_\pi \mu^r$ is reflexive if and only if λ^p or μ^r is a Montel space.*

Proof. 1) and 2). If λ^1 is reflexive, then it is a Montel space. If μ^1 is reflexive, by proposition 2 and theorems 1 and 2, $\lambda^1 \hat{\otimes}_\pi \mu^r$ has a shrinking and boundedly complete Schauder basis. Then the Frechet space $\lambda^1 \hat{\otimes}_\pi \mu^r$ is reflexive. The reciprocal statement is obvious.

3) By theorem 3, for every k, h $\in \mathbb{N}$, $\lambda^p_k \hat{\otimes}_\pi \mu^r_h$ is reflexive. As

$$\lambda^p \hat{\otimes}_\pi \mu^r = \varprojlim \lambda^p_k \hat{\otimes}_\pi \mu^r_h ,$$

the Frechet space $\lambda^p \hat{\otimes}_\pi \mu^r$ is reflexive.

4) If λ^p or μ^r is Montel, the reflexivity of $\lambda^p \hat{\otimes}_\pi \mu^r$ follows from proposition 2 and theorems 1 and 2 as in 1) and 2). Conversely, if neither λ^p nor μ^r is Montel, there are sectional subspaces F and G of λ^p and μ^r isomorphic to ℓ^p and ℓ^r respectively. As F and G are complemented in λ^p and μ^r, $\lambda^p \hat{\otimes}_\pi \mu^r$ has a subspace isomorphic to $\ell^p \hat{\otimes}_\pi \ell^r$, which by theorem 3 is not reflexive. Then $\lambda^p \hat{\otimes}_\pi \mu^r$ is not reflexive. q.e.d.

Theorem 5. *Let* λ^p, μ^r, $p \geq 1$, $r \geq 1$ *be echelon spaces. Then:*

1) $\lambda^1 \hat{\otimes}_\pi (\mu^1)^\alpha$ *is reflexive if and only if* λ^1 *and* μ^1 *are reflexive.*

2) *If* $r > 1$, $\lambda^1 \hat{\otimes}_\pi (\mu^r)^\alpha$ *is reflexive if and only if* λ^1 *is reflexive.*

3) *If* $p > 1$, $r > 1$ *and* $p > r$, $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$ *is always reflexive.*

4) *If* $p > 1$, $r > 1$ *and* $p \leq r$, $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$ *is reflexive if and only if* λ^p *or* μ^r *is a Montel space.*

Proof. 1) and 2). Using proposition 1, the proof is the same as in 1) and 2) of theorem 4.

3). As

$$E = \lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha = \varprojlim \lambda^p_k \hat{\otimes}_\pi (\mu^r)^\alpha$$

by proposition 1, it is enough to see that every $E_k = \lambda^p_k \hat{\otimes}_\pi (\mu^r)^\alpha$, $k \in \mathbb{N}$ is semi-reflexive. As λ^p_k and $(\mu^r)^\alpha$ are DF-spaces, given a bounded set M in E_k, there are bounded sets A and B in λ^p_k and $(\mu^r)^\alpha$ such that $M \subset \overline{\text{aco}}\,(A \otimes B)$. By Krein and Eberlein theorems, the proof will be complete if we show that every sequence in $A \otimes B$ has a $\sigma(E_k, E_k')$-convergent subsequence.

Let $\{x^n \otimes y^n\}_{n=1}^{\infty}$ be a sequence in $A \otimes B$ and let $h \in \mathbb{N}$ be such that $B \subset I_h((\mu_h^r)^\alpha) = (\mu_h^r)'$. As λ_k^p and μ_h^r are reflexive Banach spaces, we can choose a subsequence, that will be denoted as in the beginning, such that

$$\lim_{n \to \infty} x^n = x \in \lambda_k^p \quad \text{in} \quad \sigma(\lambda_k^p, (\lambda_k^p)') \tag{39}$$

and

$$\lim_{n \to \infty} y^n = y \in (\mu_h^r)' \quad \text{in} \quad \sigma((\mu_h^r)', \mu_h^r) \tag{40}$$

If $f \in \mathcal{B}(\lambda_k^p, (\mu^r)^\alpha) = \mathcal{L}(\lambda_k^p, \mu^r)$ (by closed graph theorem), since $p > r$, the map $I_h \circ f \in \mathcal{L}(\lambda_k^p, \mu_h^r)$ is compact by Pitt theorem; as $(\lambda_k^p)'$ is separable, by Schauder theorem on compactness of the adjoint of $(I_h \circ f)$, we obtain that

$$\{(I_h \circ f)(x^n)\}_{n=1}^{\infty}$$

converges to $(I_h \circ f)(x)$ in μ_h^r. Since $\{y^n\}_{n=1}^{\infty}$ is bounded in $(\mu_h^r)'$, by (40) we deduce from

$$< x^n \otimes y^n - x \otimes y, f > = < (x^n - x) \otimes y^n, f > + < x \otimes (y^n - y), f > =$$

$$= < y^n, f(x^n - x) > + < y^n - y, f(x) > = < y^n, (I_h \circ f)(x^n - x) > +$$

$$+ < y^n - y, (I_h \circ f)(x) >$$

that $\{x^n \otimes y^n\}_{n=1}^{\infty}$ converges to $x \otimes y$ in $\sigma(E_k, F_k')$.

4) Sufficiency: if λ^p is Montel, the proof is the same as in 1) and 2). If μ^r is Montel, we argue as in 3) with the same notations. For each $k \in \mathbb{N}$, given the sequence $\{x^n \otimes y^n\}_{n=1}^{\infty}$ in the tensor product of bounded sets

$$A \otimes B \subset \lambda_k^p \otimes (\mu^r)^\alpha,$$

being λ_k^p a reflexive Banach space and $(\mu^r)^\alpha$ a Montel space, as μ^r is separable, by Smulian theorem, we can choose a subsequence, again denoted by $\{x^n \otimes y^n\}_{n=1}^{\infty}$ such that $\{x^n\}_{n=1}^{\infty}$ converges to x in $\sigma(\lambda_k^p, (\lambda_k^p)')$ and $\{y^n\}_{n=1}^{\infty}$ converges to y in $\sigma((\mu^r)^\alpha, \mu^r)$. Since μ^r is a reflexive Frechet space, $(\mu^r)^\alpha$ is ultrabornological. Then, by closed graph theorem, every $f \in \mathcal{B}(\lambda_k^p, (\mu^r)^\alpha)$ can identified with an element of $\mathcal{L}((\mu^r)^\alpha, (\lambda_k^p)')$. Hence $\{f(y^n)\}_{n=1}^{\infty}$ converges to $f(y)$ in $(\lambda_k^p)'$. As $\{x^n\}_{n=1}^{\infty}$ is a bounded sequence in λ_k^p, from

$$<x^n \otimes y^n \quad x \otimes y, f> = <x^n \otimes (y^n - y), f> + <(x^n - x) \otimes y, f> =$$

$$= <x^n, f(y^n - y)> + <(x^n - x), f(y)>$$

we deduce that $\{x^n \otimes y^n\}_{n=1}^{\infty}$ converges to $x \otimes y$ in $\sigma(E_k, E'_k)$. The proof finishes as in 3), using proposition 1.

Necessity: Let us suppose neither λ^p nor μ^r are Montel spaces. Then there are complemented sectional subspaces F and G of λ^p and μ^r respectively such that $\lambda^p = \ell^p \oplus F$ and $\mu^r = \ell^r \oplus G$. Then, if $r' = r/(r \quad 1)$, $(\mu^r)^\alpha = \ell^{r'} \oplus G'$. As $p \leq r$, by theorem 3, $\ell^p \hat{\otimes}_\pi \ell^{r'}$ is a not reflexive subspace of $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$. Then $\lambda^p \hat{\otimes}_\pi (\mu^r)^\alpha$ is not reflexive. q.e.d.

Theorem 6. *Let* $\lambda^p, \mu^r, p \geq 1, r \geq 1$, *echelon spaces. Then:*

1) $(\lambda^1)^\alpha \hat{\otimes}_\pi (\mu^1)^\alpha$ *is reflexive if and only if* λ^1 *and* μ^1 *are reflexive.*

2) *If* $r > 1$, $(\lambda^1)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ *is reflexive if and only if* λ^1 *is reflexive.*

3) *If* $p > 1$, $r > 1$ and $p/(p \quad 1) > r$, *then* $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ *is always reflexive.*

4) *If* $p > 1$, $r > 1$ *and* $p/(p - 1) \leq r$, $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ *is reflexive if and only if* λ^p *or* μ^r *are Montel spaces.*

Proof. 1) and 2). Necessity of 1) and 2) is evident. For sufficiency, it is enough to see, by proposition 1, that $(\lambda^1)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$, $r \geq 1$, is semireflexive. Being $(\lambda^1)^\alpha$ and $(\mu^r)^\alpha$ DF-spaces, by Krein and Eberlein theorems, it suffices to prove that every sequence $\{x^n \otimes y^n\}_{n=1}^{\infty}$ in the tensor product $A \otimes B$ of the bounded sets A and B of $(\lambda^1)^\alpha$ and $(\mu^r)^\alpha$ respectively, has a weakly convergent subsequence. If λ^1 is reflexive (and μ^1 in case 1)) being λ^1 and μ^r separable spaces, by Smulian theorem we can suppose, choosing a subsequence if it is necessary, that $\{x^n\}_{n=1}^{\infty}$ is $\sigma((\lambda^1)^\alpha, \lambda^1)$-convergent to $x \in (\lambda^1)^\alpha$ and $\{y^n\}_{n=1}^{\infty}$ is $\sigma((\mu^r)^\alpha, \mu^r)$-convergent to $y \in (\mu^r)^\alpha$. But λ^1 being reflexive, λ^1 and $(\lambda^1)^\alpha$ are Montel spaces. Then $\{x^n\}_{n=1}^{\infty}$ converges to x in $\beta((\lambda^1)^\alpha, \lambda^1)$. Consequently, if $f \in \mathcal{B}((\lambda^1)^\alpha, (\mu^r)^\alpha) = \mathcal{L}((\lambda^1)^\alpha, \mu^r)$ (by closed graph theorem, being λ^1 Frechet reflexive and hence $(\lambda^1)^\alpha$ ultrabornological), we have that $\{f(x^n)\}_{n=1}^{\infty}$ has limit $f(x)$ in μ^r. Since μ^r is barrelled, $\{y^n\}_{n=1}^{\infty}$ is an equicontinuous sequence and hence $\{x^n \otimes y^n\}_{n=1}^{\infty}$ is weakly convergent to $x \otimes y$ because

$$< x^n \otimes y^n - x \otimes y, f > = < (x^n - x) \otimes y^n, f > + < x \otimes (y^n - y), f > =$$

$$= < y^n, f(x^n - x) > + < y^n - y, f(x) >$$

3) Since λ^p and μ^r are reflexive, $(\lambda^p)^\alpha$ and $(\mu^r)^\alpha$ are barrelled. Hence $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ is barrelled and we argue as in 1) and 2) with the same notations. Now, given the tensor product $A \otimes B$ of bounded sets of $(\lambda^p)^\alpha$ and $(\mu^r)^\alpha$, we choose $k \in \mathbb{N}$ such that A is bounded in the reflexive space $(\lambda_k^p)'$ and B is bounded in the reflexive space $(\mu_k^r)'$. Let $\{x^n \otimes y^n\}_{n=1}^\infty$ be a sequence in $A \otimes B$. Choosing a subsequence if it is necessary, we can suppose that $\{x^n\}_{n=1}^\infty$ has limit x in $\sigma((\lambda_k^p)', \lambda_k^p)$ and $\{y^n\}_{n=1}^\infty$ has limit y in $\sigma((\mu_k^r)', \mu_k^r)$. If

$$f \in \mathcal{B}((\lambda^p)^\alpha, (\mu^r)^\alpha) = \mathcal{L}((\lambda^p)^\alpha, \mu^r)$$

(arguing as in 1) and 2)), let f_k be the restriction of f to $(\lambda_k^p)'$. As $(\lambda_k^p)'$ is isomorphic to $\ell^{p'}$ with $p' = p/(p-1)$, μ_k^r is isomorphic to ℓ^r and $p' > r$ by hypothesis, by Pitt theorem, $I_k \circ f_k \in \mathcal{L}((\lambda_k^p)', \mu_k^r)$ is compact. Then by Schauder theorem on compactness of the adjoint of $I_k \circ f_k$ and by separability of λ_k^p, the sequence $\{(I_k \circ f_k)(x^n)\}_{n=1}^\infty$ converges to $(I_k \circ f_k)(x)$ in μ_k^r. As $\{y^n\}_{n=1}^\infty$ is a bounded sequence in $(\mu_k^r)'$, from

$$< x^n \otimes y^n - x \otimes y, f > = < (x^n - x) \otimes y^n, f > + < x \otimes (y^n - y), f > =$$

$$= < y^n, f(x^n - x) > + < y^n - y, f(x) > = < y^n, (I_k \circ f_k)(x^n - x) > +$$

$$+ < y^n - y, f(x) >$$

we deduce that $\{x^n \otimes y^n\}_{n=1}^\infty$ is weakly convergent to $x \otimes y$.

4) If λ^p is a Montel space, the proof is the same as in 1) and 2) replacing λ^1 of 1) and 2) by λ^p. Then the result is also proved if μ^r is Montel. Conversely: if neither λ^p nor μ^r are Montel spaces, there are complemented sectional subspaces F and G of λ^p and μ^r such that $\lambda^p = \ell^p \oplus F$ and $\mu^r = \ell^r \oplus G$. Hence, if $p' = p/(p-1)$ and $r' = r/(r-1)$, we have $(\lambda^p)^\alpha = \ell^{p'} \oplus F'$ and $(\mu^r)^\alpha = \ell^{r'} \oplus G'$. As $p' \leqslant r$, by theorem 3, $\ell^{p'} \hat{\otimes}_\pi \ell^{r'}$ is not a reflexive subspace of $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$. In consequence, $(\lambda^p)^\alpha \hat{\otimes}_\pi (\mu^r)^\alpha$ is not reflexive. q.e.d.

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Prof. Dr. J.A. López Molina
Cátedra de Matemáticas
E.T.S.I.A. de Córdoba.
Finca Alameda del Obispo s.n.
Córdoba. Spain.